Strathmore
UNIVERSITY

STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES

MASTER OF SCIENCE IN BIOMATHEMATICS

END OF SEMESTER EXAMINATION

BMA 8202: Partial Differential Equation Modelling in Biology

Date: 6th December, 2023 **Time: 3 Hours**

Instructions

1. This examination consists of **FIVE** questions.
2. Answer **Question ONE (COMPULSORY)** and any other **TWO** questions.

QUESTION ONE (20 MARKS)

(a) *Bacterial chemotaxis.* Keller and Segel applied the idea of attraction and repulsion in deriving their equation for bacterial chemotaxis:

$$\frac{\partial B}{\partial t} = -\frac{\partial}{\partial x}\left(\chi B \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial B}{\partial x}\right)$$

where $B(x,t)$ is the bacterial density at location x and time t while χ is the chemotactic constant. Given that bacterial flux is zero for all x, find a (nonuniform) steady-state solution of the given bacterial chemotaxis model. [5 Marks]

(b) Solve $\dfrac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0$, $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$. [5 Marks]

(c) It is given that $\varphi = \varphi(x,y)$ satisfies the partial differential equation

$$\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} = \sin x + \cos y.$$

Use the transformation equations

$$u = ax + by \quad \text{and} v = cx + dy, \quad ad - bc \neq 0$$

with suitable values of a, b, c and d, in order to determine a general solution of the above partial differential equation. [5 Marks]

(d) The Auxiliary equations (A.Es) according to Charpit's method for finding the complete integral of a nonlinear partial differential equation of the form $f(p, q, x, y, z) = 0$ is given by

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

Solve $\quad px + qy = pq \quad$ where $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$ [5 Marks]

QUESTION TWO (20 MARKS)

(a) *Density-dependent dispersal.* A model for a pair of interacting populations with densities $u(x, t)$ and $v(x, t)$, in which dispersal is a response to the total population at a location that is, to $[u(x, t)+v(x, t)]$, in one-dimension is given by

$$\begin{aligned} \frac{\partial u}{\partial t} =& k_1 \frac{\partial}{\partial x}\left[u\frac{\partial(u+v)}{\partial x}\right], \\ \frac{\partial u}{\partial t} =& k_2 \frac{\partial}{\partial x}\left[v\frac{\partial(u+v)}{\partial x}\right]. \end{aligned}$$

Find the steady states of the given density-dependent dispersal equations. [10 Marks]

(b) A homogeneous linear partial differential equation of the n^{th} with constant coefficients $(k_1, \cdots, k_n)$ is given as:

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

When $F(x, y) = x^m y^n$, where m and n are constants, a particular integral (P.I) of the give partial differential equation is of the form

$$P.I = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n.$$

Use the above formulation to solve

$$\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3.$$

[10 Marks]

QUESTION THREE (20 MARKS)

(a) Using the method of separation of variable, solve the wave equation [7 Marks]

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

(b) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by

$y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest position, find the displacement $y(x,t)$. [13 Marks]

QUESTION FOUR (20 MARKS)

(a) (i) *Model of chemotherapy of leukemia.* Consider the model due to Bischoff et al. (1971) given by

$$\frac{\partial n}{\partial t} = -v\frac{\partial n}{\partial \alpha} - \mu n,$$

What has been assumed about the rate of maturation v? [1 Mark]

(ii) Assume that cycle specificity of the drug is low, in other words, that μ is nearly independent of maturity. Take [9 Marks]

$$\mu(t) = \frac{K_1(t)c(t)}{K_2(t) + c(t)}.$$

Further assume a solution of equation in (i) is of the form $n(\alpha, t) = N_0 e^{\beta t} h(\alpha) e^{-A(t)}$. Show that

$$A(t) = \int_0^t \mu(t')\, dt$$

(b) It is given that $z = z(x,y)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = x.$$

Use the transformation equations

$$\zeta = Ax + By \quad \text{and} \eta = Cx + Dy, \quad AD - BC \neq 0$$

with suitable values of A, B, C and D, in order to determine a general solution of the above partial differential equation. [10 Marks]

QUESTION FIVE (20 MARKS)

(a) Find the solution of $\dfrac{\partial^2 u}{\partial x^2} = h^2 \dfrac{\partial u}{\partial t}$, for which $u(0,t) = u(l,t) = 0$, $u(x,0) = \sin\frac{\pi x}{l}$ by method of separation of variables separable. [7 Marks]

(b) Find the temperature distribution $u(x,t)$ in a laterally insulated copper bar of length 100cm if the initial temperature in the bar is $50\sin(\frac{\pi x}{100})°C$ and the ends of the bar are kept at $0°C$. How long will it take for the maximum temperature in the bar to fall to $20°C$. [13 Marks]

END OF PAPER