Modeling time-dependent Volatility of the USD-KSH Exchange Rate in the Presence of Speculation

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Chapter 1: Introduction

1.1 Background of the study

Financial markets serve as platforms for the exchange of diverse financial assets like stocks, bonds, currencies, derivatives, and commodities. Their core functions encompass capital allocation, price discovery, and liquidity enhancement. However, market stability is frequently disrupted by speculation. Speculation refers to trading activities driven by expectations of future short-term price changes, rather than the underlying value of an asset (Bachelier, 1900). Speculators, such as hedge funds, proprietary trading firms, retail investors, etc., analyze market sentiment, political developments, and economic forecasts to capitalize on short-term price fluctuations and generate profits. By seeking to generate short-term profit, speculators amplify market reactions to both real and perceived news, leading to enhanced market liquidity and heightened volatility (the degree of price fluctuation over time). For instance, the reelection of Donald Trump in 2024 significantly strengthened the US dollar, pushing it to a four-month high (Reuters, 2024). Similarly, the Kenyan Shilling experienced rapid devaluation during the COVID-19 pandemic, exacerbated by local political uncertainty, such as the 2022 general elections and Genz protests, as well as global events like the Russia-Ukraine War and the Israel-Hamas-Hezbollah conflict. These events triggered speculation about commodity prices, particularly food and energy, further destabilizing global and local markets (Financial Stability Report, 2024). Between 2020 and 2024, the USD-KSH exchange rate fluctuated significantly from as low as 100.3 Kenyan Shillings and as high as 160 Kenyan, settling at approximately 129.593 Kenyan Shillings on December 4, 2024 (Central Bank of Kenya, 2024). Such exchange rate fluctuations have led to capital outflows, increased debt distress, and higher public borrowing costs (Financial Stability Report, 2024).

Researchers have proposed various volatility models; the Historical Volatility Model, GARCH model, BSM model, Stochastic Volatility Models and EGARCH model, for analyzing and predicting market volatility. Among these, the EGARCH model, introduced by Nelson (1991), stands out for its ability to capture volatility clustering and respond swiftly to market changes. While the EGARCH model has been successfully applied in various contexts, such as modeling exchange rates for Tanzania by Epaphra (2016) and Kenya by Omari (2017), it exhibits notable limitations. These include computational complexity, a propensity to overfit the data, and difficulty in interpretation due to its numerous parameters and logarithmic transformations. Additionally, the EGARCH model fails to capture the mean-reverting behavior observed in exchange rates which is likely in the long-term and does not explicitly account for the impact of speculation on volatility.

To address some of these limitations, Hendricks and Ongala (2022) proposed a stochastic approach to explicitly model volatility driven by speculation. They modified the Ornstein-Uhlenbeck model by incorporating speculation parameters, resulting in the improved Ornstein-Uhlenbeck model. This model posits that volatility is time-dependent solely during speculative periods, while remaining constant outside of these periods, implying that the degree of price fluctuations during non-speculative periods is assumed to be constant, irrespective of market conditions or time. Historical evidence suggests that Volatility stabilizes downward post-speculation and is therefore not constant. As global conditions improved during the Ukraine-Russia conflict, the Central Bank of Kenya implemented measures to discourage speculative activities, such as raising interest rates to make the Kenyan Shilling more expensive to short(of Kenya, 2024). By the end of 2024, the USD/KES exchange rate had stabilized from a peak of 160 to around 129, though it remained weaker than pre-war levels. This suggests that volatility grows to a maximum during speculative periods and then decreases post-speculation until it attains a stable, constant level. While the enhanced OU model demonstrated an improved capacity to capture speculative dynamics, its assumption of constant volatility after speculative episodes restricts its ability to reflect the inherently dynamic nature of real-world market volatility that stabilizes downward, emphasizing the need for a more adaptable volatility modeling framework.

This study proposes to build on the improved Ornstein-Uhlenbeck model by relaxing the assumption of constant volatility after speculative periods. Instead, we will assume that volatility after speculation is time-variant, following an exponential decay function. This enhancement aims to address the limitations of the improved OU model, offering a more realistic and adaptable framework for analyzing speculation-driven volatility.

1.2 Basic concepts

1.2.1 Geometric Brownian Motion (GBM)

The GBM model was first introduced by Bachelier (1900) in his doctoral thesis titled "Théorie de la spéculation". He claimed that stock prices mimic an arithmetic Brownian motion, which allows for the possibility of negative prices. This was considered to be a major limitation in financial modeling Bachelier (1900). Samuelson (1965)'s seminal work established that properly anticipated prices in efficient markets would fluctuate randomly Samuelson (1965) and this meant that stock prices do not follow an arithmetic brownian motion but a GBM, leading to the introduction of the Geometric BM Model. The GBM model ensures non-negative stock prices by modeling the logarithm of prices as opposed to prices, ensuring that prices are always positive. Because of this property, the GBM model became a widely adopted model for capturing the volatility of stock prices and simulate price paths for different assets. However, the GBM model assumes a constant volatility parameter, a no market friction (market with no cost), a lognormal distribution for prices, no Jumps or discontinuities, no autocorrelation in returns. Most of these assumptions often fail to reflect the dynamic nature of real-world financial markets returns can be skewed, sudden price changes are possible, returns tend to revert to some mean, and most importantly volatility is never constant. In practice, volatility tends to fluctuate over time due to various factors, such as economic conditions, investor sentiment, and the arrival of new information (Yong, Ziaei, & Szulczyk, 2021) (Alam, Siddikee, & Masukujjaman, 2013). To address some of these limitations, researchers have developed alternative stochastic volatility models, such as the Black-Scholes, Ornstein-Uhlenbeck process among other (Yong et al., 2021). The GBM is represented by the equation shown below:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1.1}$$

where W_t represents a Brownian motion process, S_t denotes the share price, dW_t signifies the change in the Brownian motion process, and dt indicates the change in time. The parameters μ and σ measure drift and volatility, respectively. Since this is a differential equation, we can determine its solution using Ito's Lemma, which provides a powerful tool for analyzing and solving stochastic differential equations.

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dX_t^2 \tag{1.2}$$

The expression is equivalent to:

$$df(t, S_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS_t^2 \tag{1.3}$$

Replacing dS_t from equation (1.1) into (1.3), we have;

$$df(t, S_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(\mu S_t dt + \sigma S_t dW_t)^2 \tag{1.4}$$

$$df(t, S_t) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2\right)dt + \frac{\partial f}{\partial S_t}\sigma S_t dW_t \tag{1.5}$$

Equation (1.1) can also be expressed by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1.6}$$

$$d(\ln S_t) = \mu dt + \sigma dW_t \tag{1.7}$$

Let $df(t, S_t) = dln(St)$ in (1.5) and in (1.7),

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}(\ln S_t) = 0 \tag{1.8}$$

$$\frac{\partial f}{\partial S} = \frac{\partial}{\partial S}(\ln S_t) = \frac{1}{S_t} \tag{1.9}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial f}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S_t}\right) = -\frac{1}{S_t^2} \tag{1.10}$$

Plugging the results of (1.10) into (1.5);

$$df(t,S_t) = \left(\frac{\delta f}{\delta t} + \frac{\delta f}{\delta S}\mu S_t + \frac{1}{2S_t^2}\sigma^2 S_t^2\right)dt + \frac{\delta f}{\delta S}\sigma S_t dW_t \tag{1.11}$$

$$df(t,S_t) = \left(0 + \frac{1}{S_t}\mu S_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\sigma^2 S_t^2\right)\right)dt + \frac{1}{S_t}\sigma S_t dW_t \tag{1.12}$$

$$df(t,S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \tag{1.13}$$

Equation (1.13) is another Stochastic Differential Equation (SDE) of GBM. Integrating both sides of eqaution (1.13);

$$\int_0^T d\ln S_t = \int_0^T \left(\mu - \frac{1}{2}\sigma^2\right)dt + \int_0^T \sigma dW_t \tag{1.14}$$

$$\ln\left(\frac{S_T}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T \tag{1.15}$$

$$\ln\left(\frac{S_T}{S_0}\right) = \ln S_T - \ln S_0 = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T \tag{1.16}$$

$$\ln S_T = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T \tag{1.17}$$

If we exponentiate both sides, we have

$$S_T = \exp\left[\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right] \tag{1.18}$$

where S_T is the future price at T. S_T follows $\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right), \sigma^2 T\right)$

$$F(S_t)(S_T,\sigma,\mu,T) = \frac{1}{S_T\sqrt{2\pi\sigma^2 T}}\exp\left[-\frac{\left(\ln S_T - \ln S_0 - (\mu - \frac{1}{2}\sigma^2)T\right)^2}{2\sigma^2 T}\right] \tag{1.19}$$

The GBM model assumes constant parameters μ and σ (Alam et al., 2013) (Yong et al., 2021). It can yield unrealistic price predictions, increasing to improbable values or even generating negative prices (Sigman, 2006). To address these limitations, the OU process, which includes a mean-reverting parameter, was developed.

1.2.2 Ornstein-Uhlenbeck (OU) Process

The OU process is a widely employed stochastic model in finance that captures the tendency of a variable to revert to a long-term average value (Yong et al., 2021). This versatile process has found extensive application in the modeling of diverse financial time series, such as stock returns and volatility. Mathematically, the OU process is represented as:

$$dS_t = k(\mu - S_t)dt + \sigma dW_t \tag{1.20}$$

where k, μ and σ represent the rate at which price return back to the mean, mean and volatility term respectively (Doob, 1942). The OU process oscillates around the long-term mean μ, facilitated by the mean-reversion parameter k (Leung & Li, 2015). The dynamics of this oscillation around the long-term mean depends both on the short-term volatility σ and the speed of mean-reversion k (García Franco, 2003). A higher value of k suggests a faster return to the mean, while a lower value indicates a slower adjustment (MANZOOR & HAFEEZ, 2018).

The model in equation (1.20) is also presented by replacing the volatility term σdW_t from equation (1.1) to $\sigma S_t^{\gamma} dW_t$ to obtain:

$$dS_t = k(\mu - S_t)dt + \sigma S_t^{\gamma} dW_t \tag{1.21}$$

where $\gamma \in R$. Chan, Karolyi, Longstaff, and Sanders (1992) further explains when $\gamma = 0$, the original OU process is given by Leung and Li (2015) as shown below:

$$dS_t = k(\mu - S_t)dt + \sigma dW_t \tag{1.22}$$

Additionally, when $\gamma = 0$ and S_t represents short-term interest rates $S_t = R_t$, the OU process becomes a Vasicek model (Doob, 1942).

$$dS_t = k(\mu - R_t)dt + \sigma dW_t \tag{1.23}$$

When $\gamma = 1$, the OU process corresponds to (Brennan & Schwartz, 1982).

$$dS_t = k(\mu - S_t)dt + \sigma S_t dW_t \tag{1.24}$$

Finally when $\gamma = 0.5$ the OU process becomes a Cox-Ingersoll-Ross (CIR) model (Cox, Ingersoll Jr, & Ross, 1985).

$$dS_t = k(\mu - S_t)dt + \sigma \sqrt{S_t} dW_t \tag{1.25}$$

where dW_t is a Wiener process and k, μ and σ are the parameters.

The OU process incorporates a mean-reversion parameter, k, which determines the speed at which the interest rate reverts to its long-run average value, μ. This mean-reversion mechanism is a key feature of the model, as it prevents the interest rate from diverging indefinitely and ensures it remains bounded within a reasonable range. Additionally, the standard deviation factor, σ, in the Cox-Ingersoll-Ross version of the OU process plays a crucial role in preventing the possibility of negative interest rates, even for positive values of k and μ. Moreover, the CIR model imposes a condition, $2k\mu \geq \sigma^2$, which further prohibits the price from reaching zero. This condition ensures that as the price approaches zero, the volatility factor γ also becomes very small, effectively dampening the impact of random shocks and allowing the drift factor to dominate, pushing the price back towards its equilibrium level (Cox et al., 1985).

