Strathmore Institute of Mathematical Sciences
Master of Science in Statistics
End of Semester Examination
STA 8202: Stochastic Processes
Date: -/04/2024 **Duration: 3 Hours**

Instructions:

1. *Answer* ***Question 1 (Compulsory)*** *and* ***any other two*** *questions.*
2. *Show all your workings clearly in the answer sheet.*

Question 1 (30 Marks)

(a) Define the following terms as used in stochastic processes:

(i) Traffic intensity [1 Mark]

(ii) Trajectory probability [1 Mark]

(iii) Branching Process [1 Mark]

(iv) A Markov chain [1 Mark]

(b) Let X_n be the total number of children produced by the n^{th} generation. Assuming each of those children will have an average μ children of its own, and that the number of children are independent, then

$$E[X_{n+1}|X_n = x_n, \cdots, X_1 = x_1, X_0 = x_0] = \mu x_n$$

The sequence $\{X_0, X_1, \cdots\} < x_n$ is a martingale if $\mu = 1$. Prove that for any μ,

$$z_0, z_1, \cdots ; \quad z_n = \frac{x_n}{\mu^n}$$

is also a martingale. [4 Marks].

(c) Consider a steady state in performance measure of a simple queuing system *(M/M/1)* with arrival rate λ and service rate μ.
Let $P_n(t) = P$ [n customers in the system at time t]. Show that:

(i) Expected number of customers in the system, $E(n)$, is given by

$$E(n) = \frac{\lambda}{\mu - \lambda} \qquad [3\ Marks].$$

(ii) Average length of the queue, $E(m)$, is given by

$$E(m) = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

where $m = n - 1$ is the number of customers in the queue excluding the customer being served [3 Marks].

(d) A car is sent to a garage for a major overhaul. There are three operations to be carried out sequentially: *engine tune-up, air-conditioning overhaul*, and *braking system replacement*. The mean times for these three operations are 1.2, 1.5, and 2.5 hours, respectively. Assume that the times are mutually independent and there are no delays between operations. If the time to perform each operation follows an exponential distribution, what is the probability that four hours later the car is in the braking system replacement stage? [5 marks]

(e) Let $\{X_n, n \geq 0\}$ be a Markov chain with three states $0, 1, 2$ and with transition matrix given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} \end{array}$$

and the initial distribution $P_r\{X_0 = i\} = \frac{1}{3}$, $i = 0, 1, 2$.

(i) Obtain $P_r\{X_2 = 2, X_1 = 1, X_0 = 2\}$ [3 Marks]

(ii) Find the probability of obtaining the trajectory $(2, 2, 0, 0, 2)$. [2 Marks].

(f) Let Y be a random variable with probability density function given by

$$P_r\{Y = k\} = p_k, \quad k = 0, 1, 2$$

with probability generating function given by

$$\Psi(s) = \sum_{k=0}^{\infty} p_k s^k, \quad and$$

$$q_k = P\{Y > k\} = p_{k+1} + p_{k+2} + \cdots, k = 0, 1, 2, \cdots$$

with generating function given by

$$\Pi(s) = \sum_{k=0}^{\infty} q_k s^k, where \; \sum_k q_k \neq 1$$

Express $\Pi(s)$ in terms of $\Psi(s)$. [6 Marks]

Question 2 (15 Marks)

(a) Customers arrive at a facility according to a Poisson process of rate λ. Each customer pays \$1 on arrival, and it is desired to evaluate the expected value of the total sum collected during the interval $(0, t)$ discounted back to time 0. This quantity is given by

$$M = E\left[\sum_{X=1}^{X(t)} e^{-\beta W_k}\right]$$

where β is the discount rate, $W_1, W_2, \cdots$, are the arrival times, and $X(t)$ is the total number of arrivals in $(0, t)$. Evaluate the mean total discounted sum M by conditioning on $X(t) = n$. [5 Marks]

(b) Assuming *M/M/1 Queuing Model*, show that the sum of waiting time probabilities is equal to 1. [4 Marks]

(c) If a state of a Markov chain e_j is accessible from a persistent state e_i, then e_i is also accessible from e_j, and moreover e_j is persistent. Prove this assertion. [6 Marks]

Question 3 (15 Marks)

A Markov chain with non-negative integers as its state space has transition matrix given by:

$$\begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} E_0 & E_1 & E_2 & E_3 & E_4 & \cdots \end{array} \\ \left[\begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & \cdots & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots & & \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & \cdots & & \\ 0 & 0 & \frac{7}{8} & 0 & \frac{1}{8} & \cdots & & \\ 0 & 0 & 0 & \frac{15}{16} & 0 & \frac{1}{16} & 0 & \cdots \\ & \cdots & & & & & & \end{array}\right] \end{array}$$

(a) Write down the formula for the transition probabilities. [3 Marks]

(b) Let the stationary vector be $[V_0, V_1, V_2, V_3, \cdots]$. Write a difference-differential equation giving V_x in terms of V_{x-1} and V_{x+1}. [4 Marks]

(c) Find $P_{33}^{(2)}$ [3 Marks]

(d) Let $G(s) = \sum_{j=0}^{\infty} V_j s^j$. Show that $G(s) = p(s)G(\frac{1}{2}s)$, where $p(s)$ is a polynomial in s. [5 Marks]

Question 4 (15 Marks)

Consider the difference-differential equations of the *pure birth process* given by:

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), \quad n \geq 1$$

$$P_0'(t) = -\lambda_0 P_0(t), \quad n = 0$$

(a) The *Poisson process* is obtained by substituting $\lambda_n = \lambda$ in the difference-differential equations. Using *Feller's Method*, show that the mean and variance of the Poisson process is equal to λt, where λ implies birth.

[7 Marks].

(b) Substituting $\lambda_n = \lambda\left(\frac{1+an}{1+\lambda at}\right)$ in the difference-differential equations of the *pure birth process*, show that the probability that the population is of size n at time t, $P_n(t)$, for the *Polya process* is given by

$$P_n(t) = (1+\lambda at)^{-\frac{1}{a}} \binom{-\frac{1}{a}}{n} \left(\frac{-\lambda at}{1+\lambda at}\right)^n$$

Use the initial condition

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

[8 Marks].

Question 5 (15 Marks)

(a) Differentiate between the following terms as used in Markov chains:

(i) Periodicity and Ergodicity. [2 Marks]

(ii) Reducible chain and Irreducible chain. [2 Marks]

(b) Consider a homogeneous Markov chain with state space $S = \{V_1, V_2, V_3\}$ as shown below.

$$P_{ij} = \begin{matrix} & V_1 & V_2 & V_3 \\ V_1 & 0 & \frac{1}{2} & \frac{1}{2} \\ V_2 & \frac{1}{2} & 0 & \frac{1}{2} \\ V_3 & \frac{1}{2} & \frac{1}{2} & 0 \end{matrix}$$

Classify the states of the given Markov chain. [5 Marks].

(c) A gambler bets *1 US dollar* on the first game. If he wins he gets a net profit of *1 US dollar* but if he loses he suffers a loss of *negative 1 US dollar* and will bet *2 US dollars* in the next play. If he wins he will get 4 *US dollars* and hence will have made a net profit of *1 US dollar* but if he loses he will bet 4 *US dollars* on the next game and so on.

(i) Develop the gambler's net profit table if the gambler loses 5 times in a row and then wins.

[3 marks]

(ii) Using the martingale technique, develop the net-winning process of the gambler. [3 marks]