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A Comparative study on mathematical models for interest rate dynamics: a Kenyan case study.

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A Comparative Study on Mathematical Models for Interest Rate Dynamics: A Kenyan Case Study

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Declaration

I declare that this work has not been previously submitted and approved for the award of a degree by this or any other University. To the best of my knowledge and belief, the dissertation contains no material previously published or written by another person except where due reference is made in the dissertation itself.

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List of Abbreviations

ARCH: Autoregressive Conditional Heteroskedasticity

CBK: Central Bank of Kenya

CBR: Central Bank Rate

CEV: Constant Elasticity of Variance

CIR: Cox–Ingersoll–Ross

CIR SR: Cox–Ingersoll–Ross Square Root

CIR VR: Cox–Ingersoll–Ross Variable Rate

CKLS: Chan, Karolyi, Longstaff and Sanders

FRA: Forward Rate Agreements

GARCH: Generalized Autoregressive Conditional Heteroskedasticity

GMM: Generalized Method of Moments

MLE: Maximum Likelihood Estimation

OLS: Ordinary Least Squares

SDE: Stochastic Differential Equation

TB: Treasury Bill

Abstract

This dissertation calibrates equilibrium one-factor short-term interest rate models to the evolution of interest rate dynamics in Kenya. The aim of the study is to find out which one-factor short-rate model best captures the dynamics of the short-term interest rate in Kenya . Additionally, the study aims to evaluate the relationship between conditional volatility of interest rate changes and the level of interest rate. The findings of this study provide a basis for valuation of contingent claims and hedging of interest rate risk.

The data used in the study was obtained from the Central Bank of Kenya (CBK) website [1] for the period between January 2005 to July 2016. Since the short-term interest rate is unobservable in the market the 91-day Treasury Bill (TB) rate was used as its proxy. The Generalized Method of Moments (GMM) estimation technique was used to obtain the parameters for all the models under study.

Key results showed that there is weak evidence of mean reversion for all the models evaluated. Furthermore, it was established that there exists a positive relationship between interest rate volatility and the level of interest rate. The best performing model from the study is determined to be the Chan, Karolyi, Longstaff and Sanders (CKLS) model which allows the volatility of interest rate changes to be highly dependent on the level of the interest rate. This model also has the best volatility forecasting ability among the models under study. It is therefore recommended to interest rate policy makers for use in their work.

Keywords: Generalized Method of Moments (GMM), calibration, short-term interest rates

[1] https://www.centralbank.go.ke/bills-bonds/treasury-bills/

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Dedication

This is for my loving parents, Paul and Joyce Maina for believing in me and for encouraging me to push for excellence in my academics. I also dedicate this to my siblings, Florence and Daniel, for being by my side through this journey. Next, I dedicate this to my very special aunt, Lucy, for always being there through this journey. Lastly, I would like to dedicate this to my friends for always cheering me on.

Chapter 1

Introduction

1.1 Background to the study

Interest rates play a crucial role in the development of an economy. An interest rate is the rate charged to a borrower to obtain access to capital. The idea of time value of money, whereby a certain amount of money earned today is worth more than the same amount in the future, is derived from interest rates. It is critical to model and estimate interest rate dynamics for the pricing of financial assets such as bonds, options, and other derivatives. Modeling of interest rate dynamics is also crucial for interest rate risk management. According to Cuchiero (2006) managing interest rate risk presents a definite level of complexity that is best dealt with in formal mathematical modeling. Due to the importance of understanding interest rate dynamics and the extensive areas of application many models have been developed to explain the dynamic behavior of the short-term interest rate. Some of the more popular models include those by Brennan and Schwartz (1977,1979,1980), Vasicek (1977), Cox, Ingersoll, and Ross (1980, 1985), Hull and White (1990), Longstaff and Schwartz (1992) among others. Despite the existence of a large number of interest rate models very little has been done to evaluate the performance of such models in Kenya.

There are three categories of classical interest rate models namely: short-rate models, forward rate models, and market models. In this work we present a study of mathematical models of the short-rate. We focus on mathematical aspects of the models and also discretization and calibration of the models to historical data to find out which model best suits interest rates in the Kenyan market. The best model will be used to simulate and fit several distributions on the Kenyan short-term interest rates.

1.1.1 Interest rate decisions in Kenya

The Monetary Policy Committee (MPC) of the Central Bank of Kenya makes interest rates decisions in Kenya. In August 2005 the Central Bank Rate (CBR) replaced the 91-day Treasury Bill (TB) rate as the official interest rate. The Central Bank Rate is reviewed at least every two months. The rate has remained relatively stable ranging between 5.75% to 18%. In this work, however, we will use the 91-day TB rate as a proxy for the short-rate for purposes of modelling.

1.1.2 Overview of interest rate modelling

Interest rate models are fundamental to the pricing of fixed income and derivative financial instruments. Mathematical interest rate models have been derived from the need to sufficiently model and manage the risk presented by future movements in interest rates. Skovmand and Verhofen (2007) show that various interest rate models have been developed in an attempt to answer the questions of; 1) which quantities and dynamics should be modelled? 2) How should their randomness be modelled? And 3) what are the valuation consequences of the different approaches?

This research focuses on equilibrium one-factor short-rate models which are based on the presumption that interest rate changes are driven by changes in a single underlying random factor. These models were chosen because they are easy to implement as they have closed form analytic solutions. Arbitrage free models present a good class of models that allow direct modeling of observable option prices. In the Kenyan context, however, data on option prices is not available, which limits the use of arbitrage free models. Under short-rate modelling, the interest rate, r_t is modelled as a stochastic differential equation with a drift and a diffusion component. Each short-rate model specifies a different evolution for the dynamics of r_t.

1.2 Problem statement

Modeling and estimating the dynamics of interest rates is crucial in the pricing of financial instruments and managing of interest rate risk. While there exists a bewildering array of interest rate models, very little has been done to assess the performance of such models in Kenya.

This research aims to contribute to both the theory and practice of interest rate modeling in Kenya. This evaluation will involve parameter estimation of the various short-rate models to find out which model best captures the actual behavior of the short-term riskless rate.

The results of this analysis will form a good foundation for pricing and risk management for interest rate dependent securities and derivatives. This study seeks to determine whether there is a model that works best in estimating interest rates in the Kenyan market.

1.3 Research objectives

This research seeks to answer questions related to modeling of the short-rate in Kenya. The study seeks to provide an answer to the primary question of which is the best model in capturing the dynamics of the short-rate in the Kenyan economy.

1.3.1 Specific objectives

The main aims of this study can be summarised as:

- Identification of short-rate mathematical models for calibration.
- Evaluation of the relationship between conditional volatility of interest rate changes and the level of interest rate.
- Simulation of short-rates using the best model.

1.4 Significance of the study

The global financial market is largely made up of fixed-income instruments such as bonds, forward rate agreements(FRAs), swaps, caps, floors, options and swaptions which require interest rate modelling for pricing. This is in line with a report by Katie et al. (2020) which shows that the global equity market value as at 2019 stood at $95,041 Billions while the bond market value was $105,914.40 Billions. Furthermore, the global derivatives market outstanding (notional principal) stood at $654,317.70 Billions in the same period. Therefore, modelling the short-rate is an obvious concern to investors. It is also crucial for financial analysts, policymakers, and academics because interest rates affect business capital costs, household finances, government debt, and ultimately the health of the overall economy. Furthermore, the derivatives market in Kenya is at an infant stage and it will be critical to model interest rates for pricing of interest rate dependent instruments in the future.

1.5 Outline of the study

This thesis is divided into five chapters. Chapter 1 is the introduction, which provides the basic outline of the study including the background to the study, problem statement, research objectives and the significance of the study. Chapter 2 reviews literature about interest rate theories and various studies conducted on the dynamics of interest rates. Chapter 3 discusses the theoretical foundation of various short-term interest rate models, and also the calibration of the models to the empirical data. Chapter 4 is on analysis and discussions of the results obtained from calibration of the various short-rate models to the historical 91-day Treasury Bill yields data. Chapter 5 is the conclusion of the study which in addition provides suggestions for future studies.

Chapter 2

Literature review

2.1 Interest rate theories

There are several theories proposed to characterize interest rates dynamics. Generally, these theories are based on economics and are reviewed as follows.

2.1.1 Productivity theory of interest

The productivity theory of interest, expounded by Clark (1899), postulates that interest is the reward paid to capital for its productivity. Nonetheless, this theory is one sided since it overlooks capital supply. If there is ample capital supply, then, however high capital productivity is, the question of interest will not occur. Moreover, every industry has its own different capital productivity level and thus interest rates should differ from industry to industry. The theory is also flawed in the sense that it is strenuous to exactly quantify capital productivity. Finally, the theory does not account for interest paid for consumption loans.