Statistical Characteristics of the OU Model

To determine the value S_T we manipulate (1.20) to get (1.26)

$$\mathrm{d}S_t + kS_t \, \mathrm{d}t = k\mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t \tag{1.26}$$

Multiplying both sides of (1.26) by $e^{(kt)}$

$$\frac{d}{dt}\left(e^{kt}S\right) = k\mu e^{kt}dt + \sigma e^{kt}dW_t \tag{1.27}$$

Integrating from 0 to T, we get

$$e^{kT}S_T - e^0 S_0 = k\mu \frac{e^{kT} - e^{k0}}{k} + \int_0^T \sigma e^{kt} dW_t \tag{1.28}$$

Multiplying by $e^{(-kT)}$ and solving for S_T

$$S_T = S_0 e^{-kT} + \mu \left(1 - e^{-kT}\right) + \sigma \int_0^T e^{-k(T-t)} dW_t \tag{1.29}$$

Given that the integral of a deterministic function with respect to Brownian is Gaussian, S_T follows a normal distribution (Lebovits, Véhel, & Herbin, 2014)(Shahnazi-Pour, Moghaddam, & Babaei, 2021):

$$E(S_T) = E\left[S_0 e^{-kT} + \mu \left(1 - e^{-kT}\right) + \sigma \int_0^T e^{-k(T-t)} dW_t\right] \tag{1.30}$$

The expected value of a deterministic function integrated over Brownian motion is 0 (Vardar-Acar & Bulut, 2015):

$$E(S_T) = S_0 e^{-kT} + \mu \left(1 - e^{-kT}\right) \tag{1.31}$$

The variance of S_T will be then be:

$$Var(S_T) = E\left[S_T - E(S_T)\right]^2 \tag{1.32}$$

We have derived S_T and $E(S_T)$ in (1.29) and (1.31) respectively. Substituting (1.29) and (1.31) in (1.32), we get:

$$Var(S_T) = E\left[\left(S_0 e^{-kT} + \mu \left(1 - e^{-kT}\right) + \sigma \int_0^T e^{-k(T-t)} dW_t\right) - \left(S_0 e^{-kT} + \mu \left(1 - e^{-kT}\right)\right)\right]^2 \tag{1.33}$$

$$Var(S_T) = E\left[\sigma \int_0^T e^{-k(T-t)} dW_t\right]^2 \tag{1.34}$$

$$Var(S_T) = E\left[\frac{\sigma^2 e^{-2k(T-T)}}{2k}\right]\Big|_0^T \tag{1.35}$$

$$Var(S_T) = E\left[\frac{\sigma^2 e^{-2k(T-T)}}{2k} - \frac{\sigma^2 e^{-2k(T-0)}}{2k}\right] \tag{1.36}$$

$$Var(S_T) = E\left[\frac{\sigma^2}{2k}\left(1 - e^{-2kT}\right)\right] \tag{1.37}$$

$$Var(S_T) = \frac{\sigma^2}{2k}\left(1 - e^{-2kT}\right) \tag{1.38}$$

Therefore, $S_T \sim \mathcal{N}\left(S_0 e^{kT} + \mu(1 - e^{-kT}), \frac{\sigma^2}{2k}(1 - e^{-2kT})\right)$.

Given two time points T and J where $J < T$. The covariance of S_T and S_J will be:

$$\text{Cov}(S_T, S_J) = E\Big[(S_T - E[S_T])(S_J - E[S_J])\Big]$$

$$\begin{aligned}\text{Cov}(S_T, S_J) = E\Big[&S_0 e^{-kT} + \mu(1 - e^{-kT}) + \sigma\int_0^T e^{-k(T-t)}dW_t \\ &- \left(S_0 e^{-kT} + \mu(1 - e^{-kT})\right)\Big] E\Big[S_0 e^{-kJ} + \mu(1 - e^{-kJ}) \\ &+ \sigma\int_0^J e^{-k(J-u)}dW_u - S_0 e^{-kJ} + \mu(1 - e^{-kJ})\Big].\end{aligned} \tag{1.39}$$

$$\text{Cov}(S_T, S_J) = E\left[\sigma\int_0^T e^{-k(T-t)}dW_t\, \sigma\int_0^J e^{-k(J-u)}dW_u\right] = \sigma^2 e^{-k(J+T)} E\left[\int_0^T e^{kt}dW_t \int_0^J e^{ku}dW_u\right]$$

Recognizing that the covariance over disjoint periods is 0 Frankland et al. (2019) and utilizing the Ito's isometry Fujita and Kawanishi (2008), we get a deterministic integral:

$$\text{Cov}(S_T, S_J) = \sigma^2 e^{-k(J+T)} \int_0^J e^{2ku}du = \sigma^2 e^{-k(J+T)} \frac{e^{2kJ} - 1}{2k} \tag{1.40}$$

$$\text{Cov}(S_T, S_J) = \frac{\sigma^2}{2k}\left(e^{-k(J+T)+2kJ} - e^{-k(J+T)}\right) = \frac{\sigma^2}{2k}\left(e^{-k(T-J)} - e^{-k(T+J)}\right) \tag{1.41}$$

To comprehend the future behavior of the expected value and variance of S_T, we must examine the asymptotic distributions of S_T:

$$\lim_{T\to\infty} E(S_T) = \lim_{T\to\infty}\left(S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right)\right) = S_0 \lim_{T\to\infty} e^{-kT} + \mu\left(1 - \lim_{T\to\infty} e^{-kT}\right) = \mu \tag{1.42}$$

$$\lim_{T\to\infty} Var(S_T) = \lim_{T\to\infty}\left(\frac{\sigma^2}{2k}\left(1 - e^{-2kT}\right)\right) = \frac{\sigma^2}{2k}\left(1 - \lim_{T\to\infty} e^{-2kT}\right)$$

$$\lim_{T\to\infty} Var(S_T) = \frac{\sigma^2}{2k} \tag{1.43}$$

As the speed of mean reversion increases, the variance of the process tends to decrease, and vice versa. (Tsou, 2011).

Although the OU model has good characteristics, it still struggles to accurately forecast prices in the presence of market speculation (Hendricks & Ongala, 2022). To address this shortcoming by incorporating speculation parameters to the process, as demonstrated in the next section.

1.2.3 The Improved OU Process

A study conducted by Hendricks and Ongala (2022) proposed an extension of the standard OU process to incorporate the effect of speculation on the underlying asset's price dynamics. Mathematically, the revised process is represented by the equation:

$$\mathrm{d}S_t = k(\mu - S_t)\,\mathrm{d}t + f(t)\,\mathrm{d}W_t \tag{1.44}$$

where, S_t is the asset price, $\mathrm{d}S_t$ is the price variation, $\mathrm{d}t$ is the infinitesimal time increment and $\mathrm{d}W_t$ captures the random fluctuations modeled as a Brownian motion, μ is the mean and $f(t)$ is the volatility component. The improved OU model is based on the assumption that volatility is dynamic during speculation and constant at all other times:

$$f(t) = \begin{cases} \sigma, & 0 < t \leq t_1^*, \text{ Before speculation} \\ g(t), & t_1^* < t \leq t_2^*, \text{ During speculation} \\ \sigma r, & t_2^* < t \leq T, \text{ After speculation} \end{cases} \tag{Assumptions}$$

where, r belongs to the set of real numbers $r \in R, r \geq 1$ and $g(t)$ is expressed as an exponential function such that $g(t) = \sigma e^{\gamma t}$ and $\gamma \geq 0$.

Expanding equation (1.44), we have:

$$dS_t + kS_t dt = k\mu dt + f(t)dW_t \tag{1.45}$$

Multiplying both sides by e^{kt} in (1.45)

$$(dS_t + kS_t dt)e^{kt} = e^{kt}(k\mu dt + f(t)dW_t) \tag{1.46}$$

Obtaining the derivative of $e^{kt}S_t$ with respect to t using the product rule, we get the left side of (1.46). Thus,

$$\frac{d}{dt}\left(e^{kt}S_t\right) = k\mu e^{kt}dt + f(t)e^{kt}dW_t \tag{1.47}$$

By integrating both sides from 0 to T solving the resulting integrals, we obtain:

$$e^{kT}S_T - e^0 S_0 = k\mu\frac{e^{kT} - e^{k0}}{k} + \int_0^T f(t)e^{kt}dW_t \tag{1.48}$$

To isolate S_T, we multiply both sides by e^{-kT}

$$(e^{kT}S_T - e^0 S_0)e^{-kT} = e^{-kT}k\mu\frac{e^{kT} - e^{k0}}{k} + e^{-kT}\int_0^T f(t)e^{kt}dW_t \tag{1.49}$$

Solving explicitly for S_T, we get:

$$S_T = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + e^{-kT}\int_0^T f(t)e^{kt}dW_t \tag{1.50}$$

Applying the known properties and assumptions of $f(t)$ and substituting them in equation (1.50), we get:

$$S_T = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + e^{-kT}\left[\int_0^{t_1^*} \sigma e^{kt} dW_t + \int_{t_1^*}^{t_2^*} g(t) e^{kt} dW_t + \int_{t_2^*}^{T} r\sigma e^{kt} dW_t\right] \quad (1.51)$$

Replacing $g(t)$ with $\sigma e^{\gamma t}$, we have:

$$S_T = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + e^{-kT}\left[\int_0^{t_1^*} \sigma e^{kt} dW_t + \int_{t_1^*}^{t_2^*} \sigma e^{\gamma t} e^{kt} dW_t + \int_{t_2^*}^{T} r\sigma e^{kt} dW_t\right] \quad (1.52)$$

Expanding the equation (1.52), we get:

$$S_T = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + e^{-kT}\int_0^{t_1^*} \sigma e^{kt} dW_t + e^{-kT}\int_{t_1^*}^{t_2^*} \sigma e^{\gamma t} e^{kt} dW_t + e^{-kT}\int_{t_2^*}^{T} r\sigma e^{kt} dW_t \quad (1.53)$$

Taking the expectation from both sides:

$$\begin{aligned} E[S_T] &= E\left[S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right)\right] \\ &\quad + E\Bigg[\int_0^{t_1^*} \sigma e^{-k(T-t)} dW_t + \int_{t_1^*}^{t_2^*} \sigma e^{rt} e^{-k(T-t)} dW_t \\ &\quad + \int_{t_2^*}^{T} r\sigma e^{-k(T-t)} dW_t\Bigg]. \end{aligned} \quad (1.54)$$

Given that the expected value of the Brownian motion component is 0 and the expectation of a constant remains the same constant, we obtain the following result:

$$E(S_T) = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) \quad (1.55)$$

The variance of S_T will then be:

$$Var(S_T) = E\left[S_T - E(S_T)\right]^2$$

$$\begin{aligned} Var(S_T) = E\Bigg[&\Big(S_0e^{-kT} + \mu\left(1 - e^{-kT}\right)\Big) \\ &+ \int_0^{t_1^*} \sigma e^{-k(T-t)}dW_t + \int_{t_1^*}^{t_2^*} \sigma e^{rt}e^{-k(T-t)}dW_t \\ &+ \int_{t_2^*}^{T} r\sigma e^{-k(T-t)}dW_t - S_0e^{-kT} + \mu\left(1 - e^{-kT}\right)\Bigg]^2. \end{aligned} \tag{1.56}$$

$$Var(S_T) = E\left[\int_0^{t_1^*} \sigma e^{-k(T-t)}dW_t + \int_{t_1^*}^{t_2^*} \sigma e^{rt}e^{-k(T-t)}dW_t + \int_{t_2^*}^{T} r\sigma e^{-k(T-t)}dW_t\right]^2$$

$$Var(S_T) = E\left[\int_0^{t_1^*} \sigma^2 e^{-2k(T-t)}(dW_t)^2 + \int_{t_1^*}^{t_2^*} \sigma^2 e^{2rt}e^{-2k(T-t)}(dW_t)^2 + \int_{t_2^*}^{T} r^2\sigma^2 e^{-2k(T-t)}(dW_t)^2\right]$$

$$Var(S_T) = E\left[\int_0^{t_1^*} \sigma^2 e^{-2k(T-t)}dt + \int_{t_1^*}^{t_2^*} \sigma^2 e^{2rt}e^{-2k(T-t)}dt + \int_{t_2^*}^{T} r^2\sigma^2 e^{-2k(T-t)}dt\right]$$

$$Var(S_T) = \sigma^2 E\left[\int_0^{t_1^*} e^{-2k(T-t)}dt + \int_{t_1^*}^{t_2^*} e^{2rt}e^{-2k(T-t)}dt + \int_{t_2^*}^{T} r^2 e^{-2k(T-t)}dt\right]$$

$$\begin{aligned} Var(S_T) = \sigma^2 E\Bigg[&\frac{e^{-2k(T-t_1^*)} - e^{-2k(T-0)}}{2k} \\ &+ \frac{e^{2rt_2^* - 2k(T-t_2^*)} - e^{2rt_1^* - 2k(T-t_1^*)}}{2(k+r)} \\ &+ \frac{r^2e^{-2k(T-T)} - r^2e^{-2k(T-t_2^*)}}{2k}\Bigg]. \end{aligned} \tag{1.57}$$

$$
\begin{aligned}
Var(S_T) = \sigma^2 E\Bigg[& \frac{e^{-2k(T-t_1^*)} - e^{-2kT}}{2k} \\
& + \frac{e^{2rt_2^*-2k(T-t_2^*)} - e^{2rt_1^*-2k(T-t_1^*)}}{2(k+r)} \\
& + \frac{r^2 e^{-2k(0)} - r^2 e^{-2k(T-t_2^*)}}{2k}\Bigg]
\end{aligned} \tag{1.58}
$$

$$
\begin{aligned}
Var(S_T) = \frac{\sigma^2}{2} E\Bigg[& \frac{e^{-2kT+2kt_1^*} - e^{-2kT}}{k} \\
& + \frac{e^{2rt_2^*-2kT+2kt_2^*} - e^{2rt_1^*-2kT+2kt_1^*}}{(k+r)} \\
& + \frac{r^2 - r^2 e^{-2kT+2kt_2^*}}{2k}\Bigg]
\end{aligned} \tag{1.59}
$$

