Strathmore Institute of Mathematical Sciences
Master of Science in Statistics
End of Semester Examination
STA 8202: Stochastic Processes

Date: Wednesday, 3^{rd} May, 2023 **Time: 5:00 - 8:00 Pm**

Instructions:

1. Answer **Question 1** and **any other two** questions.
2. Show all your workings clearly in the answer sheet.

Question 1 (30 Marks)

(a) Define the following terms as used in stochastic processes:

(i) Hidden Markov Chain. [1 Mark]

(ii) Counting process. [1 Mark]

(iii) Irreducible Markov chain. [1 Mark]

(b) Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β. Let the process be in state 0 when it rains and state 1 when it does not rain.

(i) Write down the transition probabilities of preceding 2-state Markov chain. [1 Mark]

(ii) If $\alpha = 0.7$ and $\beta = 0.4$, obtain the probability that it will rain four days from today given that it is raining today. [3 Marks]

(c) Consider a production process that in each period is either in a good state (state 1) or in a poor state (state 2). If the process is in state 1 during a period, then, independent of the past, with probability 0.9 it will be in state 1 during the next period and with probability 0.1 it will be in state 2. Once in state 2, it remains in that state forever. Suppose that a single item is produced each period and that each item produced when the process is in state 1 is of acceptable quality with probability 0.99, while each item produced when the process is in state 2 is of acceptable quality with probability 0.96.

(i) If the status, either acceptable or unacceptable, of each successive item is observed, while the process states are unobservable, then the preceding is a hidden Markov chain model. The signal is the status of the item produced, and has value either a, or u, depending on whether the item is acceptable or unacceptable. Write down the signal probabilities and transition probabilities of the underlying Markov chain. [3 Marks]

(ii) Let X_n be the current state of the Markov chain and suppose that $P\{X_1 = 1\} = 0.8$. Given that the successive conditions of the first 3 items produced are a, u, a:

(a) What is the probability that the process was in its good state when the third item was produced? [3 Marks]

(b) What is the probability that X_4 is 1? [3 Marks]

(c)) What is the probability that the next item produced is acceptable? [3 Marks]

(d) Let $\{X_n, n \geq 0\}$ be a Markov chain with three states $0, 1, 2$ and with transition matrix given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$

and the initial distribution $P_r\{X_0 = i\} = \frac{1}{3}$, $i = 0, 1, 2$. Obtain:

(i) $P_r\{X_1 = 1/X_0 = 2\}$ [1 Mark]

(ii) $P_r\{X_2 = 2, X_1 = 1/X_0 = 2\}$ [2 Marks]

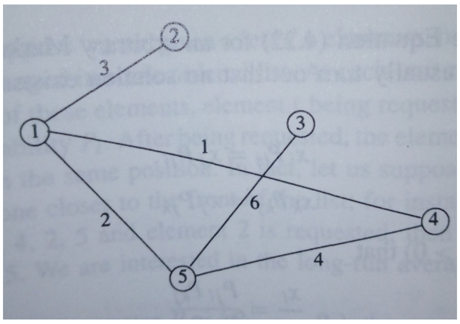
(iii) $P_r\{X_2 = 2, X_1 = 1, X_0 = 2\}$ [2 Marks]

(iv) Find the probability of obtaining the trajectory $(2, 1, 0, 0, 2)$. [2 Marks]

(e) Let $S^n = (s_1, s_2, \cdots, s_n)$ be the random vector of the first n signals. For a fixed sequence of signals $s_1, s_2, \cdots, s_n$, let $S_k = (s_1, s_2, \cdots, s_k)$, $k \leq n$. Determine the conditional probability of the Markov chain state at time n given that $S^n = s_n$. [4 Marks]

Question 2 (15 Marks)

(a) Consider an arbitrary connected graph given below, having a number w_{ij} associated with arc (i, j) for each arc. Consider a particle moving from node to node in this manner. If at any time the particle resides at node i, then it will next move to node j with probability P_{ij} where $P_{ij} = \frac{w_{ij}}{\sum_j w_{ij}}$ and where w_{ij} is 0 if (i, j) is not an arc.