2.1.2 Abstinence or waiting theory of interest

The abstinence or waiting theory of interest was best expressed by Senior and Whately (1938). In this theory capital is the result of saving and when people save they "abstain" from instant consumption. Consequently, to encourage saving some form of incentive must be offered as a reward for waiting. The waiting theory, therefore, postulates that interest is the compensation for abstinence from immediate consumption. The theory, however, does not account for capital productivity. It is also subjective since the real cost of saving cannot be measured.

2.1.3 Classical theory of interest

The classical theory of interest, propounded by Marshall (1961), posits that interest rate is determined by the demand and supply of capital. Demand for capital is inversely proportional to the rate of interest while supply of capital is directly proportional to the rate of interest. The equilibrium interest rate is thus determined at the point which demand and supply of capital intersect. This theory is narrow in its scope since it only takes into account capital used for production ignoring consumption loans.

2.1.4 Liquidity preference theory of interest

The liquidity preference theory, advance by Keynes et al. (1971), postulates that the rate of interest is the compensation for surrendering liquidity. Liquidity is defined as the convenience of holding cash. According to Keynes et al. (1971) people yearn for liquidity for different motives. These motives are transaction motive, precautionary motive, and speculative motive. The relationship between preference of liquidity and the rate of interest is inverse. On the other hand, money supply is not affected by interest rate and thus remains constant in the short run. The interaction between money supply and liquidity preference determine the rate of interest. Nonetheless, this theory ignores the element of saving without which there can be no liquidity to part with. It also ignores productivity of capital and concentrates only on the short-run.

2.2 Term structure of interest rates

The term structure of interest rates is defined as the relationship between yields of bonds with different maturity terms. When interest rates of such bonds are plotted against their terms the resulting graph is the yield curve. According to Mishkin (2007) the yield curve can be catalogued as downward sloping, flat or upward sloping reflecting the market's future interest rates expectation and monetary policy conditions.

When the yield curve is sloping upward, the short-term interest rates, such as 91-day Treasury bill rate, are below the long-term rates, such as 10 years government bond. When the yield curve is sloping downward, the short-term interest rates are above the long-term rates. When the curve is flat, the short-term and long-term interest rates are equal. Various theories have been developed to explain the shape of the yield curve. The three main theories are the expectations hypothesis, the segmented market theory and the liquidity premium theory which will be explained below.

2.2.1 Expectations hypothesis

This hypothesis posits that the rate of interest on a long-term bond will be equal to the average of the short-term interest rates expected to occur over the life of the long-term bond. Therefore there exists a positive correlation between the yield curve and the short-term interest rates. When the yield curve is sloping upward the expectation is that short-rates will grow in the future and vice-versa. Given that long-term interest rates are presently above the short-rates, the average of future short-rates is expected to grow. When the yield curve is flat the short-rates are not expected to change in the future. The expectations hypothesis also explains the mean-reverting nature of interest rates. Under the expectations hypothesis framework short-term interest rates have more volatility than long-term rates. The major drawback of the expectations hypothesis is that it fails to explain the upward slope of the yield curve.

2.2.2 Segmented market theory

The segmented market theory postulates that credit markets are segmented, disjointed and distinct. Mishkin (1990) interprets this to mean that the interest rate on each bond with a different maturity is determined by the demand and supply of that bond with no dependence on expected returns on other bonds with other maturities. This theory states that investors are risk-averse such that they prefer short-term bonds therefore explaining why yield curves are usually upward-sloping.

2.2.3 Liquidity premium theory

This theory combines the expectations hypothesis and the segmented market theory explained above. It states that the interest rate on a long-term bond will equal an average of short-term interest rates expected to occur over the life of the long-term bond plus a premium that acts in response to demand and supply conditions for that bond. The theory supposes that investors are risk-averse and thus they will demand a premium for long-term bonds as reward for interest rate risk.

2.3 Short-term interest rates modelling

The most common approach to short-rate modelling is to assume that the short-term interest rates follow a continuous time Markov process. Merton (1973) was the first to propose a general stochastic process as a model for the short-rate. His model belongs to the Gaussian sub-class of affine models and the short-term rate follows an arithmetic Brownian motion.The model assumes a constant risk premium and allows for negative values of interest rates. The Vasicek (1977) model was the first to make a substantial impact on modelling of interest rates and as such it holds a special place in the interest rate term structure literature. The model depends on an arbitrage argument and makes assumptions about the stochastic evolution of interest rates by exogenous specification of the short-term interest rate process.

Dothan (1978) introduces a model that relates the volatility of interest rate movements more strongly to the level of interest rates. Under this model the term structure is a monotonically decreasing function of time to maturity, an increasing function of interest rates and a decreasing convex function of volatility. Brennan and Schwartz (1980) extend Dothan's model by adding a mean reverting term. In this model the distribution for the short-rate is unknown and contingent claim prices must be computed using numerical methods. Rendleman (1980) introduce a model that assumes that the short-rate follows a Geometric Brownian Motion with constant drift and diffusion parameters. This model is alternatively known as the Geometric Brownian motion (GBM) model.

A later approach used by Cox et al. (1985) specifies an equilibrium economy which forms the foundation for specification of the model. The model assumes stochastic evolution of exogenous state variables and investor preferences. The equilibrium economy assists in endogenous derivation of the form of the short-rate and hence the prices of contingent claims. Bond prices are derived from exogenous specification of the economy which is provided for by production opportunities, investor tastes and beliefs about future states of the world making the CIR a complete equilibrium model. Ho and Lee (1986) create an arbitrage-free term structure model by using the existing term structure as an input. In their model the short-rate is related to forward rates with an additional random factor and it is presented as a binomial tree. By introducing a time-dependent drift Hull and White (1990) extend the models of Vasicek and CIR to be free of arbitrage. This culminates to a model that is consistent with current market prices of bonds. Black et al. (1990) developed a model that takes as inputs both the existing term structure of zero-coupon yields and the term structure of yield volatilities for the same bonds using a binomial lattice framework. As the model alters the implied interest rate distribution matches an observed interest rate volatility curve at each time step.

2.4 Empirical literature

In their paper Chan et al. (1992) evaluate and analyze a range of continuous time of short-rate models using the Generalized Method of Moments to find out which model best fits the short term interest rate data. They use one-month Treasury bill yield data covering the period June 1964 to December 1989. They observe that the most effective models in capturing the short-rate dynamics are those that allow for high sensitivity of interest rate changes volatility to the level of interest rate. In their study several reputable short-rate models exhibit poor results due to their restrictions on the volatility of the term structure. For the models that capture the conditional volatility of the short-rate process the authors find no evidence of a structural shift in October 1979. They demonstrate that the results of their analysis have necessary ramifications for the use of various short-rate models in hedging of interest rate risk and also in pricing of interest rate contingent claims.

Ait-Sahalia (1996) investigate whether short term and long term interest rates follow a continuous time Markov process given that only discrete-time rates are available. The authors are motivated by the need to affirm, or disqualify the prevailing technique of modeling interest rate factors as continuous-time diffusions. To do this they test an important and sufficient restriction on diffusions conditional densities at the sampling interval of the observed data. They find that for the period 1857 to 1995 neither short-term nor long-term interest rates follow Markov processes individually but jointly they form a Markovian system. They also demonstrate that the yield curve slope is both a diffusion and a univariate Markov process.

Chapman and Pearson (2001) present an extensive review of term structure models. In their study they establish that mean reversion is weak at best for a wide array of interest rate models. They also find that there is no consensus on whether mean reversion is stronger for very low or very high interest rates. Furthermore, they find out that within normal interest rate ranges there exists a positive correlation between volatility of interest rates and the level of interest rates. They recommend more research to be carried out so as to figure out which interest rate model is best.

Nowman (2011) uses Gaussian estimation econometric methods to evaluate the stochastic differential equation models for the interest rate dynamics of the United Kingdom bond market. His work used monthly data over the period from 1970 to 2010 utilizing a variety of maturities. The results of single and two equation models Gaussian estimates suggest that the volatility of rates depend on the level of rates across the maturities. Moreover, the study observes that there is no empirical evidence of mean reversion in the interest rates in the market. In addition, the popular CIR-SR and CKLS models have empirical support in the United Kingdom bond market.