Factorizing similar terms

$$
Var(S_T) = \frac{\sigma^2 e^{-2kT}}{2} E\left[\frac{e^{2kt_1^*} - 1}{k} + \frac{e^{2rt_2^*+2kt_2^*} - e^{2rt_1^*2kt_1^*}}{(k+r)} + r^2 \frac{e^{2kT} - e^{2kt_2^*}}{2k}\right] \tag{1.60}
$$

$$
Var(S_T) = \frac{\sigma^2 e^{-2kT}}{2} \left[\frac{e^{2kt_1^*} - 1}{k} + \frac{e^{2rt_2^*+2kt_2^*} - e^{2rt_1^*2kt_1^*}}{(k+r)} + r^2 \frac{e^{2kT} - e^{2kt_2^*}}{2k}\right] \tag{1.61}
$$

$$
Var[S_T] = \underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left(e^{2kt_1^*} - 1\right)}_{\text{term1}} + \underbrace{\frac{\sigma^2 e^{-2kT}}{2(r+k)}\left(e^{2t_2^*(r+k)} - e^{2t_1^*(r+k)}\right)}_{\text{term2}} + \underbrace{\frac{\sigma^2 e^{-2kT}}{2k} r^2 \left(e^{2kT} - e^{2kt_2^*}\right)}_{\text{term3}} \tag{1.62}
$$

Note that this is a result from the derivation done in this study and is not consistent with the variance formula derived in the study by Hendricks and Ongala (2022) which is given below:

$$Var[S_T] = \frac{1}{2}\left[\underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left(e^{2kt^*}-1\right)}_{\text{term1}}+\underbrace{\frac{\sigma^2 e^{-2kT}}{r+k}\left(e^{2t^*(r+k)}\right)}_{\text{term2}}+\underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left(r^2\left(e^{-2kT}-e^{2kt^*}\right)\right)}_{\text{term3}}\right] \tag{1.63}$$

Therefore, the improved OU model is normally distributed with:

$$E[S_T] = S_0 e^{-kT} + \mu\left(1-e^{-kT}\right)$$

$$Var[S_T] \neq \frac{1}{2}\left[\underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left(e^{2kt^*}-1\right)}_{\text{term1}}+\underbrace{\frac{\sigma^2 e^{-2kT}}{r+k}\left(e^{2t^*(r+k)}\right)}_{\text{term2}}+\underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left(r^2\left(e^{-2kT}-e^{2kt^*}\right)\right)}_{\text{term3}}\right]$$

But with variance given by:

$$Var[S_T] = \underbrace{\frac{\sigma^2 e^{-2kT}}{2k}\left(e^{2kt_1^*}-1\right)}_{\text{term1}}+\underbrace{\frac{\sigma^2 e^{-2kT}}{2(r+k)}\left(e^{2t_2^*(r+k)}-e^{2t_1^*(r+k)}\right)}_{\text{term2}}+\underbrace{\frac{\sigma^2 e^{-2kT}}{2k}r^2\left(e^{2kT}-e^{2kt_2^*}\right)}_{\text{term3}}$$

The first term represents the volatility before the onset of speculation. The second term corresponds to the volatility observed during the speculative period. Finally, the third term accounts for the change in volatility caused by speculation, which takes place between the time t_2^* and the final time T.

The variance of the Improved OU model steadily declined toward zero, indicating that incorporating speculation in the model helps in reducing volatility. While the standard OU model also exhibits volatility stabilization over time, the Improved model achieves this at a faster rate (Hendricks & Ongala, 2022).

1.3 Problem Statement

Kenya has recently experienced pronounced exchange rate fluctuations, more so than ever before in its history and this was mostly driven by speculation, especially during events of uncertainty (Central Bank of Kenya, 2024). Such fluctuations of the exchange rate accentuated capital outflow, elevated debt distress and increased cost of public borrowing (Financial Stability Report, 2024). The need to model speculation arises from its profound impact on the financial stability of Kenya and by the fact that since Kenya shilling has experienced its highest pressure ever against the US dollar, no research has been done to model this unprecedented volatility.

The need to effectively capture volatility driven by speculation is immense, as evidenced by the gap in the existing literature. The Geometric Brownian motion model, though foundational in stock pricing, has limitations such as the assumptions of constant volatility and drift, which may not align with real-world dynamics. To address the issue of unbounded deviation from the mean, researchers have developed models like the Ornstein-Uhlenbeck and Vasicek models, but these have their own drawbacks, such as the possibility of generating negative returns and the inability to accurately capture the term structure of interest rates. Subsequent attempts to overcome these limitations, such as the CIR model and the Hull-White model, have also faced shortcomings, such as ensuring price stationarity.

The key shortcoming of these models in the context of modeling volatility in the presence of speculation is that they do not explicitly account for volatility driven by speculative activities, despite recent events suggesting the significant impact of such activities. While Hendricks and Ongala (2022) successfully improved the OU model by incorporating speculation parameters in their study, their study assumed constant volatility after the speculation period, which is inaccurate due to the expected interventions from governments to stabilize the exchange rate. The assumption of constant volatility after speculation is therefore impractical in the context of modeling the USD-KSH exchange rate, which is prone to interventions by the Central Bank of Kenya to stabilize. The improved OU model may have limited practical application as it fails to capture the stabilization stage that follows the speculation period.

1.4 Objectives of the study

The primary goal of this research paper is to modify the OU model to enable it to better capture the dynamics of speculative market behavior. Specifically, this study aims:

1. To modify the OU model with time-variant volatility in the presence of speculation.
2. To assess the statistical properties of the modified OU model.
3. To test the performance of the modified model using KSH/USD exchange rate data before and during the volatile period of the Russia-Ukraine war.

1.5 Significance of the Study

The findings of this study will benefit a variety of stakeholders, including government entities and academics, by providing insights on how to enhance the IOU model's relevance in exchange rate volatility modeling. In Kenya, the unpredictable high volatility of the USD/KSH exchange rate has eroded market participants' confidence and exacerbated the country's economic recession (Financial Stability Report, 2024). This research aims to offer valuable insights on predicting the USD/KSH exchange rate during periods of stability and heightened market turbulence.

Chapter 2: Literature review

2.1 General approaches for modeling volatility

Various modeling techniques have been developed to address the crucial task of forecasting volatility. A prominent and widely employed framework is the autoregressive conditional heteroskedastic model introduced by (Engle & Ng, 1993). Building upon Engle's seminal ARCH model, subsequent models have been proposed to capture diverse aspects of volatility dynamics. As suggested by Y.-c. Chen and Rogoff (2003), the variance in one period can depend on past variances and disturbances. These volatility models are extensively utilized across different fields of econometrics, particularly in financial time series analysis, such as exchange rate modeling (Epaphra, 2016) (Omari, 2017).

Let Y_t represent the price of an asset over the period from $t-1$ to t. The past information set, $F(t-1)$ contains all observed values of relevant variables up to time $t-1$. The conditional expectation and variance of Y_t given $F(t-1)$ are defined by C. W. Chen and Watanabe (2019) as the expected return and volatility at the point of investment decision. These are denoted by m_t and h_t, respectively:

$$m_t = E(Y_t \mid F(t-1)) \quad \text{and} \quad h_t = \text{Var}(Y_t \mid F(t-1)).$$

ϵ_t, can be expressed as:

$$\epsilon_t = Y_t - m_t,$$

This serves as a measure of news impact or speculation at time t. A Positive ϵ_t indicates favorable news, whereas a negative ϵ_t signals unfavorable news, with volatility generally increasing in response to bad news and decreasing with good news. Furthermore, a high absolute value of ϵ_t suggests that the news is particularly "noteworthy" or "influential," as it causes a significant unexpected shift in price.

As proposed by Engle and Ng (1993), the conditional variance h_t can be modeled as a function of the lagged ϵ_t, indicating that volatility can be forecasted based on past news. The most comprehensive model developed by Engle and Ng (1993) is the p-th order autoregressive conditional heteroskedasticity model, which captures this relationship.

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \epsilon^2(t-i) \tag{2.1}$$

where the constant parameters $\alpha_1, \ldots, \alpha_p$ and ω capture the influence of past return shocks on the current volatility. Typically, the impact of older news diminishes, with more recent shocks having a greater effect on present volatility than older ones. In an ARCH model, news arriving more than p periods ago does not affect the current volatility. Bollerslev (1986) builds upon the ARCH framework by introducing the Generalized ARCH(p) model to the GARCH(p, q) model, expressed as:

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \epsilon^2(t-i) + \sum_{j=1}^{q} \beta_j h(t-1) \tag{2.2}$$

The GARCH model extends the ARCH process by allowing past return shocks to influence current volatility, with their effects diminishing exponentially over time. Empirical studies indicate that the GARCH family of models has achieved considerable success, with the GARCH model being a preferred choice in practical applications. However, these parametric models exhibit certain limitations in capturing important data features. One notable shortcoming is the leverage or asymmetric effect, first identified by Black (1976) and subsequently corroborated by various other studies (Dzieliński, Rieger, & Talpsepp, 2018). As proposed by (Y.-c. Chen & Rogoff, 2003), the ARCH framework provides one approach to account for the distributional properties of exchange rate changes. Nonetheless, these models present additional drawbacks, such as the negative correlation between current returns and future returns volatility observed by Black, and the potential explosion of conditional moments in the GARCH model even when the process is stationary and ergodic, as noted by (Nelson, 1991). Another limitation of the GARCH model is the requirement for non-negativity constraints on the conditional volatility equation parameters to ensure non-negative conditional volatility. To address these limitations, Nelson (1991) developed an exponential GARCH as an alternative.

$$\log(h_t) = \omega + \beta \log(h_{t-1}) + \sum_{i=1}^{p} \alpha_i \epsilon^2(t-i) + \sum_{j=1}^{q} \beta_j h(t-1) \tag{2.3}$$

$$\log(h_t) = \omega + \beta \log(h_{t-1}) + \gamma \cdot \frac{\epsilon_t - 1}{\sqrt{h_{t-1}}} + \alpha \frac{\epsilon_t - 1}{\sqrt{h_{t-1}}} - \sqrt{\frac{2}{\pi}} \tag{2.4}$$

where ω, β, γ, and α are constant parameters. The EGARCH model exhibits asymmetry because it includes the standardized lagged residual, $\frac{\epsilon_t - 1}{\sqrt{h_t - 1}}$. By comparing the GARCH and EGARCH models, a framework for analyzing how news influences conditional heteroskedasticity is provided. By keeping information dated $t - 2$ and earlier constant, the relationship between $\epsilon_t - 1$ and h_t. According to Asemota and Ekejiuba (2017), the EGARCH model by Nelson (1991) allows for asymmetric effects between positive and negative asset returns and specifies volatility in the form of a logarithmic transformation. Furthermore, Asemota and Ekejiuba (2017) accounts for asymmetric effects between positive and negative asset returns by applying a logarithmic transformation. Additionally, they highlight that assuming normality in financial time series modeling may impact the robustness of parameter estimates. A fat-tailed error distribution, as opposed to a normal one, can better capture the kurtosis commonly seen in financial data (Nelson, 1991). Malmsten (2004) argued that the first-order EGARCH model, when assuming normally distributed errors, lacks sufficient flexibility to adequately capture both the kurtosis and autocorrelation observed in stock returns.

2.2 Stochastic approaches for modeling volatility

While the ARCH and GARCH models employ parametric specifications to model time-varying volatility, there are also stochastic approaches that aim to capture the inherent randomness of asset price fluctuations. Stochastic volatility models describe frameworks where the variance of a process is randomly distributed. These models are widely applied in mathematical finance for assessing derivative securities, including options (Gatheral, 2011). The term "stochastic volatility models" originates from their approach to modeling the underlying security's volatility as a random process. This process is governed by state variables such as the price level of the underlying security, the tendency of volatility to revert to a long-term mean, and the variance of the volatility process itself, among other factors (Gatheral & Jacquier, 2014).

