STRATHMORE UNIVERSITY

INSTITUTE OF MATHEMATICAL SCIENCES

BBS ACTUARIAL SCINCE

END OF SEMESTER EXAMINATION

BSA 3218 ACTUARIAL MODELING 2

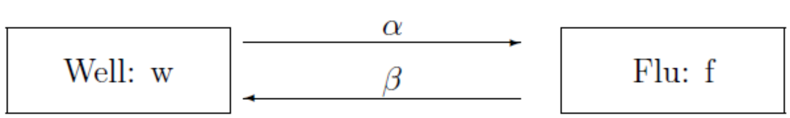
DATE: 10th July 2017 Time: 2 Hours

Instructions

1. This examination consists of **FIVE** questions.
2. Answer **Question ONE (COMPULSORY)** and any other **TWO** questions.

Question 1:

Consider the two-state model for influenza or flu for males aged between 30 and 40 described in the diagram. Deaths are ignored.



(a) Explain the difference between ${}_tp_x^{\overline{ww}}$ and ${}_tp_x^{ww}$. Which is larger? (3)

(b) Let $y(t) = {}_tp_x^{\overline{ww}}$. Find the Kolmogorov forward equation for $y(t)$. Write down the solution to this equation. (3)

(c) Let $z(t) = {}_tp_x^{ww}$. Find the Kolmogorov forward equation for $z(t)$ in terms of $z(t)$ only. (3)

(d) Show algebraically that $e_x = p_x(1+e_{x+1})$.

Hence

Given that $e_{50} = 30$ and $\mu_{50+t} = 0.005$ for $0 \le t \le 1$, what is the value of e_{51}?

(e)

In a mortality investigation covering a 5-year period, where the force of mortality can be assumed to be constant, there were 46 deaths and the population remained approximately constant at 7,500. Estimate the force of mortality.

Find a 95% confidence interval for the force of mortality

(f) A life insurance company has a small group of policies written on impaired lives and has conducted an investigation into the mortality of these policyholders. It is proposed that the crude mortality rates be graduated for use in future premium calculations.

Discuss the suitability of two methods of graduation that the insurance company could use. [3]

(g)

(i) Prove that, under Gompertz's Law, the probability of survival from age x to age $x + t$, ${}_t p_x$, is given by:

$$ {}_t p_x = \left[\exp\left(\frac{-B}{\ln c} \right) \right]^{c^x (c^t - 1)} . \qquad [3]$$

For a certain population, estimates of survival probabilities are available as follows:

$${}_1 p_{50} = 0.995$$
$${}_2 p_{50} = 0.989 .$$

(ii) Calculate values of B and c consistent with these observations. [3]

(iii) Comment on the calculation performed in (ii) compared with the usual process for estimating the parameters from a set of crude mortality rates. [3]

(h) It is intuitively sensible to think that mortality is a smooth function of age. But can you think of any reasons for mortality not to be smooth at certain ages?

(i)

A student has said "If the data includes the whole population, there is no need to graduate the crude rates because there will be no sampling errors". Discuss briefly.

(j) For a given set of data you have calculated crude estimates $\{\hat{\mu}_{x+½}\}$ for ages $x = 30, 31, ..., 79$ under the Poisson model. You have graduated the rates assuming that the underlying force of mortality follows Makeham's law $\mu_x = A + Bc^x$.

You now wish to assess the adherence of the graduation to the observed data. State the test statistic and the form of the test.

Question 2 (20 marks)

Let T_x be a random variable denoting future lifetime after age x, and let T be another random variable denoting the lifetime of a new-born person.

(i) (a) Define, in terms of probabilities, $S_x(t)$, which represents the survival function of T_x.

(b) Derive an expression relating $S_x(t)$ to $S(t)$, the survival function of T.

[2]

(ii) Define, in terms of probabilities involving T_x, the force of mortality, μ_{x+t}.

[1]

The Weibull distribution has a survival function given by

$$S_x(t) = \exp\left(-(\lambda t)^{\beta}\right),$$

where λ and β are parameters ($\lambda, \beta > 0$).

(iii) Derive an expression for the Weibull force of mortality in terms of λ and β. [3]

(iv) Sketch, on the same graph, the Weibull force of mortality for $0 \le t \le 5$ for the following pairs of values of λ and β:

$\lambda = 1, \ \beta = 0.5$
$\lambda = 1, \ \beta = 1.0$
$\lambda = 1, \ \beta = 1.5$

[4]

Data are available from a small portfolio of impaired lives, and show the time in months until a claim is made; the + indicates that there was no claim made under the policy and the policy was censored.

Male	1	1+	2+	2+	5	7	12
Female	2	4+	6	7+	10+	18+	21

(v) The following R-code fits the Kaplan-Meier model to the data on males only

```
library(survival)
Time.Male = c(1,1,2,2,5,7,12)
Censor.Male = c(1,0,0,0,1,1,1)
summary(survfit(Surv(Time.Male, Censor.Male) ~ 1))
```

Part of the output from the above code is

time	n.risk	n.event	survival
1	7	1	0.857
5	3	1	0.571
7	2	1	0.286
12	1	1	0.000

Let $S_m(t)$ denote the probability that the time to claim for a male exceeds t.

(a) Use the above output to estimate $S_m(6)$.

(b) State the formula for the Kaplan-Meier estimate of $S_m(6)$ and use it to verify your answer to (i). (5)

Question 3 (20 marks)

A portfolio, $\mathcal{P}$, of insured lives is divided into three groups, A, B and C. The force of mortality at time t of a life in $\mathcal{P}$ is denoted $\lambda(t; \boldsymbol{z})$ where $\boldsymbol{z}' = (z_B, z_C)$ is an indicator vector for group membership; group A is taken as the reference group. Suppose that $\lambda(t; \boldsymbol{z})$ follows the Cox proportional hazards model with regression coefficients $\boldsymbol{\beta}' = (\beta_B, \beta_C)$. Let $\lambda_A(t)$ be the force of mortality for group A with a similar notation for groups B and C.