Ben Salah and Abid (2012) perform a study using American 91-day Treasury bill rate as a proxy for the short-rate to establish the best short-rate model. They investigate eight various short-rate models with the goal of scrutinizing the effect of mean reversion, the importance of particular aspects of the stochastic process of the short-term interest rates and the susceptibility of the volatility to the level of interest rate. Due to parametric specification of the mean and volatility of the diffusion process none of the models under study reproduce the actual path of the short-rates. Furthermore, there is high sensitivity of volatility to the level of interest rates. The results also demonstrate that the effect of mean reversion is not statistically significance.

Chakroun and Abid (2014) establish a methodology for the estimation of the interest rate yield curve in the Tunisian bond market using Treasury bond prices for the period from 14 July 2004 to 10 September 2012 . This market is considered to be an illiquid market since it has a low trading volume. To solve the problem of missing observations they implement the cubic spline interpolation method. Their study estimates and examines the performance of the CIR SR and Vasicek models in predicting the dynamics of the interest rate yield curve using the maximum likelihood estimation (MLE) and the ordinary least squares (OLS) methods. The results demonstrate the accuracy of the cubic spline method in constructing the average interest rate yield curve. Moreover, parameter estimates of the models generate an upward sloping yield curves and in particular the Vasicek model replicates the stylised facts of the short-rates better in the Tunisian bond market.

In their study Zhao and Wang (2017) investigate the performance of the CIR and the Vasicek short-rate models in the United Kingdom, the United States, and New Zealand. Their goal is to establish whether each of the countries under study can have its own short-rate model that best depicts the behaviour of the historical short-term interest rate data. They carry out GMM estimation on the historical data of the 91-day Treasury bill yield rate from the United Kingdom, the 91-day Treasury bill secondary market rate from the United States, and the 30-day bank bill yield rate from New Zealand. Depending on goodness-of-fit test they conclude that the CIR model is a better fit for the real data than the Vasicek model. They also establish that both the CIR and Vasicek models can mirror the dynamics of the interest rates in long term but they are unable to predict the same dynamic movement.

Interest rate modeling in Kenya so far has focused on more descriptive approaches. Few models have focused on empirical evaluation of interest rate models using market data.

To make sense of interest rate dynamics, Ngugi (2001) concentrates on the factors influencing the interest rate spread in Kenya's banking sector. Her work focused on the widening of the interest rate spread in the period after liberalization and she demonstrates that the spread increases significantly due to poor policy as well as economic conditions. Furthermore, the study also establishes that interest rate spread varies due to bank attempts to maintain vulnerable profit margins. To curb this variation in the interest spread the study recommends reform of the legal system and also introduction of fiscal and monetary policies that target financial markets development.

Olweny (2011) focuses on the relationship between short-rate volatility and the level of interest rates in Kenya. To perform the analysis he uses the 91-day Treasury bill rates from August 1991 to December 2007. The findings of the study indicate that there is positive correlation between volatility and the level of the short-term interest rate. He also establishes that when it comes to modeling the volatility of short-term interest rates in Kenya, GARCH models perform better than ARCH models. This is because they are capable of capturing the crucial phenomena of volatility clustering reported in several financial time series inclusive of short-rates.

Caporale and Gil-Alana (2016), while studying Kenyan interest rate dynamics, investigate the stochastic properties of the interest rate spread to provide handy information about the effects of shocks and proper policy responses. To perform the analysis they use data on four bank interest rates (deposits, savings, lending and overdraft) together with the 91-day Treasury bill rates from July 1991 to August 2010. The results of the study demonstrate that all series investigated are nonstationary. The orders of integration for the examined models are equal to or higher than 1 when using parametric methods and slightly smaller than 1 when using semiparametric techniques. The results also suggest that, regardless of the estimation method used, the T-bill rate is likely to exhibit mean reversion, whereas evidence for mean reversion is weak for the commercial bank rates.

In his study Chelimo (2017) uses the ordinary least squares (OLS) regression to calibrate the Vasicek short-rate model to the evolution of interest dynamics in Kenya. The study uses both a single-state and a multi-state model in a Hidden Markov context. The 3-month Treasury bill rate is used as a proxy for the short-term interest rate for calibration. The findings of the study demonstrate that an increase in the number of states results in the increase in the mean reversion parameter. This suggests that model parameters become more stable with introduction of regimes. Furthermore, volatility is not affected by the level of the interest rate. The study therefore suggests that interest rate models should incorporate regime switches.

2.5 Summary of literature review

The objective of this study is to provide an answer to the primary question of which is the best model in capturing the dynamics of the short-term riskless rate in the Kenyan economy. The specific models under study include those by Merton (1973), Vasicek (1977), Dothan (1978), Brennan and Schwartz (1980), Rendleman (1980), Cox (1975), Cox, Ingersoll, and Ross (1980,1985), and Chan, Karolyi, Longstaff, and Sanders (1992).

Past studies by Chan et al. (1992), Chapman and Pearson (2001), Nowman (2011), and Ben Salah and Abid (2012) show that there exists a positive correlation between volatility of interest rates and the level of interest rates. According to studies by Chapman and Pearson (2001), Nowman (2011), and Ben Salah and Abid (2012) there is weak evidence of mean reversion for a wide array of interest rate models.

Furthermore, Nowman (2011) finds that the popular CIR SR and CKLS models have empirical support in the United Kingdom bond market. In their study Zhao and Wang (2017) investigate the performance of the CIR SR and the Vasicek short-rate models in the United Kingdom, the United States, and New Zealand. Depending on goodness-of-fit test they conclude that the CIR SR model is a better fit for the real data than the Vasicek model. On the other hand Chakroun and Abid (2014) find that the Vasicek model replicates the stylised facts of the short-rates better than the CIR SR model. in the Tunisian bond market.

In Kenya interest rate modeling has so far focused on more descriptive approaches with few studies focusing on empirical evaluation of interest rate models using market data. Olweny (2011) finds that there is positive correlation between volatility and the level of the short-term interest rate in the 91-day Treasury bill rates from August 1991 to December 2007. In his study Chelimo (2017) uses the ordinary least squares (OLS) regression to calibrate the Vasicek short-rate model to the evolution of interest dynamics in Kenya incorporating regime switches. The results of the study suggest that model parameters become more stable with introduction of regimes. Furthermore, volatility is not affected by the level of the interest rate. The study therefore suggests that interest rate models should incorporate regime switches.

It is clear from the above studies that little has been done in terms of empirical analysis of short-term interest rate models in Kenya. In particular no study has been conducted to evaluate the ability of short-rate models to capture the dynamics of the short-term riskless rate. This is the research gap that this study seeks to address.

Chapter 3

Research methodology

3.1 Data collection

The main data collection method for this study will be through secondary means particularly the Central Bank of Kenya website. This dissertation uses historical T-bills rates for the period January 2005 to July 2016. The rationale for selecting this time period is because the rates in the period before 2005 were highly volatile.

3.2 Data validity and reliability

The data used in this study is obtained directly from the Central Bank of Kenya website. It is monthly and presented as percentage per annum. Therefore, the reliability and validity of the data are beyond question.

3.3 Short-rate models

Key concepts on the short-rate are expounded upon following derivations and relationships that are available in Brigo and Mercurio (2007), Björk (2009) and Musiela and Rutkowski (2005a).

A risk-free security, compounded continuously at a risk-free rate considered to be the short-rate is defined as follows: Let r_t denote the riskless rate of borrowing or lending at time t over the infinitesimal time interval $[t, t + dt]$. The rate r_t is assumed to be an adapted[2] process on a filtered probability space $(\Omega, \mathcal{F}, \mathrm{P}, (\mathcal{F}(t)_{0<t<T}))$ for some $T > 0$, with almost all sample paths

[2]An adapted process is one that cannot "see into the future". An easy interpretation is that X is adapted if and only if, for every realisation and every t, X_t is known at time t. In other words a stochastic process process X is adapted if X_t is an $\mathcal{F}_t$-measurable random variable for each time $t \geq 0$.

integrable on $[0, T]$. The value of the bank account at $t > 0$, defined as $B_t = B(t, \omega)$, evolves for almost all $\omega \in \Omega$ according to the differential equation

$$dB_t = r_t B_t dt, \text{ with } B_0 = 1. \tag{1}$$

Consequently,

$$B_t = \exp\left(\int_0^t r_s ds\right) \quad \text{for all} \quad t \in [0, T]. \tag{2}$$

The general form of the model for the dynamics of the short-rate is as follows:

$$dr_t = \mu(r, t)dt + \sigma(r, t)dW_t. \tag{3}$$

In this study, equilibrium one-factor short-rate models will be used. The drift and volatility of these models depend only on interest rates and not time. The dynamics for the short-rate of these equilibrium models can be nested within the following stochastic differential equation:

$$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^\gamma dW_t. \tag{4}$$

where $-\beta$ is the speed of mean reversion, α is the product of β and the long-run equilibrium interest rate θ, γ dictates the sensitivity of the change in interest rates to the current level, and σ is a scaling parameter.