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.5}$$

where μ represents the constant drift rate, σ denotes the constant volatility, and W_t is a standard Wiener process with a mean of 0 and a standard deviation of 1. The explicit solution to this stochastic differential equation is as follows:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] \tag{2.6}$$

Hagan (2002) indicates that stochastic volatility models are unable to account for long-observed characteristics of the implied volatility surface, such as the volatility smile and skew. These phenomena demonstrate that implied volatility tends to vary with strike price and expiration. However, this model has limitations in capturing the dynamics of the financial market. The main drawback is that the model assumes a constant volatility over time, which is inconsistent with empirical observations of time-varying volatility.

2.2.1 Black-Scholes Model

The Black-Scholes-Merton (BSM) model, proposed by the eponymous academics in 1973, provides a foundational framework for comprehending the intricate relationship between commodity prices and option pricing. This model is extensively employed to estimate the fair market value of European-style options, thus serving as a basis for pricing various financial instruments. As noted by Mandelbrot and Mandelbrot (1997), the Black–Scholes or Black–Scholes–Merton model is a mathematical framework used to represent the dynamics of financial markets containing derivative investment instruments. Lorig and Sircar (2016) further explicate that in the BSM, the price S_t follows a GBM, as described by the following equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{2.7}$$

In the standard Black-Scholes framework, the price movements of an underlying asset are modeled using geometric Brownian motion, where the parameter μ represents the constant drift rate, and σ denotes the constant volatility. A key result from Black and Scholes (1973) is that the non-arbitrage price of an option depends only on the parameter σ and not on μ. However, assuming constant volatility within the Black-Scholes model is a limitation, as empirical studies indicate that volatility is actually time-varying. To address this issue, stochastic volatility models have been introduced, treating volatility as a random process. Despite these improvements, the gap between observed market prices and those predicted by the BBM model has long been acknowledged in equity option pricing research (Badescu, Kulperger, & Lazar, 2008). Additionally, Haug and Taleb (2011) criticized the BSM for emphasizing dynamic hedging over risk, arguing that this approach aims to align the model more closely with neoclassical economic theory. They also note that Boness (1964) had previously published a formula identical to the Black-Scholes call option pricing equation. Furthermore, Derman* and Taleb (2005) have criticized the concept of dynamic hedging, and stated that similar models to BSM had been proposed by other researchers prior to Black and Scholes. Stewart (2012) have argued that the Black-Scholes equation provided the "mathematical justification for trading" and was a contributing factor to the financial crisis of 2007-2008. This, along with factors such as financial mismanagement, political incompetence, distorted incentives, and weak regulatory oversight, contributed to the crisis.

2.2.2 Heston Model

To address the limitations of the BSM, Heston (1993) introduced a stochastic volatility model that assumes the variance of the underlying asset follows a square-root diffusion process. Chakrabarti and Santra (2017) compared the Heston and BSM models and found that the Heston model performed better on when the investment period was at least 90 days.

$$dS_t = \mu S_t dt + \sigma S_t dW_t^S \tag{2.8}$$

The Heston Model posits that the instantaneous variance σ follows a stochastic process Heston (1993), a key characteristic that distinguishes stochastic volatility models from the Black-Scholes model, which assumes constant volatility Chakrabarti and Santra (2017). Additionally, the Heston Model is a type of volatility smile model, which graphically depicts the relationship between option strike prices and their corresponding volatilities for a given expiration date, showing increased volatility as the strike price deviates from the current price (Borland, 2007). The Heston Model exhibits several distinctive features: it incorporates the potential correlation between the underlying asset's price and its volatility, it models volatility as mean-reverting, it provides a closed-form solution, and it does not necessitate that the asset price adheres to a lognormal probability distribution (Heston, 1993).

2.2.3 Constant Elastic Volatility (CEV) Model

The CEV model is another influential stochastic volatility framework that proposes a power law relationship between the asset price and its volatility, diverging from the mean-reverting square-root process of the Heston model as first discovered by John (n.d.) and subsequently validated by other scholars (Beckers, 1980). The CEV model aims to capture both stochastic volatility and the leverage effect, making it a popular choice among financial practitioners, particularly for modeling equities and commodities. The model mathematically describes the association between volatility and price, incorporating stochastic volatility with a leverage effect:

$$dS_t = \mu S_t dt + \sigma S_t^{\gamma} dW_t \tag{2.9}$$

The CEV model effectively captures the dynamic association between asset prices and volatility. The model describes the evolution of the price over time, S_t, where the drift parameter μ governs the overall trend. Crucially, the CEV framework incorporates parameters σ and γ that regulate the intricate relationship between volatility and price movements. Conceptually, the CEV model can accommodate scenarios where volatility rises concurrently with increasing prices, as well as situations where volatility tends to increase as prices decline. However, some scholars argue that the CEV model should not be categorized as a true stochastic volatility model, as it does not employ an independent stochastic process to model volatility, and is instead considered a local volatility model. A key characteristic of the CEV model is the parameter γ which determines the relationship between price and volatility. When $\gamma < 1$, the model captures the leverage effect, a common phenomenon in equity markets where stock volatility rises as prices decline. Conversely, in commodity markets, $\gamma > 1$ is frequently observed, indicating the inverse leverage effect, where price increases are associated with rising volatility (Hendricks & Ongala, 2022).

2.3 Ornstein–Uhlenbeck (OU) Model

The OU process is a stochastic process with applications in financial mathematics and the physical sciences, originally used to model the velocity of a massive Brownian particle under the influence of friction (Doob, 1942). This process was discovered by Uhlenbeck and Ornstein (1930) and hence the name. Ahmad (1988)'s study indicates that OU process is one of several approaches used to model interest rates, currency exchange rates, and commodity prices stochastically. Doob (1942) The following equation is an example of an OU model:

$$dS_t = k(\mu - S_t)dt + \sigma S_t^{\gamma} dW_t \tag{2.10}$$

where the parameter μ represents the equilibrium or mean value supported by fundamental factors, the parameter k denotes the speed of reversion to μ, and σ captures the volatility (Ahmad, 1988) (Griffin & Steel, 2010)(Masoliver & Perelló, 2006). The Equation was enhanced by Chan et al. (1992) through the replacement of the volatility term with $\sigma S_t^{\gamma} dW_t$, where σ is a parameter that governs the potential leverage effect between the asset price and its volatility. This change captures the potential leverage effect γ between the asset price and its volatility. The generalized OU model has been widely applied in various contexts as a superior alternative to the classical OU process, as it is a more flexible and adaptive model that can allow for time varying or state dependent coefficients, making it better suited to capture the non-Gaussian behavior and heavy tails observed in empirical financial data (da Fonseca, Figueiredo, de Castro, & Mendes, 2013). As further explained by Chan et al. (1992) and Leung and Li (2015), when solving for $\gamma = 0$, the OU process becomes the original OU process given by $dS_t = k(\mu - S_t)dt + \sigma dW_t$ which corresponds to the Vasicek model if S_t represents short term interest rates instead of the price. Additionally, if solved for $\gamma = 1$, the process becomes state dependent and correspond to (Brennan & Schwartz, 1982): $dS_t = k(\mu - S_t)dt + \sigma S_t dW_t$ (Doob, 1942). Finally, according to Cox et al. (1985), when solving for $\gamma = 0.5$, the OU process becomes a CIR model given by: $dS_t = k(\mu - S_t)dt + \sigma(\sqrt{S_t})dW_t$ (Hendricks & Ongala, 2022). The CIR model addresses the limitations of the Classical OU and Vasicek model, which can generate negative outputs. Introducing the squared root of state-dependent volatility in CIR models it ensures that the model will not generate negative outputs, which is plausible for many applications in finance. We derived statistical characteristics extensively in Chapter 1.

2.4 Gap in Literature

All volatility models reviewed assumed constant volatility at some point with the exception of the study by Hendricks and Ongala (2022), which explicitly modeled speculation. Although Hendricks and Ongala (2022) successfully incorporated speculation parameters in the OU model, the assumptions of constant volatility after speculation fail in the context of the volatile USD-KSH exchange rate. The KSH is a floating currency, and the Central Bank of Kenya (CBK) would take measures to prevent devaluation beyond the true value. This is why volatility may not be constant after speculation since the CBK would seek to intervene and stabilize the KSH to its true value. Therefore, a model that incorporates a time-variant volatility function post-speculative period may better capture the dynamics of the stabilization stage and is needed. This study proposes relaxing the assumption of constant volatility after speculation made by Hendricks and Ongala (2022) in developing the improved OU model.

Chapter 3: Methodology

3.1 Introduction

We will commence with a depiction of the data sources and characteristics. Subsequently, a descriptive analysis was undertaken to discern salient features, including outliers and shocks. Moreover, the statistical methodologies and tools utilized in the analysis are explicated. The chapter also revisits the OU model and its underlying properties. Lastly, the proposed model and its validation process are elaborated.

3.2 Data Description

This investigation will utilize secondary daily data sourced from the Yahoo Finance foreign exchange platform. The data consist of prices in Kenyan shillings at which 1 USD is traded in the foreign exchange market on a given day. The data span the period from January 1, 2019 to March 25, 2025, selected to capture fluctuations of USD-KSH during both tranquil and turbulent periods, such as the recent devaluation of the Kenyan shilling due to the Russia-Ukraine war, the COVID-19 pandemic, Kenya politics (Elections, Genz protests, etc). We shall work with the closing price at the end of the business day instead of the mean of the highest or lowest price prices. This means ensures our daily data is always recorded daily at a fixed time. The data will undergo cleaning and analysis employing all in the R programming language. We shall use the differenced data to model volatility.

3.2.1 Assumption

Many factors affect the level of speculation-driven volatility. This study focuses on the impact of the Ukraine-Russia war as the sole factor influencing speculation-driven volatility. It is assumed that the speculative period begins with the start of the Ukraine-Russia war in February 2022 and ends when the effects of this news event are normalized, even though the war may not have concluded entirely by that point. Since speculation is defined to be short-term and about expetation, speculation must have ended even as the war continues because the market has adapted and the war is no longer new information upon which speculators could use to make short-term profit.

3.3 Proposed Model

We propose the following model:

$$dS_t = (k - \mu)S_t dt + \sigma_t dW_t \tag{3.1}$$

Where:

- S_t is the USD/KSH exchange rate at time t,
- W_t is a Wiener process at time t,
- μ is the mean, assumed to be constant,
- k is the speed at which the process revert back to the mean
- σ_t is the volatility at time t and is assumed to be inherently changing with time as is the case in reality.

$$\sigma_t = \begin{cases} \sigma & 0 < t \leq t_1, \text{ Constant before speculation} \\ \sigma e^{\beta t} & t_1 < t \leq t_2, \text{ exponentially growing during speculation} \\ \sigma e^{\beta t_2} e^{-\alpha t} & t_2 < t \leq T, \text{ exponentially decaying after speculation} \end{cases} \tag{New Assumptions}$$

where, $\beta \geq 0$, $\alpha \geq 0$ and t_2 is a constant.

3.4 Validation of the model

The model validation process will entail simulating price trajectories using both the improved OU model and the proposed model in this study. These simulated paths will undergo a comparative evaluation with assessment of their statistical properties; the expected means and variances.

Chapter 4: Analysis

4.1 Description of the Data

We visualize the data described in chapter 3. Our data is a time series spanning from January 2019 to March 2025 as depicted by figure 4.1.