Using time reversibility equations, show that:

(i) $$\pi_i = \frac{\sum_j w_{ij}}{\sum_i \sum_j w_{ij}}$$ [6 Marks]

hence find

(ii) $\pi_1, \quad \pi_2, \quad \pi_3, \quad \pi_4, \quad \pi_5$ [5 Marks]

(b) Let the $t-step$ transition probability P_{ij}^t be the probability that a process in state i will be in state j after t additional transitions, that is

$$P_{ij}^t = P\{X_{t+k} = j / X_k = i\}, \quad t \geq 0, \quad i, j \geq 0$$

Show that

$$P_{ij}^{t+m} = \sum_{k=0}^{\infty} P_{ik}^t P_{kj}^m, \quad t, \ m \geq 0, \forall \, i, j$$

using the $Chapman-Kolmogorov$ method; where P_{ij}^{t+m} is the probability of transiting from i to j in $t + m$ steps. [4 Marks]

Question 3 (15 Marks)

(a) Classify the states for the infinite Markov chain given below. [10 Marks]

$$
\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \\ E_n \end{array}
\begin{array}{c}
\begin{array}{cccccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & \cdots & E_n \end{array} \\
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots & \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots & \\
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \cdots & \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \cdots & \\
\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \cdots &
\end{bmatrix}
\end{array}
$$

(b) A problem of interest to sociologists is to determine the proportion of society that has an upper-or-lower class occupation. One possible mathematical model would be to assume that transitions between social classes of successive generations in a family can be regarded as transitions of a Markov chain. That is, we assume that the occupation of a child depends only on his or her parent's occupation. Let us suppose that such a model is appropriate and that the transition probability matrix is given by

$$
P = \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}
$$

that is, for instance, suppose that the child of a middle-class worker will attain an upper-, middle- or lower-class occupation with respective probabilities 0.05, 0.70, 0.25. Obtain the limiting probabilities, π_i, for the social classes.

[5 Marks]

Question 4 (15 Marks)

(a) A DNA nucleotide has any of 4 values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability $1 - 3\alpha$, and if it does change then it is equally likely to change to any of the other 3 values, for some $0 < \alpha < \frac{1}{3}$.

(i) Show that $P_{1,1}^n = \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n$. [3 Marks]

(ii) What is the long-run proportion of time the chain is in each state? [4 Marks]

(b) Suppose that a production process changes states in accordance with an irreducible, positive recurrent Markov chain having transition probabilities $P_{ij}, i, j = 1, 2, 3, 4$ given below. Suppose also that states 1 and 2 are acceptable (up) and states 3 and 4 are unacceptable (down).

$$P_{ij} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

(i) Obtain the limiting probabilities, $\pi_i, i = 1, 2, 3, 4$ [3 Marks]

(ii) Find the rate of breakdowns (the rate at which it enters unacceptable state from an acceptable one). [3 Marks]

(iii) Find the proportion of time the system is up. [2 Marks]

Question 5 (15 Marks)

(a) Let $S(t)$ denote the price of a security at time t. A popular model for the process $\{S(t), t \geq 0\}$ supposes that the price remains unchanged until a " shock " occurs, at which time the price is multiplied by a random factor. If we let $N(t)$ denote the number of shocks by time t, and let X_i denote the i^{th} multiplicative factor, then this model supposes that

$$S(t) = S(0) \prod_{i=1}^{N(t)} X_i$$

where $\prod_{i=1}^{N(t)} X_i$ is equal to 1 when $N(t) = 0$. Suppose that the X_i are independent exponential random variables with rate μ; that $\{N(t), t \geq 0\}$ is a Poisson process with rate λ; that $\{N(t), t \geq 0\}$ is independent of the X_i; and that $S(0) = s$.

(i) Find $E[s(t)]$ [5 Marks]

(ii) Find $E[s^2(t)]$ [4 Marks]

(b) Individuals join a club in accordance with a Poisson process with rate λ. Each member must pass through k consecutive stages to become a full member of the club. The time it takes to pass through each stage is exponentially distributed with rate μ. Let $N_i(t)$ denote the number of club members at time t who have passed through exactly i stages, $i = 1, \cdots, k-1$. Also let $\vec{N}(t) = (N_1(t), N_2(t), \cdots, N_{k-1}(t))$.

(i) Is $\{N(t), t \geq 0\}$ a continuous-time Markov chain? [1 Mark]

(ii) If the answer in (i) above is yes, give the infinitesimal transition rates. That is, for any $n = (n_1, \cdots, n_{k-1})$, give the possible next states along with their infinitesimal rates. [5 Marks]