The models to be evaluated are derived from equation (4) through parametric restrictions. They are presented in table 1 below:

Table 1: Parameter restrictions imposed by alternative short-rate models.

Model	**Dynamics**	**Restrictions**	**Degrees of freedom**
1. CKLS	$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^\gamma dW_t$	Unrestricted	0
2. Merton	$dr_t = \alpha dt + \sigma dW_t$	$\beta = 0$, $\gamma = 0$	2
3. Vasicek	$dr_t = (\alpha + \beta r_t)dt + \sigma dW_t$	$\gamma = 0$	1
4. CIR SR	$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^{\frac{1}{2}} dW_t$	$\gamma = \frac{1}{2}$	1
5. Dothan	$dr_t = \sigma r_t dW_t$	$\alpha = \beta = 0$, $\gamma = 1$	3
6. GBM	$dr_t = \beta r_t dt + \sigma r_t dW_t$	$\alpha = 0$, $\gamma = 1$	2
7. Brennan-Schwartz	$dr_t = (\alpha + \beta r_t)dt + \sigma r_t dW_t$	$\gamma = 1$	1
8. CIR VR	$dr_t = \sigma r_t^{\frac{3}{2}} dW_t$	$\alpha = \beta = 0$, $\gamma = \frac{3}{2}$	3
9. CEV	$dr_t = \beta r_t dt + \sigma r_t^\gamma dW_t$	$\alpha = 0$	1

Model 1 is an "unrestricted" version of the single-factor interest rate diffusion processes in discrete time, estimated by Chan et al. (1992). Model 2 is used to derive a model of discount bond prices by Merton (1973). Model 3 is known as the Ornstein-Uhlenbeck process derived by Vasicek (1977). It is used in deriving an equilibrium model of discount bond prices. Model 4 is the square root (SR) process, which appears in Cox et al. (1985). Model 5 is used by Dothan (1978) in valuing discount bonds. Model 6 is the geometric Brownian motion (GBM) process developed by Rendleman (1980). Model 7 is used by Brennan and Schwartz (1980) in deriving a numerical model for convertible bond prices. Model 8 is introduced by Cox et al. (1980) in their study of variable-rate (VR) securities. Model 9 is a version of the constant elasticity of variance (CEV) process developed by Cox (1975).

3.4 Forward rate models

Chiarella et al. (2016) define the forward rate by considering an investor who holds a bond maturing at T_1 and asking what return he or she would earn between T_1 and $T_2 (> T1)$,if he or she contracted now at time t. The required rate of return is the forward rate $f(t, T_1, T_2)$ given by

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1} \ln \left[\frac{P(t, T_1)}{P(t, T_2)} \right], \tag{5}$$

where $P(t, T_1)$ denotes the price at time t of a pure discount bond paying 1 at time T_1 and $P(t, T_2)$ denotes the price at time t of a pure discount bond paying 1 at time T_2.

The instantaneous forward rate, defined by $f(t, T) \equiv f(t, T, T)$, is the instantaneous rate of return the bond holder can earn by extending the investment an instant beyond T. The relationship between the price of the pure discount bond and the forward rate can be expressed as follows:

$$P(t, T) = e^{-\int_t^T f(t,s)ds}. \tag{6}$$

According to Heath et al. (1990) the dynamics of the forward rate $f(t, T)$ are

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t, \tag{7}$$

where α and σ are adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively. For any fixed maturity date T, the initial condition $f(0, T)$ is determined by the current value of the continuously compounded forward rate for the future date T that prevails at time 0.

Musiela and Rutkowski (2005b) postulate that the short-term interest rate satisfies $r(t) =$

$f(t,t)$ for every t and consequently, the value of the bank account equals

$$B(t) = \exp\left(\int_0^t f(u,u)du\right). \tag{8}$$

The drawback of instantaneous forward rates is the fact that they are not observable in the market. This property makes both calibration of the forward rate models and pricing of complex derivatives difficult. This study, however, is not concerned with the calibration of forward rate models.

3.5 Calibration of the models

Several methods of estimation such as the Maximum Likelihood Estimation (MLE) can be used for calibration of the models . It is natural to estimate parameters using the MLE method by assuming the diffusion part of the models follow a Markovian process and also assuming prior knowledge of the distribution. However, the distribution of the diffusion part of most of the nested models under study is unknown and with development of research on interest rate models it is convenient to use the Generalized Method of Moments (GMM). The GMM is used by many scholars, such as Chan et al. (1992) and Hansen and Scheinkman (1995) to estimate parameters.

This study therefore incorporates the Generalized Method of Moments (GMM) developed by Hansen (1982) to calibrate the short-rate models and examine their explanatory power for the dynamics of the short-term interest rates. This method is advantageous in that it does not require that the distribution of interest rate changes be normal and it can produce unbiased parameter estimates. It also produces consistent parameters in the sense that having a sufficient number of observations, the estimator will converge in probability to the true value of the parameter. Moreover, it is an asymptotically efficient estimator for weights that are chosen optimally. This means that (asymptotically) it has minimum variance among all estimators .

3.5.1 Generalized Method of Moments estimation

To estimate parameters of the short rate models using GMM we have to discretize the SDE. Chan et al. (1992) applied Euler discretization scheme on equation (4) and obtained:

$$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{t+1}, \tag{9}$$

$$\varepsilon_{t+1} = \sigma r_t^{\gamma} \mathcal{N}(0,1), \tag{10}$$

$$\text{where } \mathrm{E}[\varepsilon_{t+1}] = 0 \text{ and } \mathrm{E}[\varepsilon_{t+1}^2] = \sigma^2 r_t^{2\gamma}. \tag{11}$$

Assuming that the error term ε_{t+1} is uncorrelated with the explanatory variable r_t and utilizing equations (11) we can derive four moment functions:

$$f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ r_t \end{bmatrix} = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1} r_t \\ \varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma} \\ (\varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma}) r_t \end{bmatrix}. \tag{12}$$

The moment functions are constructed so that

$$\mathrm{E}[f_t(\theta)] = 0. \tag{13}$$

Equations (13) are known as moment conditions of moments $f_t(\theta)$. The idea of GMM is to fit these moment conditions in our data set. The corresponding sample moments are defined as:

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} f_t(\theta), \tag{14}$$

where T is the number of observations. Equations (14) are the natural sample counterparts for the population moments $\mathrm{E}[f_t(\theta)]$. Reparametizing equation (9) and employing time step Δt turns equations (9) and (10) into:

$$r_{t+1} = a + b r_t + \varepsilon_{t+1}, \tag{15}$$

$$\varepsilon_{t+1} = \sigma r_t^{\gamma} \sqrt{\Delta t} \mathcal{N}(0,1), \tag{16}$$

where $a = \alpha \Delta t$ and $b = 1 + \beta \Delta t$. The sample moments are as follows:

$$g_T(\theta) = \begin{bmatrix} \Sigma_{t=1}^{T}(r_{t+1} - a - b r_t) \\ \Sigma_{t=1}^{T}((r_{t+1} - a - b r_t)^2 - \sigma^2 r_t^{2\gamma} \Delta t) \\ \Sigma_{t=1}^{T}(r_{t+1} - a - b r_t) r_t \\ \Sigma_{t=1}^{T}((r_{t+1} - a - b r_t)^2 - \sigma^2 r_t^{2\gamma} \Delta t) r_t \end{bmatrix}. \tag{17}$$

The GMM objective function is defined as:

$$J_T = g_T'(\theta) W_T g_T(\theta), \tag{18}$$

where W_T is a positive-definite weighting matrix. Parameter estimates are found by solving

$$\hat{\theta} = \arg\min_{\theta} J_T. \tag{19}$$

GMM sets the weighted sum of squared sample moments as close to zero as possible since their population counterparts are constructed to be equal to zero. The weighting matrix tells us how much attention to pay to each moment. The CKLS model has four parameters and four moment conditions thus it is exactly identified and as such we pick an arbitrary positive-definitive W_T. This means that we can set all sample moments exactly equal to zero.

The rest of the models have less parameters than moment conditions meaning they are over-identified. Therefore, it is not possible to set all sample moment conditions equal to zero. To obtain asymptotically efficient estimates we set

$$W_T = \hat{S}^{-1}, \tag{20}$$

where $\hat{S}$ is an estimate of the spectral density matrix of population moment functions. This choice of the weighting matrix secures the smallest asymptotic covariance matrix of the estimate of θ.