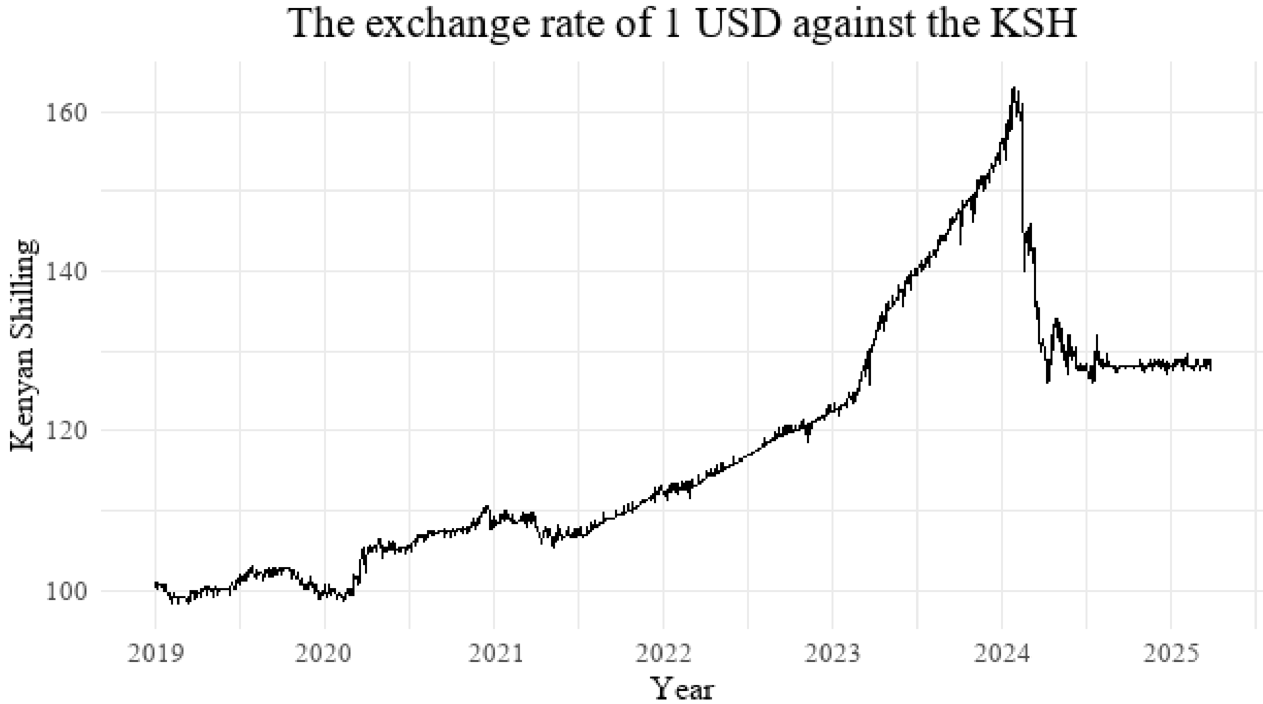

Figure 4.1: USD-KSH Exchange Rate (2014–2024)

The KSH has over time depreciated against the USD. There have been very high fluctuations, ranging from a minimum buying price of 1 USD at 98.29 KSH on February 7, 2019, to a maximum of 163 KSH per USD on January 26, 2025.

Our data had 2 weekdays with missing exchange rates, which we imputed. On average, we have 261 trading days per year in our data set which is more than the commonly assumed 252 trading days. This assumption is attributed to the fact that a calendar year consists of 365 days, 52 weeks, 104 weekends, and 10 public holidays, ultimately resulting in 252 trading days. The discrepancy is the fact that our data does not assume any public holiday. Our data has a total of 1624 trading days.

The data exhibits a distinct pattern resembling a mountain, with the base situated in 2021 and the peak attained in January 2024, followed by a subsequent decline. This trend serves as evidence of heightened speculative activity, which commences in 2021 and experiences exponential growth until 2024, subsequently undergoing a sharp decline. The minimum exchange rate in 2021 was 105.5, recorded on May 10, 2021 (614th trading day). The exchange rate then climbed to the maximum of 163 on January 26, 2024 (1322nd trading day), followed immediately by a sharper and quicker decay which stabilizes at approximately 129 in mid-2024.

The data suggest that speculation was sustained for over 2 years, from March 2021 to January 2024, exhibiting an exponential growth pattern during this period. Following this speculative phase, an exponential decay in volatility is observed from January 2024 until March 2025.

4.2 Numerical simulation

In this section, we run the numerical simulation using the exact solutions of the modified OU and standard OU models.

4.2.1 Modified OU Model

In this segment, we specified the modified OU model, derived its statistical properties, calibrated it to the USD-KSH exchange rate data, simulated its probability distribution, and forecasted the price trajectories for the subsequent 252 trading days.

Model specification

We defined the modified OU as:

$$dS_t = (k - \mu)S_t dt + \sigma_t dW_t \tag{4.1}$$

$$\sigma_t = \begin{cases} \sigma & 0 < t \leq t_1, \text{ Constant before speculation} \\ \sigma e^{\beta t} & t_1 < t \leq t_2, \text{ exponentially growing during speculation} \\ \sigma e^{\beta t_2} e^{-\alpha t} & t_2 < t \leq T, \text{ exponentially decaying after speculation} \end{cases}$$

We can then develop equation (4.1) as:

$$dS_t + kS_t dt = k\mu dt + \sigma_t dW_t \tag{4.2}$$

Multiplying both sides by e^{kt} in (4.2)

$$(dS_t + kS_t dt)e^{kt} = e^{kt}(k\mu dt + \sigma_t dW_t) \tag{4.3}$$

Obtaining the derivative of $e^{kt}S_t$ with respect to t using the product rule, we get the left side of (4.3). Thus,

$$\frac{d}{dt}\left(e^{kt}S_t\right) = k\mu e^{kt}dt + \sigma_t e^{kt}dW_t \tag{4.4}$$

By integrating both sides from 0 to T solving the resulting integrals, we obtain:

$$e^{kT}S_T - e^0 S_0 = k\mu\frac{e^{kT} - e^{k0}}{k} + \int_0^T \sigma_t e^{kt}dW_t \tag{4.5}$$

To isolate S_T, we multiply both sides by e^{-kT}

$$(e^{kT}S_T - e^0 S_0)e^{-kT} = e^{-kT}k\mu\frac{e^{kT} - e^{k0}}{k} + e^{-kT}\int_0^T \sigma_t e^{kt}dW_t \tag{4.6}$$

Solving explicitly for S_T, we get:

$$S_T = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + e^{-kT}\int_0^T \sigma_t e^{kt}dW_t \tag{4.7}$$

Applying the known properties and assumptions of σ_t and substituting them in equation (1.50), we get:

$$S_T = S_0 e^{-kT} + \mu\left(1 - e^{-kT}\right) + e^{-kT}\left[\int_0^{t_1} \sigma e^{kt}dW_t + \int_{t_1}^{t_2} \sigma e^{\beta t}e^{kt}dW_t + \int_{t_2}^{T} \sigma e^{\beta t_2}e^{-\alpha t}\sigma e^{kt}dW_t\right] \tag{4.8}$$

$$S_T = S_0 e^{-kT} + \mu(1 - e^{-kT}) + \int_0^{t_1} \sigma e^{-k(T-t)}\, dW_t + \int_{t_1}^{t_2} \sigma e^{\beta t}e^{-k(T-t)}\, dW_t + \int_{t_2}^{T} \sigma e^{\beta t_2}e^{-\alpha t}e^{-k(T-t)}\, dW_t \tag{4.9}$$

Where T is the terminal time and 0 is the start time

Statistical characteristics

$$E[S_T] = E\left[S_0e^{-kT}+\mu(1-e^{-kT})+\int_0^{t_1} \sigma e^{-k(T-t)}\, dW_t+\int_{t_1}^{t_2} \sigma e^{\beta t}e^{-k(T-t)}\, dW_t+\int_{t_2}^{T} \sigma e^{\beta t_2}e^{-\alpha t}e^{-k(T-t)}\, dW_t\right] \tag{4.10}$$

Since $dW_t \sim \mathcal{N}(0, dt)$ the expectation of a wiener process is zero, then

$$E[S_T] = E\left[S_0e^{-kT} + \mu(1-e^{-kT})\right] + 0 + 0 + 0 \tag{4.11}$$

The expected value of a constant remain that constant

$$E[S_T] = S_0e^{-kT} + \mu(1-e^{-kT}) \tag{4.12}$$

From determistic statistics, we know that:

$$var[S_T] = E\left[S_T - E(S_T)\right]^2$$

$$\begin{aligned} \text{var}[S_T] = E\Bigg[& S_0e^{-kT} + \mu(1-e^{-kT}) \\ & + \int_0^{t_1} \sigma e^{-k(T-t)}\, dW_t \\ & + \int_{t_1}^{t_2} \sigma e^{\beta t}e^{-k(T-t)}\, dW_t \\ & + \int_{t_2}^{T} \sigma e^{\beta t_2}e^{-\alpha t}e^{-k(T-t)}\, dW_t \\ & - S_0e^{-kT} + \mu(1-e^{-kT})\Bigg]^2 \end{aligned} \tag{4.13}$$

$$var[S_T] = E\left[\int_0^{t_1} \sigma e^{-k(T-t)}\, dW_t + \int_{t_1}^{t_2} \sigma e^{\beta t}e^{-k(T-t)}\, dW_t + \int_{t_2}^{T} \sigma e^{\beta t_2}e^{-\alpha t}e^{-k(T-t)}\, dW_t\right]^2 \tag{4.14}$$

Applying the Itô's Isometry: variance of an Itô integral equals the integral of the square of the integrand. We get

$$var[S_T] = E\left[\int_0^{t_1} \sigma^2 e^{-2k(T-t)}\, dt + \int_{t_1}^{t_2} \sigma^2 e^{2\beta t} e^{-2k(T-t)}\, dt + \int_{t_2}^{T} \sigma^2 e^{2\beta t_2} e^{-2\alpha t} e^{-2k(T-t)}\, dt \right] \quad (4.15)$$

Taking all constant terms outside

$$var[S_T] = E\left[\sigma^2 e^{-2kT} \int_0^{t_1} e^{2kt}\, dt + \sigma^2 e^{-2kT} \int_{t_1}^{t_2} e^{2(\beta+k)t}\, dt + \sigma^2 e^{-2kT} e^{2\beta t_2} \int_{t_2}^{T} e^{2(k-\alpha)t}\, dt \right] \quad (4.16)$$

Evaluating the integrals:

$$var[S_T] = E\left[\sigma^2 e^{-2kT} \left[\frac{e^{2kt}}{2k} \right]_0^{t_1} + \sigma^2 e^{-2kT} \left[\frac{e^{2(\beta+k)t}}{2(\beta+k)} \right]_{t_1}^{t_2} + \sigma^2 e^{-2kT} e^{2\beta t_2} \left[\frac{e^{2(k-\alpha)t}}{2(k-\alpha)} \right]_{t_2}^{T} \right] \quad (4.17)$$

$$var[S_T] = \sigma^2 e^{-2kT} E\left[\frac{1}{2k} \left(e^{2kt_1} - e^{2k0} \right) + \frac{1}{2(\beta+k)} \left(e^{2(\beta+k)t_2} - e^{2(\beta+k)t_1} \right) + \frac{e^{2\beta t_2}}{2(k-\alpha)} \left(e^{2(k-\alpha)T} - e^{2(k-\alpha)t_2} \right) \right] \quad (4.18)$$

$$var[S_T] = \sigma^2 e^{-2kT} E\left[\frac{1}{2k} \left(e^{2kt_1} - 1 \right) + \frac{1}{2(\beta+k)} \left(e^{2(\beta+k)t_2} - e^{2(\beta+k)t_1} \right) + \frac{e^{2\beta t_2}}{2(k-\alpha)} \left(e^{2(k-\alpha)T} - e^{2(k-\alpha)t_2} \right) \right] \quad (4.19)$$

$$var[S_T] = \sigma^2 e^{-2kT} \left[\frac{1}{2k} \left(e^{2kt_1} - 1 \right) + \frac{1}{2(\beta+k)} \left(e^{2(\beta+k)t_2} - e^{2(\beta+k)t_1} \right) + \frac{e^{2\beta t_2}}{2(k-\alpha)} \left(e^{2(k-\alpha)T} - e^{2(k-\alpha)t_2} \right) \right] \quad (4.20)$$

Rewriting the variance in terms of the different speculation periods:

$$var[S_T] = \underbrace{\frac{\sigma^2 e^{-2kT}}{2k} \left(e^{2kt_1} - 1 \right)}_{\text{Variance before speculation}} + \underbrace{\frac{\sigma^2 e^{-2kT}}{2(\beta+k)} \left(e^{2(\beta+k)t_2} - e^{2(\beta+k)t_1} \right)}_{\textbf{Variance during speculation}} + \underbrace{\frac{\sigma^2 e^{-2kT} e^{2\beta t_2}}{2(k-\alpha)} \left(e^{2(k-\alpha)T} - e^{2(k-\alpha)t_2} \right)}_{\textbf{Variance after speculation}} \quad (4.21)$$

Let Δt represent the smallest change in time.This means that $T = (t + \Delta t)$ and $t = \Delta t$. We can then measure the smallest in change in S_T by redefine equation (4.9), to start at time t

and end at time $t + \Delta t$. We get

$$S_{t+\Delta t} = S_t e^{-k\Delta t} + \mu(1 - e^{-k\Delta t}) + \int_0^{t_1} \sigma e^{-k\Delta t}\, dW_t + \int_{t_1}^{t_2} \sigma e^{\beta t} e^{-k\Delta t}\, dW_t + \int_{t_2}^{t+\Delta t} \sigma e^{\beta t_2} e^{-\alpha t} e^{-k\Delta t}\, dW_t \tag{4.22}$$

For the right-hand side of equation(4.22), we replaced T by Δt since $S_{t+\Delta t}$ started at S_t and not S_0

$$E[S_{t+\Delta t}] = E\left[S_t e^{-k\Delta t} + \mu(1 - e^{-k\Delta t}) + \int_0^{t_1} \sigma e^{-k\Delta t}\, dW_t + \int_{t_1}^{t_2} \sigma e^{\beta t} e^{-k\Delta t}\, dW_t + \int_{t_2}^{t+\Delta t} \sigma e^{\beta t_2} e^{-\alpha t} e^{-k\Delta t}\, d \tag{4.23}$$

Since $dW_t \sim \mathcal{N}(0, dt)$ the expectation of a constant is that constant, then

$$E[S_{t+\Delta t}] = S_t e^{-k\Delta t} + \mu(1 - e^{-k\Delta t}) \tag{4.24}$$

The get the variance of $S_{t+\Delta t}$, we replace T by $t + \Delta t$ in the derived variance in equation(4.21).

$$var[S_{t+\Delta t}] = \sigma^2 e^{-2k\Delta t} \left[\frac{\left(e^{2kt_1} - 1\right)}{2k} + \frac{\left(e^{2(\beta+k)t_2} - e^{2(\beta+k)t_1}\right)}{2(\beta + k)} + \frac{e^{2\beta t_2}}{2(k - \alpha)} \left(e^{2(k-\alpha)(t+\Delta t)} - e^{2(k-\alpha)t_2}\right) \right] \tag{4.25}$$

Therefore, $S_{t+\Delta t} \sim \mathcal{N}(equation(4.24), equation(4.25))$ and $S_{t+\Delta t}$ can be simulated as:

$$S_{t+\Delta t} = E[S_{t+\Delta t}] + \sqrt{var[S_{t+\Delta t}]}\, \mathcal{N}(0, 1) \tag{4.26}$$

$$\begin{aligned} S_{t+\Delta t} &= S_t e^{-k\Delta t} + \mu(1 - e^{-k\Delta t}) \\ &+ \sqrt{\sigma^2 e^{-2k\Delta t} \left[\frac{e^{2kt_1} - 1}{2k} + \frac{e^{2(\beta+k)t_2} - e^{2(\beta+k)t_1}}{2(\beta + k)} + \frac{e^{2\beta t_2}\left(e^{2(k-\alpha)(t+\Delta t)} - e^{2(k-\alpha)t_2}\right)}{2(k - \alpha)} \right]}\, \mathcal{N}(0, 1) \end{aligned} \tag{4.27}$$

Modeling assumptions

- We assume that change occurs daily. Then $\Delta t = 1$ day
- In estimating σ, we are assuming equal probabilities of the process being in any state (before, during and after speculation) and approximate volatility at $t = T$
- For modeling volatility we calculate the differenced data as $ln(\frac{S_t}{S_{t-1}})$ which delete an observation.

- We use the least square method and a linear model to calibrate parameters.

Our data is a sequence of observations taken daily, which fits the definition of a daily univariate time series data. A univariate time series. Therefore Equation (4.19) can be likened to an Autoregressive process of order 1 AR(1).

In our case the mean is zero, so the AR(1) is

$$X_{t+1} = C + \theta_1 X_t + \cdots + e_{t+1} \qquad \text{(zero mean} \quad \text{and constant C)} \tag{4.28}$$

Likening terms:

$$\begin{aligned} X_{t+1} &= S_{t+\Delta t} \\ X_t &= S_t \\ \theta_1 &= e^{-k\Delta t} \\ C &= \mu(1 - e^{-k\Delta t}) \\ e_t &= \sqrt{\sigma^2 e^{-2k(t+\Delta t)} \left[\frac{e^{2kt_1} - 1}{2k} + \frac{e^{2(\beta+k)t_2} - e^{2(\beta+k)t_1}}{2(\beta+k)} + \frac{e^{2\beta t_2} \left(e^{2(k-\alpha)(t+\Delta t)} - e^{2(k-\alpha)t_2} \right)}{2(k-\alpha)} \right]} \end{aligned}$$

Estimation of Time t_1, t_2, T

- We assumed that t_1 is the smallest recorded exchange rate at the beginning of the exponential growth period (speculation period). The exponential growth started in 2021, and the smallest recorded exchange rate was on the 10th of May, day 614

- We assume that t_2 is the largest recorded exchange rate at the peak of the speculation period. The peak of exponential growth was around the end of 2023 and the beginning of 2024. The highest recorded rate was on January 26, 2024, which corresponds to trading day 1322 in our data.

- T was assumed to be the day of the last record in the data set, which was March 25, 2025, which correspond to trading day 1623 in our data.

Estimation of β

$$e^{\beta(t_2 - t_1)} = \frac{S_{t_2}}{S_{t_1}}$$

$$\beta = ln\left(\frac{S_{t_2}}{S_{t_1}}\right) * (t_2 - t_1)$$

Estimation of α

$$e^{-\alpha(T-t_2)} = \frac{S_T}{S_{t_2}}$$

$$\alpha = -\frac{ln(\frac{S_T}{S_{t_2}})}{(T-t_2)}$$

Estimation of σ

$$\sigma = \frac{e_t}{\sqrt{\frac{e^{-2kT}}{2}\left[\frac{e^{2kt_1}-1}{k} + \frac{e^{2(\beta+k)t_2}-e^{2(\beta+k)t_1}}{\beta+k} + \frac{e^{\beta t_2}(e^{-2(k-\alpha)T}-e^{-2kt_2})}{k-\alpha}\right]}}$$

Calibration Results

The initial parameters after fitting the data to the suggested model are:

- k (the Speed of Mean Reversion) is: 0.001613126
- μ (the Long-Term Mean) is: 128.6385
- σ (The Volatility) is: 0.03967876
- $t1$ (the beginning of speculation) is 614, which is on May 10, 2021
- $t2$ (the end of speculation) is 1322, which is on January 26, 2024
- T(The total number of trading days) is: 1623, which is on March 25, 2025
- β (exponential growth rate during speculation) is 0.0006144622
- α (the exponential decay after speculation) is 0.000777202
- e_t the error term of the AR(1) is: 0.8167957

We can also annualize key parameters :

- $k = 0.4065078$
- $\sigma = 1.187064$

Distributions

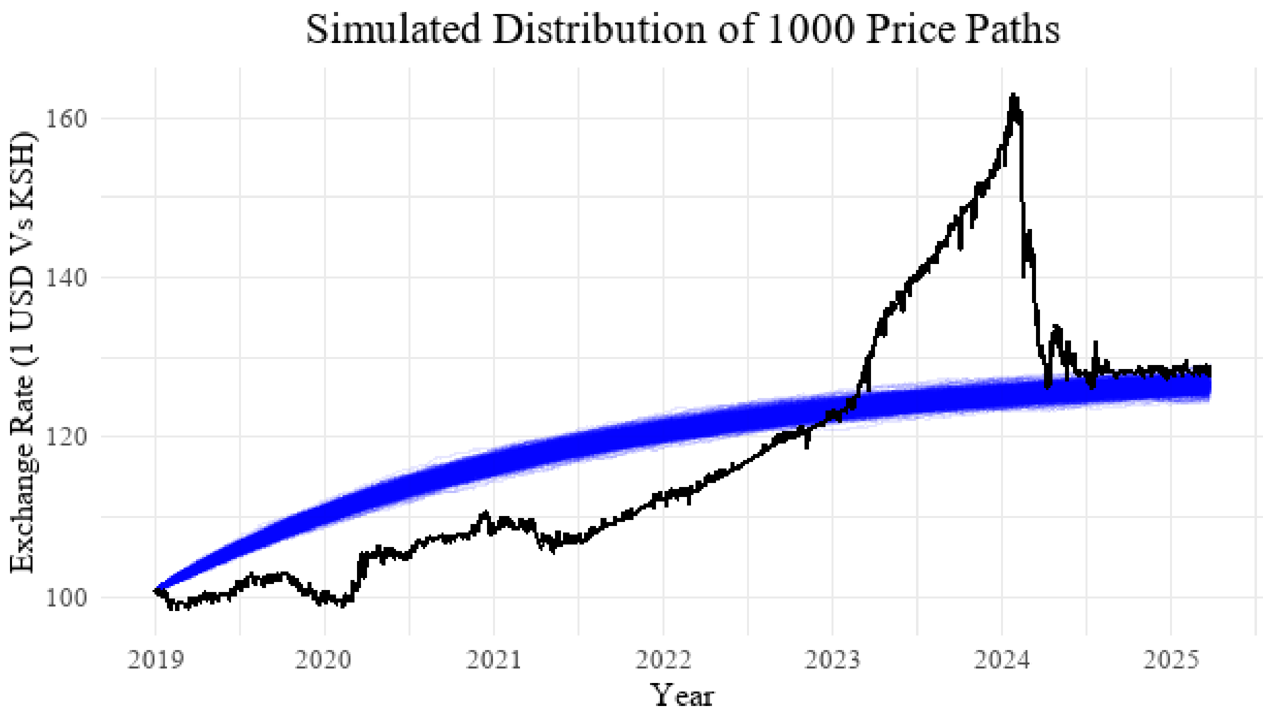

Figure 4.2: Simulated 1000 Price Paths

Figure 4.2 illustrates 1000 simulated price paths. The model stabilizes around the long-term mean 128.6385 at T. It shows the expected mean of the 1000 simulated paths. It supports the trends of the simulated 1000 paths. The curse blue lines similar to the simulated expected mean in figure 4.6, and the line in blue shows the actual data. The finding is consistent with the expectation set by the modified OU model by assuming that as time tends to infinity, the expected prices tend to μ. This is true independent of where the process begins as evidenced in figure 4.3

The process takes longer than T to eventually converge. The starting point is matters in determining how long the process takes before it stabilizes.

The analysis of the rolling volatility over 30 days window in figure 4.4 shows that the USD-KSHS exhibits a mean-reverting behavior as Volatility tends to spike and then fall. From it, we may categorize 3 main periods:

1. Periods of moderate Volatility $0.05 < \sigma < 0.10$ was mostly observed around 2019–2021, the volatility hovers around 0.05–0.1, indicating a calm market. During this period however, there is an unusual higher spike (0.15) in early 2020 and may be a consequence of the COVID pandemic.