The spectral density matrix is defined as:

$$S = \sum_{j=-\infty}^{\infty} \mathrm{E}[f_t(\theta) f'_{t-j}(\theta)]. \tag{21}$$

The spectral density matrix allows for serial correlation and heteroskedasticity in the observations of the moments function. A popular consistent estimate of the spectral density matrix is the Newey-West estimator given by:

$$\hat{S} = \hat{S}_0 + \sum_{j=1}^{\infty}(1 - \frac{j}{k+1})(\hat{S}_j + \hat{S}'_j), \tag{22}$$

where

$$\hat{S}_j = \frac{1}{T}\sum_{t=j+1}^{T} f_t(\theta) f'_{t-j}(\theta).$$

For over-identified models, GMM is a two stage estimator. We proceed as follows:

1. We minimize (18) using identity weighting matrix ($W_T = I$). This means that we consider all moments equally important. We plug estimated parameter vector θ into (22) and inverse to get W_T.

2. Again, we minimize (18), but this time using the W_T from the first step.

3.5.2 Testing the significance of individual parameters

Under the exactly identified GMM, we have asymptotic normality for the estimated parameters:

$$\sqrt{T}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, (d' S^{-1} d)^{-1}), \tag{23}$$

where d is the Jacobian of the population moments vector $\mathrm{E}[f_t(\theta)]$. For constructing the test statistics, we substitute in the corresponding sample moments and evaluate them at the estimated parameters. For S, we substitute in our estimate (22) of the spectral density matrix, that is our weighting matrix W_T.

3.5.3 Overall test of model adequacy

If the model is over-identified, GMM offers an overall test of the model by testing whether the "extra" sample moments are sufficiently close to zero relative to their distribution. Hansen's (1982) J-test of over-identifying restrictions is the primary tool for determining whether the data are consistent with the hypothesised short-rate model.

We consider two hypotheses for this test

- $H_0 \ : g_T(\theta) = 0$ (the null hypothesis that the model is "valid"), and
- $H_1 \ : g_T(\theta) \neq 0$ (the alternative hypothesis that the model is "invalid", the data does not come close to meeting the restrictions)

Under hypothesis H_0 the test statistic is asymptotically χ^2 distributed with $m - p$ degrees of freedom:

$$T \times J_T(\hat{\theta}) = T \times g_T^{'}(\hat{\theta}) W_T(\hat{\theta}) g_T(\hat{\theta}) \xrightarrow{d} \chi^2_{m-p}, \tag{24}$$

where m is a number of moment conditions and p is a number of parameters.
Under the alternative hypothesis H_1, the J-statistic is asymptotically unbounded:

$$J_T(\hat{\theta}) \xrightarrow{p} \infty. \tag{25}$$

To perform the test we compute the value of J from the data and compare it with (for example) the 0.95 quantile of the χ^2_{m-p} distribution:

- H_0 is rejected at 95% confidence level if $J > q_{0.95}^{\chi^2_{m-p}}$,
- H_0 cannot be rejected at 95% confidence level if $J < q_{0.95}^{\chi^2_{m-p}}$.

If the null hypothesis is rejected then the underlying model that generated the system of moment conditions is declared invalid.

3.6 Simulation

In this study we will use the Euler approximation developed by Euler (1845) to simulate short-rates using the best model. The Euler approximation method considers a stochastic process X_t

satisfying the stochastic differential equation of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{26}$$

where the initial value X_0 is fixed. Given the functions $\mu(t, X_t)$ and $\sigma(t, X_t)$, the stochastic process X_t is a solution of the SDE (26) if X_t solves the integral equation

$$X_t = X_0 + \int_0^t \mu(t, X_s)ds + \int_0^t \sigma(t, X_s)dW_s. \tag{27}$$

However, in practice not many SDEs are explicitly solvable like the above one and therefore sometimes we are unable to get an analytical solution X_t to a given SDE. But we could always get approximate numerical solution of the given SDE. Define $\hat{X}$ which is the time discretized approximation to X. We also discretize the time interval $[0, T]$ by letting $\Delta t = T/N$ and $t_n = n\Delta t, n = 0, 1, 2, \cdots, N$. The exact solution on the time grid $0 = t_0 < t_1 < \cdots < t_N =$ T where $\hat{X}_0 = X_0$ would be

$$X_{t_{n+1}} = X_{t_n} + \int_{t_n}^{t_{n+1}} \mu(t, X_s)ds + \int_{t_n}^{t_{n+1}} \sigma(t, X_s)dW_s. \tag{28}$$

For $i = 0, 1, \cdots, N-1$ the Euler approximation on the time grid $0 = t_0 < t_1 < \cdots < t_N =$ T is given by

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + \mu(t_i, \hat{X}_{t_i})[t_{i+1} - t_i] + \sigma(t_i, \hat{X}_{t_i})\sqrt{t_{i+1} - t_i} Z_{i+1}, \tag{29}$$

where Z is a standard normal random variable. Therefore, the Euler approximation of the nested short rate models SDE,

$$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^\gamma dW_t,$$

is given by

$$\hat{r}_{t_{i+1}} = \hat{r}_{t_i} + (\alpha + \beta \hat{r}_{t_i})[t_{i+1} - t_i] + \sigma \hat{r}_{t_i}^\gamma \sqrt{t_{i+1} - t_i} Z_{i+1}. \tag{30}$$

The Euler approximation scheme is weakly convergent with order 1 and strongly convergent with order 0.5.

Chapter 4

Analysis and discussion

4.1 Data description

This research uses the 91-day Treasury bill rate as a proxy for the instantaneous short rate to estimate the parameters of single factor short rate models. Chapman et al. (1999) show that parameter estimate errors resulting from using the 30-day or 91-day Treasury bill yield data as a proxy for the unobservable short-rate are economically insignificant. The sample is a data-set of 139 monthly observations covering the period from January 2005 to July 2016. The data is obtained from the Central Bank of Kenya website.

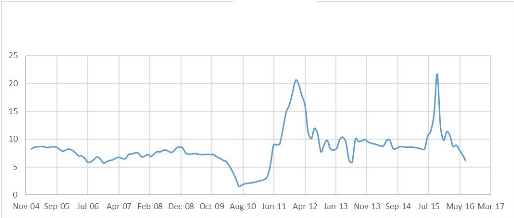
4.2 Data analysis

The descriptive statistics of the data are presented in table 2 below:

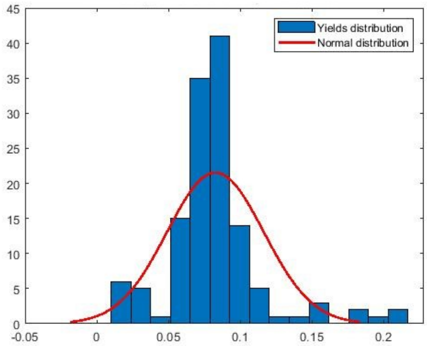
Table 2: Summary statistics of monthly Treasury bill rates and rate changes from January 2005 to July 2016.

Variables	**N**	**Mean**	**Std**	**Skewness**	**Kurtosis**	**Min**	**Median**	**Max**
r_t	139	8.2963	3.3122	1.3471	4.0450	1.6000	8.1500	21.6500
$r_{t+1} - r_t$	138	-0.0152	1.5045	-0.9184	13.6322	-9.3100	0.0000	7.0400

Table 2 above cites a variety of descriptive statistics. The short-rates vary from 1.6% to 21.65% within the covered period. The unconditional mean and standard deviation are 8.2963% and .3122% respectively. Table 2 also shows that the skewness of the short-rate is non-zero and positive, therefore it is not normally distributed and skewed to the right. The skewness of the

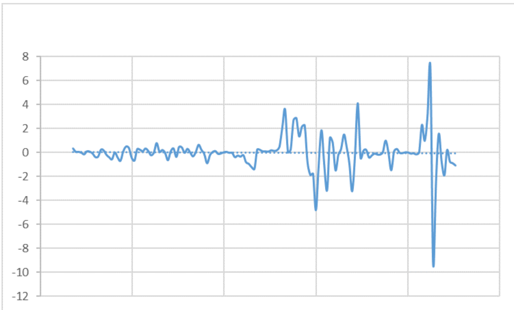


(a)

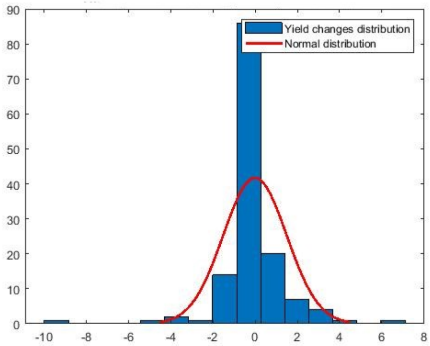


(b)