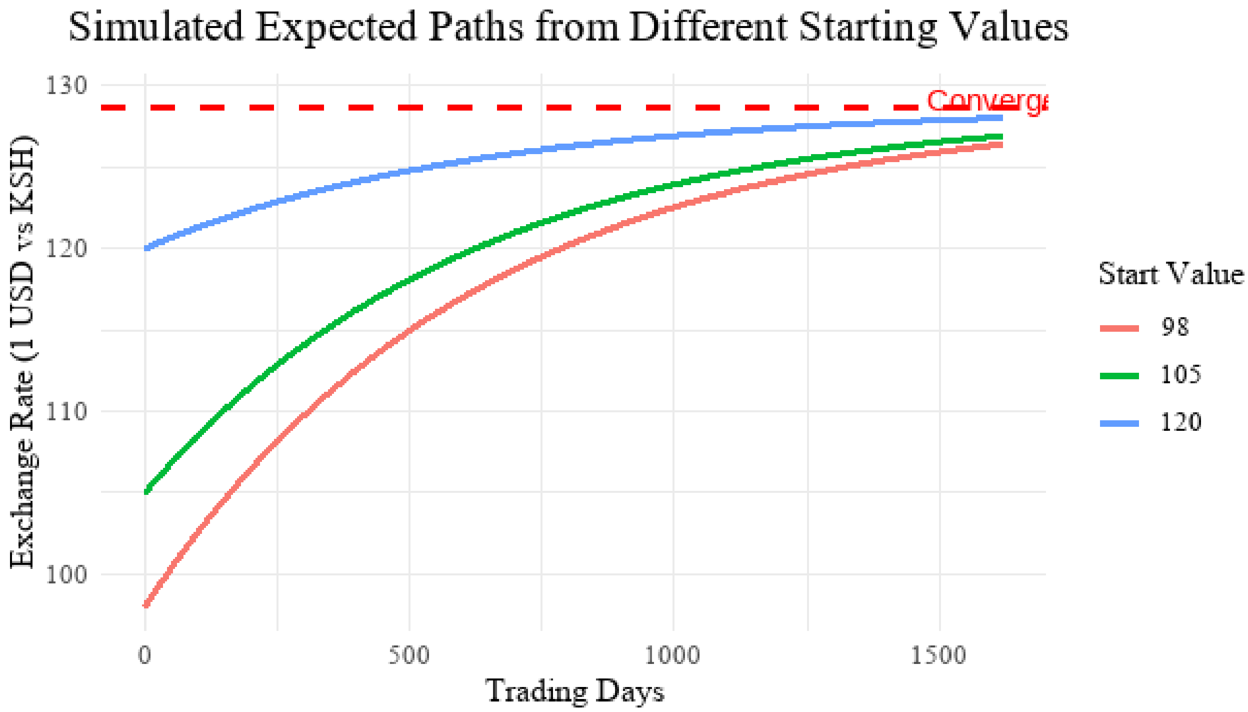

Figure 4.3: Simulated E[St] from different starting points

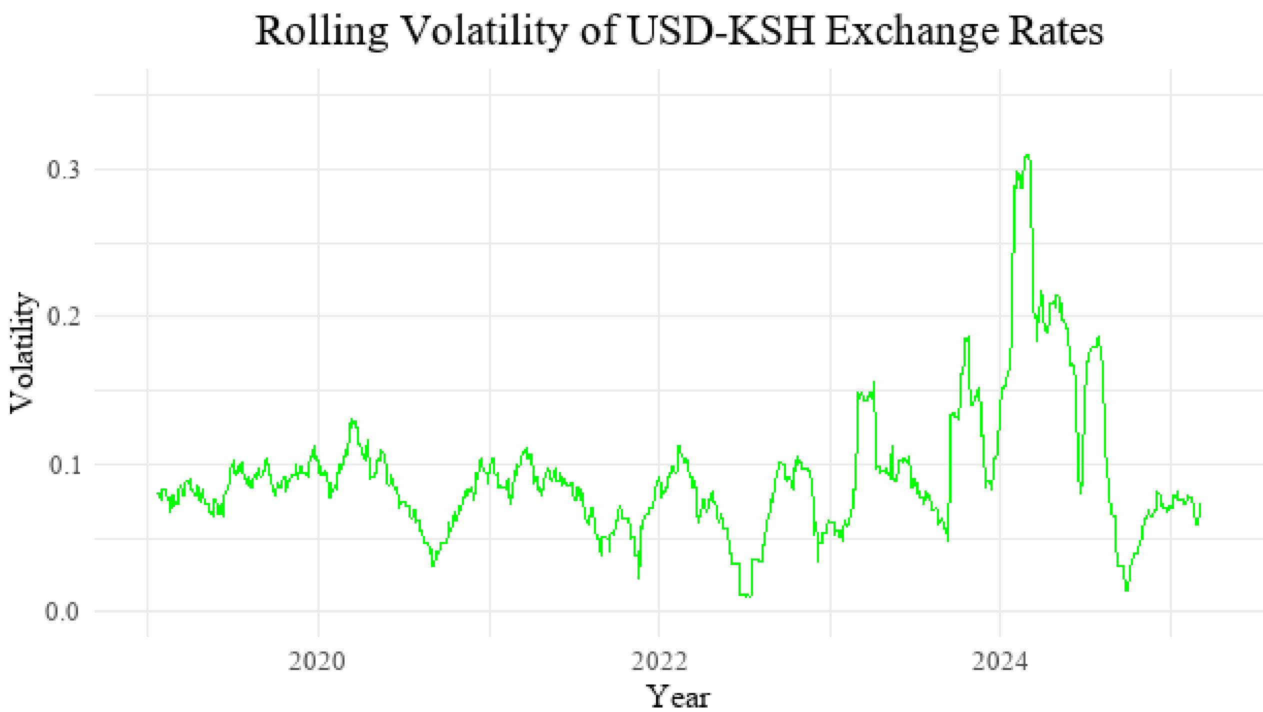

Figure 4.4: Simulated 30 days $\sqrt{var[St]}$

2. Periods of High Volatility $\sigma \geq 0.1$: Spikes around late 2023 to early 2024 exceed 0.3, meaning the highest and longest market volatility in any year. All the years had a short and less intense wave of speculation except 2023 and 2024 which had longer and very intense 2 waves of speculation.

3. Periods of low volatility $\sigma \leq 0.05$ were very short and rare; only lasting about a month

and observed 3 times in the last 5 years.

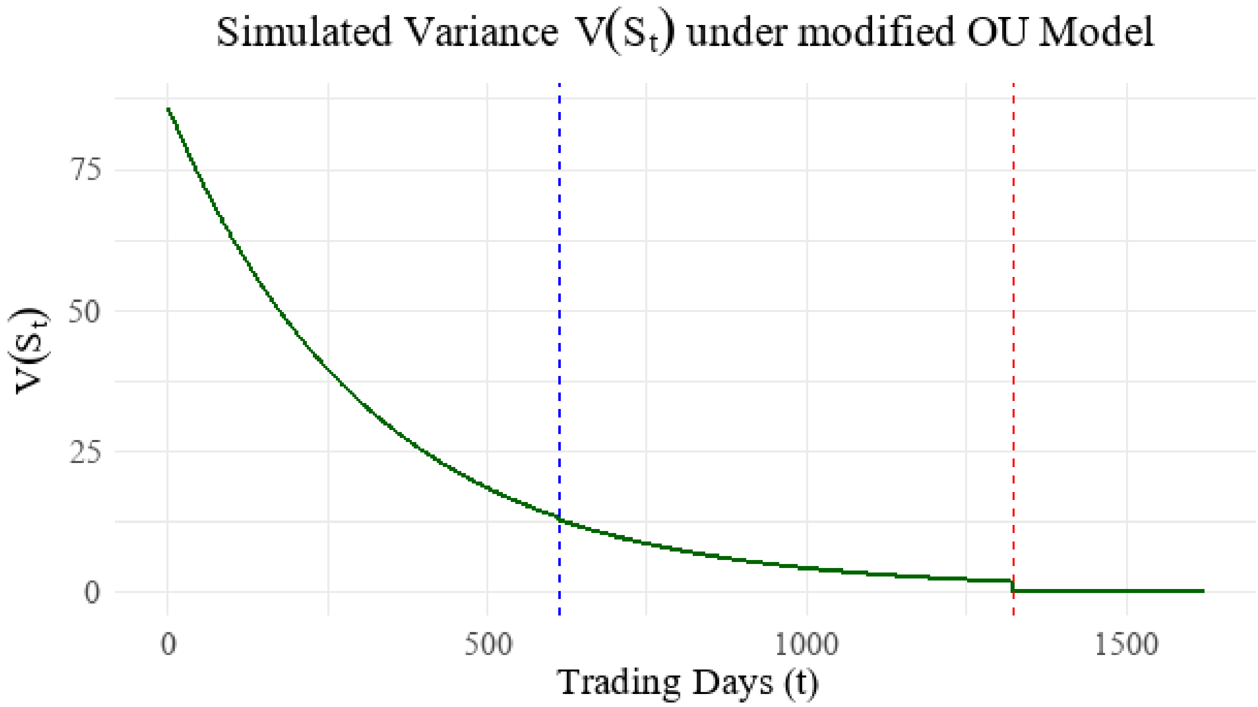

Figure 4.5: Simulated $var[S_T]$

Figure 4.5 shows the evidence that the variance of the modified OU model stabilizes downward. The blue and the red dotted lines indicate t_1 and t_2 respectively.

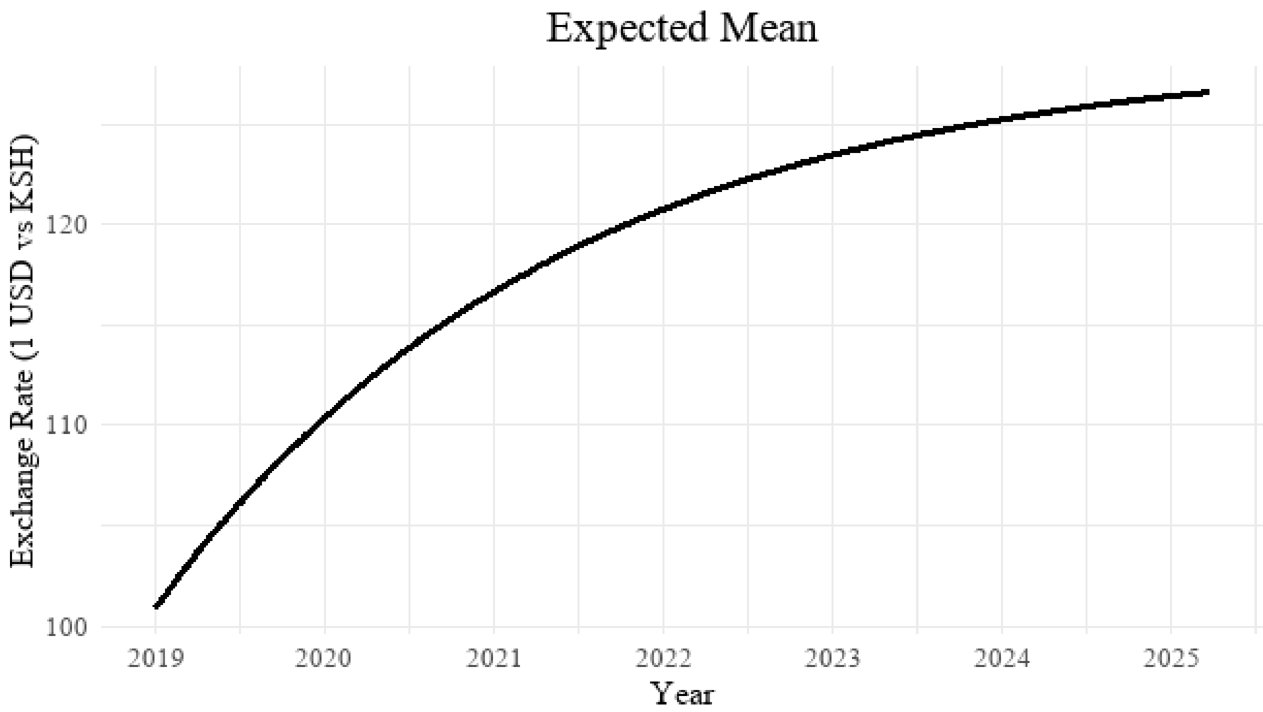

Figure 4.6: Simulated $E[S_T]$ of the modified OU model

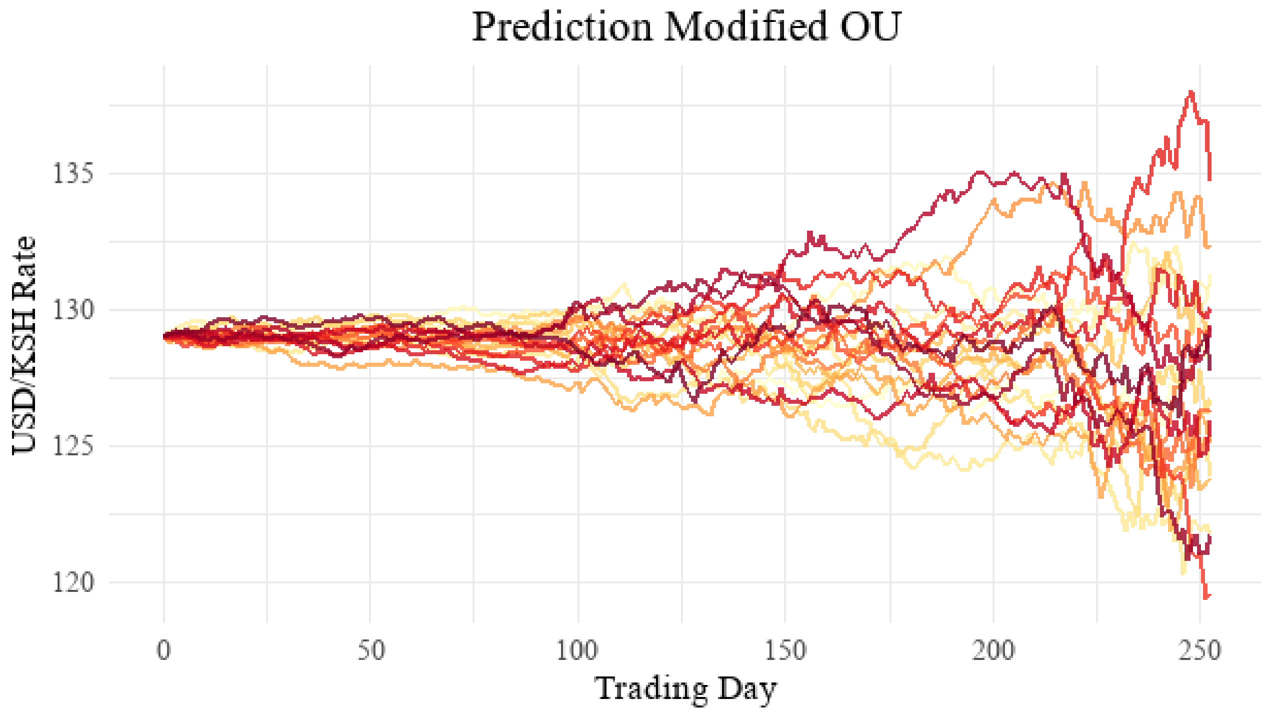

Figure 4.7: 20 sample Paths Prediction of the modified OU model

Prediction

Figure 4.7 is the prediction of the next fluctuation of prices starting from the end value S_T which was on March 25, 2025. This is done using the annualized parameters calibrated onto the data. The sample paths spread out further as we predict further into the future.