Figure 1: Time series plot and Histogram of 91-day T-bill rates



(a)

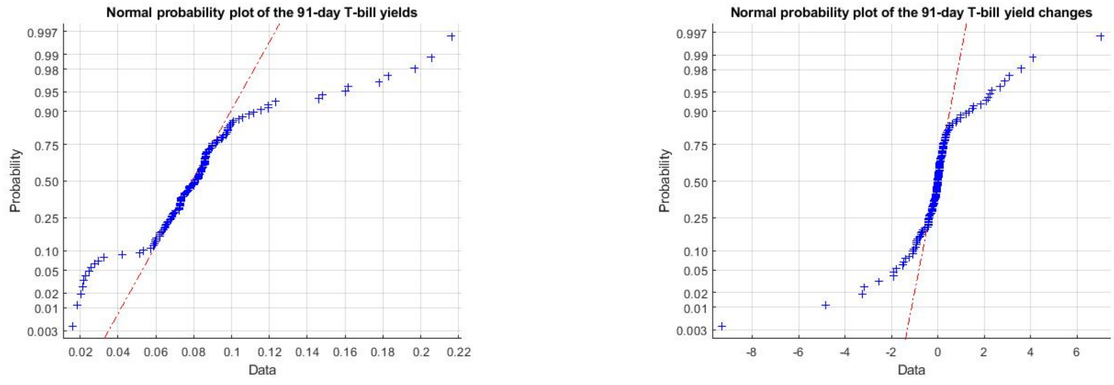


(b)

Figure 2: Time series plot and Histogram of 91-day T-bill yield changes

rate changes is non-zero and negative, therefore it is is not normally distributed and skewed to the left. Moreover, kurtosis is more than 3, meaning that the distribution of the data is leptokurtic categorized by heavy tails on either side, indicating large outliers.

To intuitively show the characteristics of the short-rate we plot the histogram of the data and the basic trend of both the T-bill rates and the yield monthly changes. The time series plot of the short-rates seems to suggest that they are mean reverting to the level of between 5% to 10% p.a. The histogram shows that the short-rate is skewed to the right. The plot of the rate changes shows that variation in interest rates increases with increase in the short-rate. Its histogram shows that the variation in short-rates is slightly skewed to the left. Normal probability plots of the two datasets also confirm that they are not normally distributed.



(a) (b)

Figure 3: Normal probability plots of the of 91-day T-bill rates and yield changes

4.3 Empirical results

The results of estimated parameters (α, β, σ^2, γ) for each model are shown in table 3 below. GMM asymptotic standard errors are calculated using the heteroskedasticity-and-autocorrelation consistent covariance matrix of Andrews (1991), with t-statistics reported in parentheses. Overidentified models are tested for adequacy using the J-test of Hansen (1982) and is reported with p-value in parentheses.

Table 3: GMM estimates of short-rate models.

Model	α	β	σ^2	γ	$J_T(\hat{\theta})$ **(p-value)**	d.f.	R_1^2	R_2^2
CKLS	0.0991	-1.2145	5.0435	1.6356			0.0499	0.2081
	(1.64)	(-1.61)	(0.51)	(2.74)				
Merton	0.0031	0.0000	0.0012	0.0000	1.9851	2	0.0000	0.0000
	(0.23)		(1.79)		(0.3706)			
Vasicek	0.0605	-0.8115	0.0012	0.0000	1.8692	1	0.0223	0.0000
	(1.31)	(-1.33)	(1.82)		(0.1716)			
CIR SR	0.05429	-0.7133	0.0170	0.5000	1.7892	1	0.0172	0.0028
	(1.21)	(-1.22)	(2.14)		(0.1810)			
Dothan	0.0000	0.00000	0.2063	1.0000	2.1112	3	0.0000	0.0200
			(2.64)		(0.5497)			
GBM	0.0000	0.0054	0.2041	1.0000	2.3165	2	0.0000	0.0195
		(0.03)	(2.54)		(0.3140)			
Brennan-Schwartz	0.0613	-0.7512	0.2259	1.0000	1.1393	1	0.0191	0.0239
	(1.29)	(-1.25)	(2.72)		(0.2858)			
CIR VR	0.0000	0.0000	2.2008	1.5000	2.0204	3	0.0000	0.0943
			(3.30)		(0.5682)			
CEV	0.0000	-0.0010	3.6715	1.6196	2.2218	1	0.0000	0.1222
		(-0.01)	(0.32)	(1.66)	(0.1361)			

4.4 Model comparison

Table 3 gives the report of parameter estimates, asymptotic t-statistics, and J-statistics with their p-values for the nine models. Each of the models imposes restriction(s) on the parameters of the short-rate model thereby affecting estimates of other parameters. The R_j^2 statistics are computed as the proportion of total variation of actual rate changes for $j = 1$ and the volatility explained by the respective predictive value for $j = 2$ for every model.

4.4.1 Significance of parameters

All parameter estimates have a t-statistic value marked in the parentheses in table 3 above. The critical value for our observation sample at 0.05 significance level is 1.960. Based on the results of GMM estimation there is little evidence of statistical significance of the parameters α and β in all the models since the t-statistics are below the critical value at 95% confidence level. This gives weak evidence of mean reversion for all the models. All σ^2 parameter estimates of all the models except the CKLS model are statistically significant. The γ parameter estimate under the CKLS is the most highly significant of all the parameter estimates. This parameter determines the level of heteroskedasticity in interest rate changes.

4.4.2 Mean reversion

We can convert the estimated parameters (α, β , σ^2, and γ) into the economically interesting θ, κ, and σ by letting

$$dr_t = k(\theta - r_t)dt + \sigma r_t^{\gamma} dW_t \tag{31}$$

where $\theta = \frac{\alpha}{\kappa}$ is the long-run mean, $\kappa = -\beta$ is the speed of mean reversion, $\frac{-1}{\beta}$ is the period for reversion to mean in the absence of noise and σ is the volatility. Models with restriction of 0 on parameters α and/or β imply no mean reversion. This means that the only mean reverting models are CKLS, Vasicek, CIR SR, and Brennan-Schwartz.

Table 4: Mean reversion parameters of the short-rate models.

Model	θ	κ	σ	γ	Average conditional volatility
CKLS	8.1614	1.2145	2.2458	1.6356	1.1971
Vasicek	7.4565	0.8115	0.0341	0.0000	09838
CIR SR	7.6105	0.7133	0.1304	0.5000	10640
Brennan-Schwartz	8.1655	0.7512	0.4753	1.0000	101404

Table 4 above shows that CKLS model has long-run mean, $\theta = 8.1614\%$, speed of adjustment, $\kappa = 1.2145$, volatility, $\sigma = 2.2458$, conditional volatility, $\gamma = 1.6356$ and average conditional volatility = 1.1971%. The model has a period of 0.8 years for reversion to mean. The Vasicek model has long-run mean, $\theta = 7.4565\%$, speed of adjustment, $\kappa = 0.8115$, volatility, $\sigma = 0.0341$, conditional volatility, $\gamma = 0.0000$ and average conditional volatility = 0.9838%. This model has a period of 1.2 years for reversion to mean. CIR SR model has long-run mean,

$\theta = 7.6105\%$, speed of adjustment, $\kappa = 0.7133$, volatility, $\sigma = 0.1304$, conditional volatility, $\gamma = 0.50000$ and average conditional volatility = 1.0640%. The model has a period of 1.4 years for reversion to mean. Finally, the Brennan-Schwartz model has long-run mean, $\theta = 8.1655\%$, speed of adjustment, $\kappa = 0.7512$, volatility, $\sigma = 0.4753$, conditional volatility, $\gamma = 1.0000$ and average conditional volatility = 1.1404%. This model has a period of 1.3 years for reversion to mean. Therefore, the CKLS model has the strongest mean reversion qualities among the four mean-reverting models.

4.4.3 Model specification

Hansen's (1982) J-test provides a goodness-of-fit test by allowing for the ability to test the specification of the models. If the model is misspecified and/or some of the moment conditions do not hold, then the J-statistic will be large relative to a χ^2 random variable with $m - p$ degrees of freedom. As shown in table 3, the J-test for goodness-of-fit suggest that all models are correctly specified. The critical values at 95% confidence level for 1, 2 and 3 degrees of freedom are 3.841, 5.991 and 7.815 respectively. The short-rate models have $J_T(\theta)$ values below the critical values and therefore, they cannot be rejected at 95% confidence level. All these models cannot be rejected even at 90% confidence level.

4.4.4 Volatility

To further evaluate the relative performance of the models under study, we test their power to forecast interest rate changes. Additionally, their power to forecast squared interest rate changes is also tested. The squared interest rate changes provide simple ex post measures of interest rate volatility. Table 3 shows that the models vary in their explanation power for interest rate changes. This explanation power is given by the coefficient of determination, R^2 measure. The first R^2 metric specifies the fit of the models under study to the actual rate changes while the second R^2 measure describes the fit to volatility of rate changes.

The Merton, Dothan, GBM, CIR VR and CEV models have no explanatory power for interest rate changes. Their R_1^2 measures are zero since the parameters α and/or β are restricted to zero. Therefore, the level equation cannot be estimated. The remaining models have R_1^2 measures varying from 0.0172 to 0.0499. The CKLS model explains 4.99% of the total variation in yield changes emerging the best in that category. Based on R_2^2 values, the best volatility model is the CKLS model. The model captures 20.81% of the total variation in volatility. The R_2^2 values for the Merton and Vasicek models are zero implying constant volatiltiy of interest rate changes. To further demonstrate these results we compare the actual volatility versus forecasted volatility for all the short-rate models and plot them below. The CKLS model is plotted against all the other models.

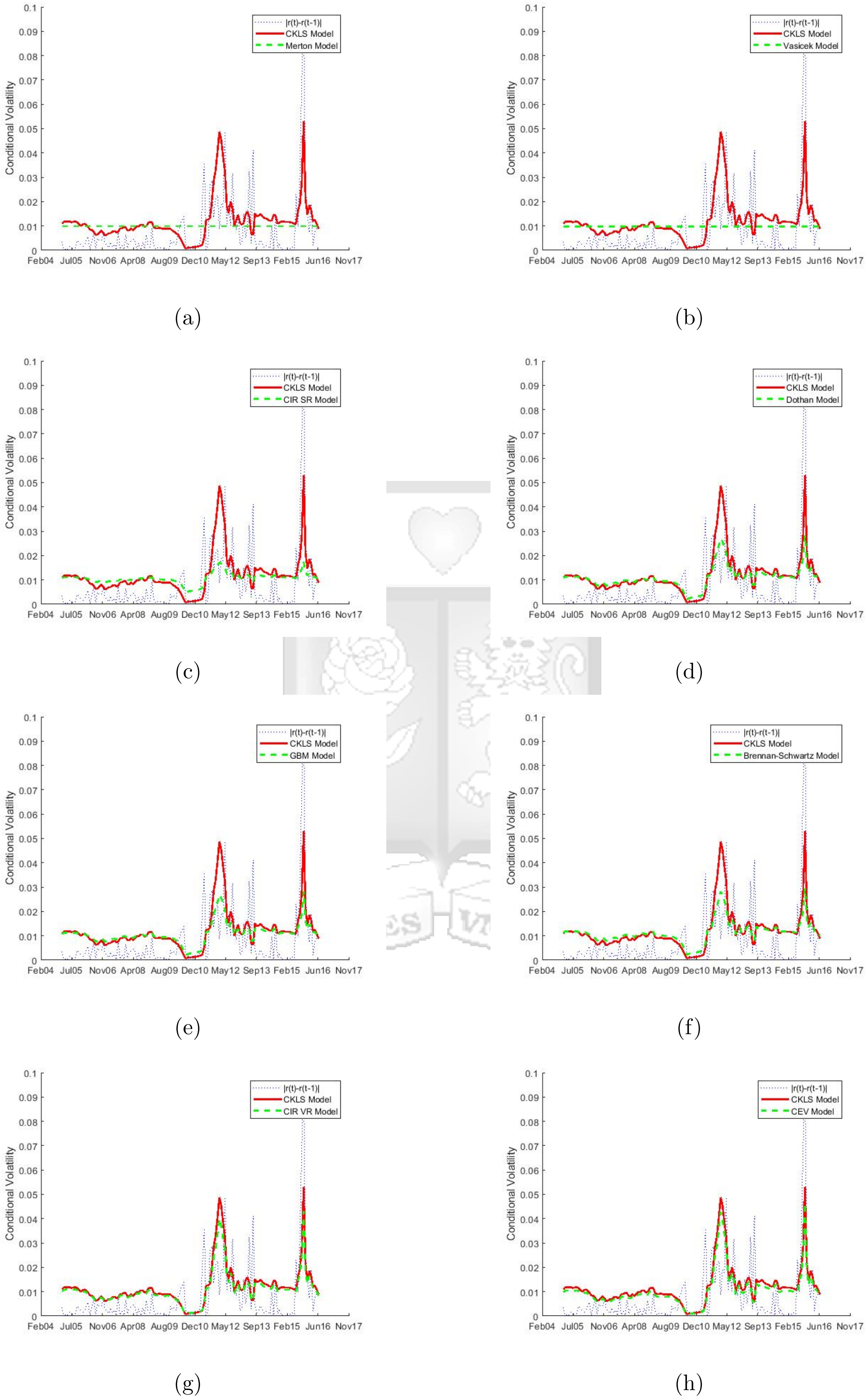


Figure 4: Forecasts of monthly ex post volatility of the short-rate using models under study

4.5 Simulation results

The 91-day T-bill rates are plotted together with the simulated CKLS trajectories obtained using the Euler approximation method. The simulation results are presented in the figure below:

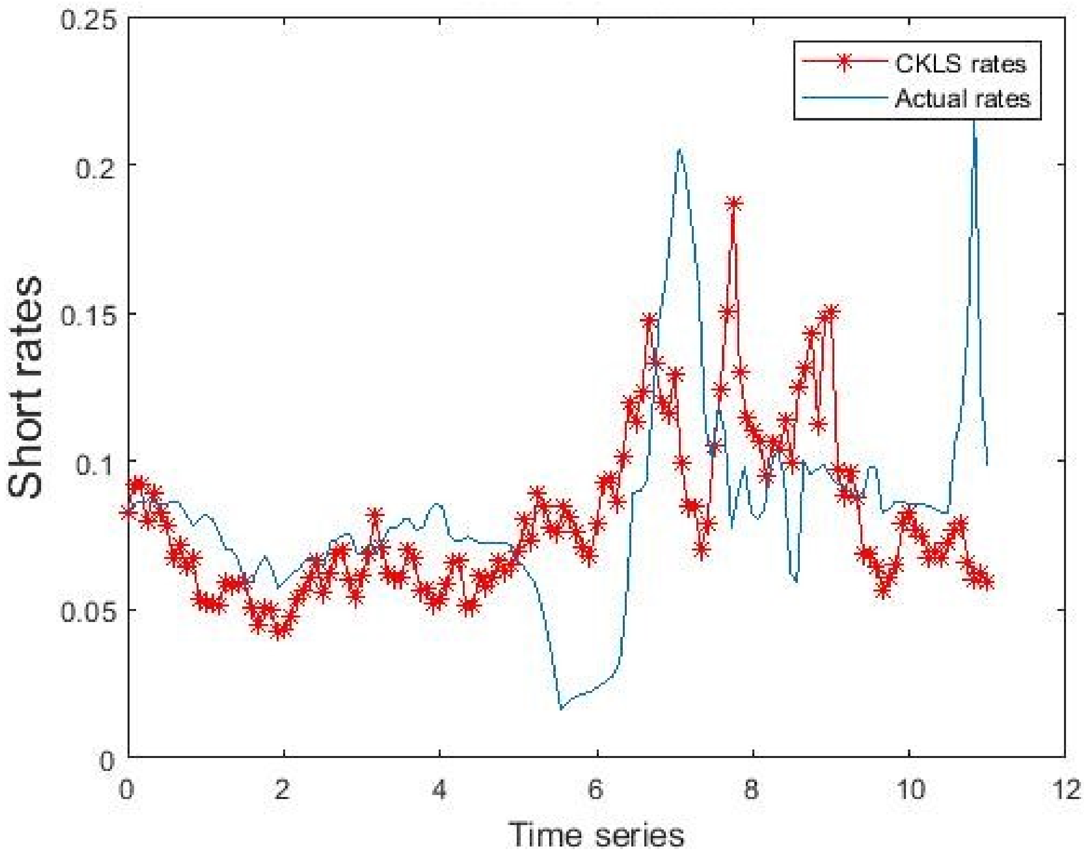


Figure 5: T-bill rates and the simulated CKLS rates by using the Euler approximation method

From figure 5 above we can see that the CKLS simulated rates share the properties of the 91-day T-bill rates data of non-negative interest rates and that the fluctuation of data is larger for larger interest rates. From normal probability plot we find out that the CKLS simulated rates are not normally distributed. This is similar to the actual 91-day T-bill rates.