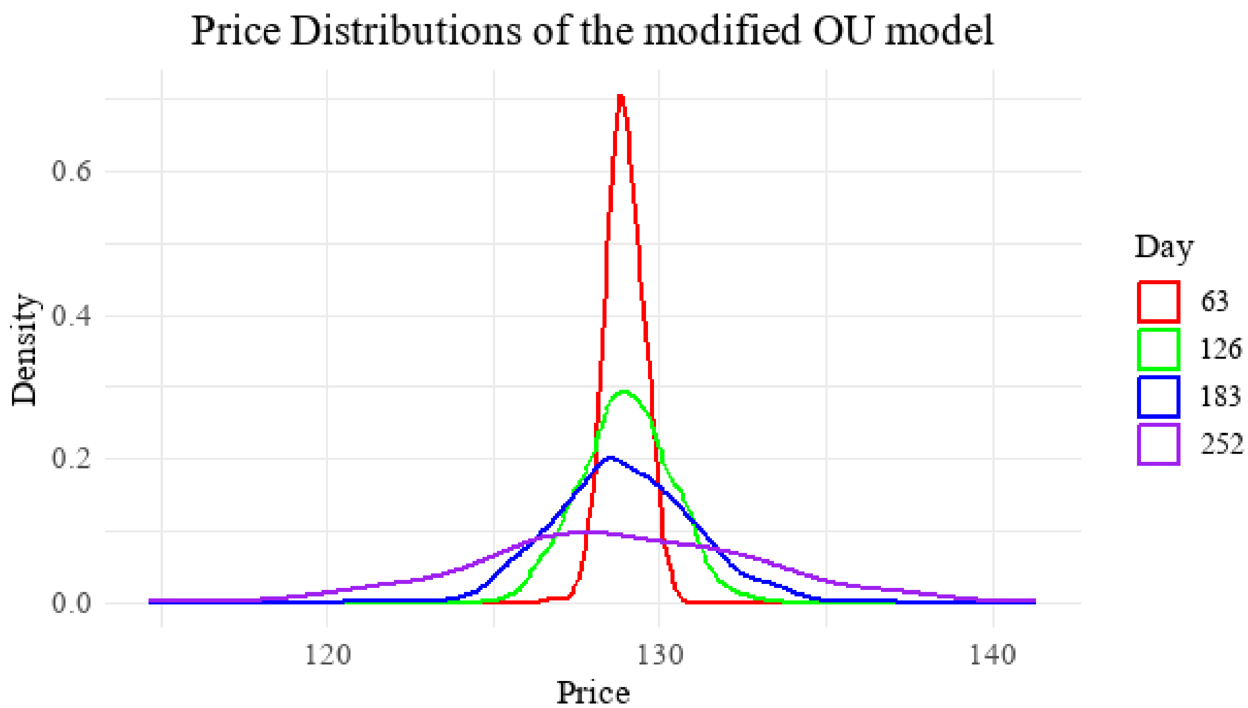

Figure 4.8: Uncertainty of Modified OU as time increases

The longer we predict, the higher the likelihood of errors as evidenced the growing fat tails of the distribution. So the modified model should be used for short-term predictions.

4.2.2 OU Model

Statistical properties

The following was derived in chapter one:

$$S_T \sim \mathcal{N}\left(S_0 e^{kT} + \mu(1-e^{-kT}), \frac{\sigma^2}{2k}(1-e^{-2kT})\right).$$

$$E[S_{t+\Delta t}] = S_{t-1}e^{k\Delta t} + \mu(1-e^{-k\Delta t}) \tag{4.29}$$

$$var[S_{t+\Delta t}] = \frac{\sigma^2}{2k}(1-e^{-2k(t+\Delta t)}) \tag{4.30}$$

$$S_{t+\Delta t} = S_{t-1}e^{k\Delta t} + \mu(1-e^{-k\Delta t}) + \sqrt{\frac{\sigma^2}{2k}(1-e^{-2k(t+\Delta t)})}\mathcal{N}(0,1) \tag{4.31}$$

Calibration

Recalling the AR(1) and likening terms:

$$X_{t+1} = C + \theta_1 X_t + \cdots + e_{t+1} \quad \text{(zero mean and constant C)} \tag{4.32}$$

$$C = \mu(1-e^{-k\Delta t})$$

$$e_t = \frac{\sigma^2}{2k}(1-e^{-2k(t+\Delta t)})$$

Since the smallest change is a day, we get:

$\mu = 128.6385$, $k = 0.001613126$ and $\sigma = 0.03967876$

Distributions

Since theoretically the difference between the modified OU model and the standard OU model is in the random term and not the drift term, their expected means are equal but the variances are different. We only need the distribution of the variance.

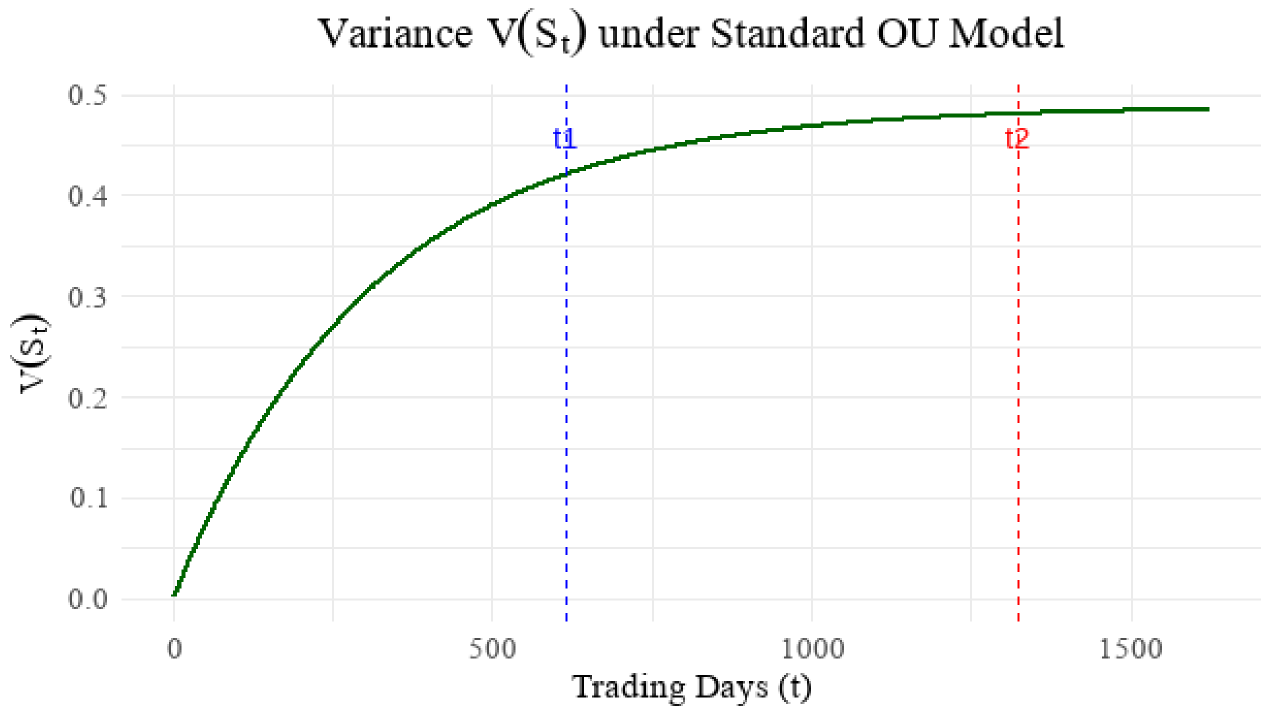

Figure 4.9: Simulated $var[S_T]$

The variance of the OU model stabilizes upward. t_1 and t_2 show the beginning and end of the speculation period, respectively.

4.3 Dicussion

It has been observed that the variance of the modified model stabilizes downward to zero unlike the variance of the OU model, which stabilizes upward. However, their means stabilize towards the long-term mean μ.

It is also important to note the difference between the simulated and actual prices, which are very distant especially in 2024 where volatility is at its peak. This may mean that the initial parameters did not maximize the volatility functions. So it is important to study how the model behaves as we change different parameters. The model did not converge to μ in the short term, possibly due to a smaller mean reversion parameter but it converges ultimately.

The modified OU model offers enhanced value compared to the standard OU model by incorporating a time-varying volatility function. This modification allows the model's variance to decrease over time, in contrast with the standard OU model, where the variance typically increases.

Chapter 5: Conclusion

5.1 Review and evaluation of the objectives

We have successfully modified the standard OU model by incorporating time-dependent volatility during and after speculative periods. We successfully derived the analytical expressions for the mean and variance of the modified OU model. We then calibrated the parameters of this time-dependent volatility model using the actual USD-KSH exchange rate data observed during the speculative period associated with the Ukraine-Russia war and others. We have evaluated the statistical characteristics of the modified OU model in contrast to the standard OU model and discovered that the means of both models tend to converge towards the long-term mean. However, the variance of the modified OU model exhibits a downward stabilizing trend, while the variance of the standard OU model demonstrates an upward stabilizing trend.

Limitation

- Preidentifying the onset of a speculation period. High speculative periods are typically recognized retrospectively, as evidenced by the 2007-2008 crisis, the COVID-19 pandemic, and the recent instability of the Ksh/USD exchange rate. Consequently, the modified OU model is likely to be utilized only when high volatility has already commenced.
- Sensitivity of the models to parameter changes. The simulated prices get further away from the actual prices during the year 2024, suggesting that the model the selected parameters may have maximized the volatility function σ_t.
- Possibility of negative variance if $k > \beta$ which is impractical, as negative variance is mathematically infeasible.
- The modified OU model assumes a single wave of speculation. However, the empirical evidence suggests that there have been multiple waves of speculation between the year 2019 and 2025. This discrepancy between the model's assumption and the observed data may have adversely impacted the model's accuracy, as it was predicated on a single wave of speculative activity in its specification.

- High volatility may be caused by other factors, not necessarily speculation. This study assumed speculation to be the only factor influencing the volatility of the USD-KSH

5.1.1 Recommendation

The next study may consider:

- Evaluating the performance of the modified OU model on short-term data with a single speculative episode.
- Integrating a Markov chain framework with transition probabilities to allow for dynamic shifts between volatility regimes within the modified OU model.
- Comparing the performance of the modified OU model to other established volatility models, such as the Improved OU and EGARCH, to assess its relative effectiveness in capturing volatility dynamics.

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Appendix A: Similarity Report

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Appendix B: Ethical Clearance Confirmation

Strathmore
UNIVERSITY

3rd April 2025

Mr Irenee Ireme,
irenee.vunabandi@strathmore.edu

Dear Mr Irenee,

RE: Modeling Time-Dependent Volatility of the USD-KSH Exchange Rate in the Presence of Speculation

This is to inform you that SU-ISERC has reviewed and **approved** your above **SU-masters** proposal. Your application reference number is **SU-ISERC2796/25.** The approval period is from **3rd April 2025 to 2nd April 2026.**

This approval is subject to compliance with the following requirements:

i. Only approved documents including (informed consents, study instruments, MTA) will be used.
ii. All changes including (amendments, deviations, and violations) are submitted for review and approval by SU-ISERC.
iii. Death and life-threatening problems and serious adverse events or unexpected adverse events whether related or unrelated to the study must be reported to SU-ISERC within 72 hours of notification.
iv. Any changes anticipated or otherwise that may increase the risks or affected safety or welfare of study participants and others or affect the integrity of the research must be reported to SU-ISERC within 72 hours.
v. Clearance for the export of biological specimens must be obtained from relevant institutions.
vi. Submission of a request for renewal of approval at least 60 days prior to the expiry of the approval period. Attach a comprehensive progress report to support the renewal.
vii. Submission of an executive summary report within 90 days of completion of the study to SU-ISERC.

Before commencing your study, you will be expected to obtain a research license from National Commission for Science, Technology, and Innovation (NACOSTI) https://research-portal.nacosti.go.ke/ and obtain other clearances needed.

Yours sincerely,

Mr Ambrose Rachier,
Chairperson; SU-ISERC