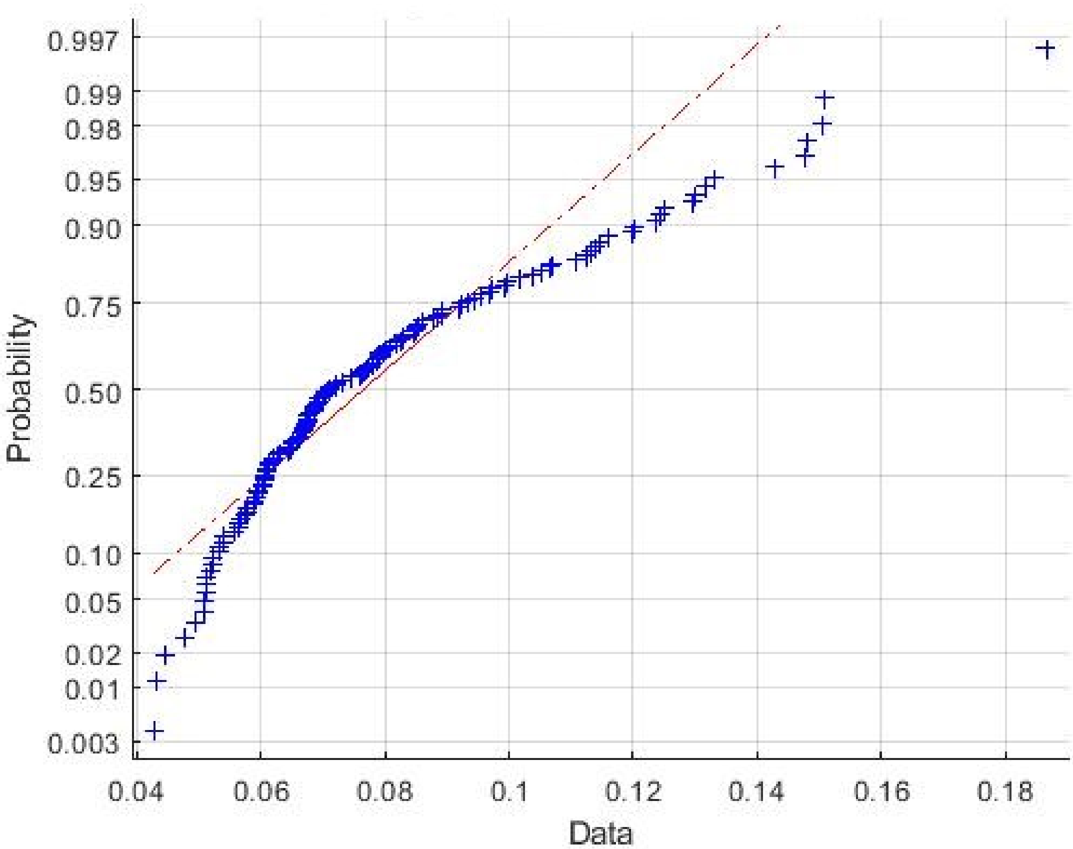


Figure 6: Normal probability plot of the simulated CKLS rates

Finally, we plot a histogram of the simulated rates and fit several distributions to better visualize the data. Just like the actual rates the CKLS simulated rates are skewed to the right.

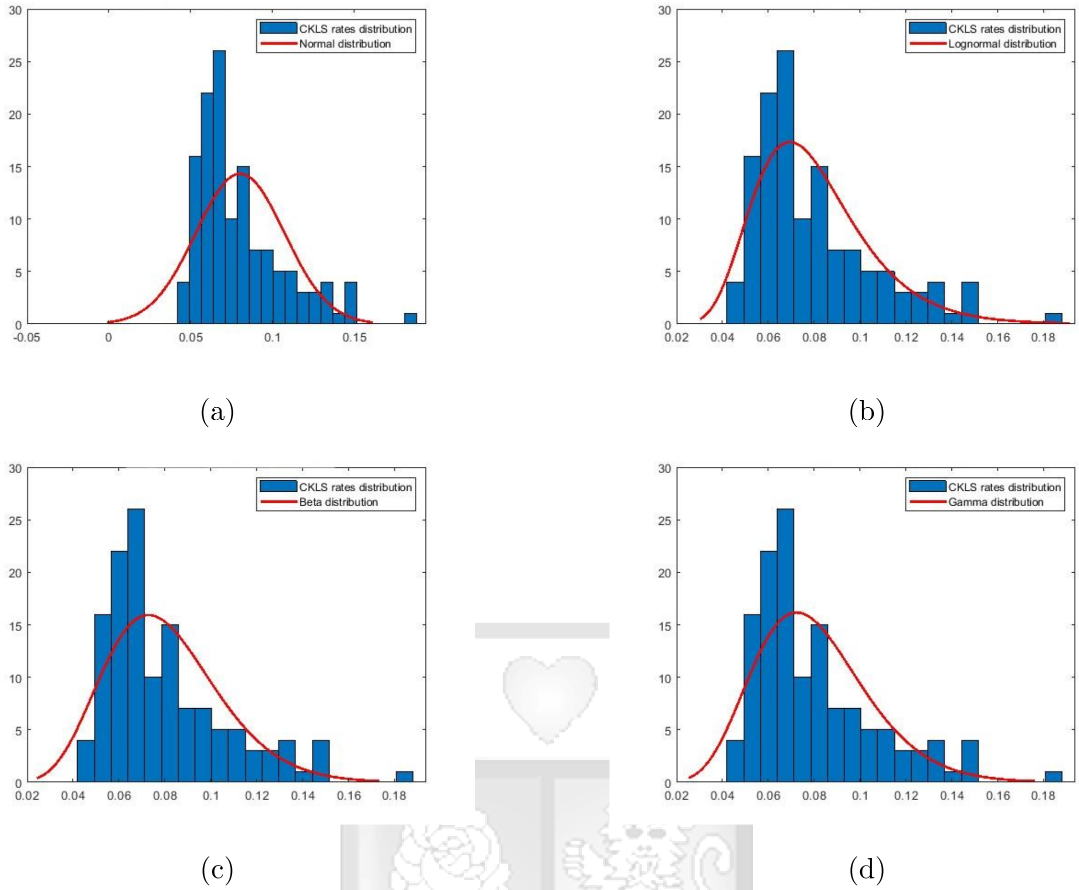


Figure 7: Histogram of CKLS simulated rates fitted with several distributions

Chapter 5

Conclusion

This research has discussed the calibration of various equilibrium one-factor short-rate models to the Kenyan 91-day Treasury bill yield data. The Generalized Method of Moments (GMM) has been used for estimation of parameters. From the t-statistics of the parameters α and β there is weak evidence of mean reversion for all the models evaluated. This raises questions over the ability of GMM estimation method to detect mean reversion. The goodness-of-fit test indicate that the CKLS, Merton, Vasicek, CIR SR, Dothan, GBM, Brennan-Schwartz, CIR VR and CEV models cannot be rejected at 95% confidence level. However, the model that best fits the Kenyan short-term 91-day Treasury bill yield data is the CKLS model. This model has a γ parameter estimate of 1.6356 which is much higher than the value used in most of the other models. Particularly, six of the models under study suggest that $0 \leq \gamma \leq 1$. This parameter is highly significant in the CKLS model since it has a t-statistic value of 2.74 This implies that the conditional volatility of the CKLS process is highly dependent on the level of the interest rate. Furthermore, the CKLS model has the best volatility forecasting ability among the models under study. The results also confirm that interest rate volatility is dependent on the level of interest rate.

5.1 Recommendations for further research

One of the most important features of the term structure is the dependence of its volatility on the level of interest rate. Therefore, future work on term structure models in the Kenyan context should focus on this dependency since interest rate volatility is of fundamental importance in valuing contingent claims and hedging interest rate risk. Furthermore, the results of this research can be extended and applied to different multi-factor models of interest rates with implications on bond and option pricing.

This study focused on calibration of equilibrium one-factor short-rate models using the GMM estimation method. Other methods of estimation should be used and compared in future studies. Moreover, future studies should evaluate the ability of multi-factor models to capture the dynamics of the Kenyan short-term interest rates. Furthermore, they should consider using a variety of econometric techniques including efficient method of moments and nonparametric estimators. This research relied on the assumption of constant volatility which is not always realistic. For that reason, future studies should consider modelling stochastic volatility. Data analysis also showed that it has leptokurtic behaviour. This kind of behaviour is often modelled using Jump models. On that account, Jump models should be used to capture these stylised facts. Other features of financial variables such as regime switches could also be modelled by future studies.

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