(a) Write down expressions for $\lambda_A(t)$, $\lambda_B(t)$ and $\lambda_C(t)$ in terms of a baseline hazard $\lambda_0(t)$ and the regression coefficients $\boldsymbol{\beta}$. (2)

(b) The log of the force of mortality of all three groups is plotted against t. What property does the graph have? (1)

(c) Suppose initially that there are three lives in each group and that the first death occurs in group C at time $t = 3$. Previously, a life in group A had been censored at $t = 1$ and a life in group B had been censored at $t = 2$. What is the contribution to the partial likelihood of this death? (3)

(d) Suppose that the two censored lives in (c) had both been censored at time $t = 2$. What difference, if any, does this change in the data make to the partial likelihood? (1)

(e) What general property of the partial likelihood does (d) illustrate? (1)

(f) Suppose the Cox proportional hazards model applies to the full data and that the `Male` group is taken as the reference category, i.e., $\lambda_F(t) = \lambda_M(t)\exp(\beta)$ in the standard notation.

(i) Find the partial loglikelihood $\ell(\beta)$.

(ii) The following additional R-code fits the Cox model to the data.

```
Time.Female = c(2,4,6,7,10,18,21)
Censor.Female = c(1,0,1,0,0,0,1)
Time = c(Time.Male, Time.Female)
Censor = c(Censor.Male, Censor.Female)
Sex <- factor(c(rep(1,7), rep(2,7)))
summary(coxph(Surv(Time, Censor) ~ Sex))
```

Part of the output from the above code is as follows.

```
  n= 14
         coef  exp(coef) se(coef)
Sex2 -0.619      0.539    0.716
```

Use this output to write down the maximum likelihood estimate of β together with an estimate of its standard error. Hence conduct a z-test (also known as a Wald test) of $H_0 : \beta = 0$ v $H_1 : \beta \neq 0$; state your conclusion carefully.

(10)

Question 4 (20 marks)

The table gives the central exposed to risk, E_x^c, the number of deaths, d_x, and $\log \hat{\mu}_x$ for a set of male assured lives, $\mathcal{S}$, for ages $x = 60, \ldots, 72$. Here $\hat{\mu}_x$ is the usual maximum likelihood estimate of the force of mortality at age $x + \frac{1}{2}$.

Age x	E_x^c	d_x	$\log \hat{\mu}_x$	$\dot{d}_x$	z_x
60	52250	314	-5.11	*	-0.02
61	47400	312	-5.02	320.7	*
62	44860	352	-4.85	341.3	0.58
63	42920	334	-4.86	367.1	-1.73
64	38920	405	-4.57	374.3	1.59
65	24950	257	-4.58	269.8	-0.78
66	16900	206	-4.41	205.5	0.03
67	14990	212	-4.26	204.9	0.50
68	13550	235	-4.05	208.3	1.85
69	12160	223	-4.00	210.1	0.89
70	10950	200	-4.00	212.7	-0.87
71	9770	217	-3.81	213.4	0.25
72	8770	195	-3.81	215.4	-1.39

The table is graduated by fitting a Poisson generalised linear model, giving a fitted graduation of the form $\log \dot{\mu}_x = -12.145 + 0.1172x$. The table also shows the fitted number of deaths, $\dot{d}_x$, and the standardised residuals, z_x, from this model.

(a) Plot $\log \hat{\mu}_x$ against age. (2)

(b) Add a plot of the fitted force of mortality to the plot in (a) and hence comment on the suitability of the model. (3)

(c) The graduated deaths, $\dot{d}_x$, and the standardised residuals, z_x, are shown in the table. Calculate the missing values of $\dot{d}_x$ and z_x (indicated by $*$) and hence plot the standardised residuals against age. Comment on the success of the graduation. (6)

(d) Use the values in the table to carry out the standardised deviations test of the suitability of the graduation; use four equal area cells for the test. (4)

(e) An actuary suggests graduating the table by minimising

$$S(\alpha, \beta) = \sum_x w_x (\log \hat{\mu}_x - \alpha - \beta x)^2.$$

Explain briefly the role of the weights w_x in the above expression and use the Δ-method to suggest suitable values for the weight function w_x. (4)

Question 5 (20 marks)

A census type investigation into the mortality of a certain class of policyholders was conducted. The definition of age x for the census data is: age x nearest birthday; the definition of age x for those who died is: age x next birthday. Part of the data is given below.

	Census data			Total
Age	1/1/02	1/1/03	1/1/04	Deaths
50	5451	4515	5934	70
51	6002	5534	4428	76
52	5789	6087	5172	83

(a) Use these data to estimate μ_{51} and from this deduce an estimate of q_{51} (give your answers to 5 decimal places). At what exact ages do these estimates apply? Any assumptions made should be clearly stated. (8)

(b) (i) Estimate q_{51} directly using an appropriate estimate of the initial exposed to risk E_{51}.

(ii) Explain briefly which of the two methods used in parts (a) and (b)(i) is preferable for estimating q_{51}. Give reasons for your answer.

(4)

(c) Suppose now that additional census data are available, and a new estimate $\tilde{\mu}_{51} = 0.00751$ is obtained based on this additional information. Compute an approximate standard error for $\tilde{\mu}_{51}$ and hence comment on the difference between the two estimates of μ_{51}. (3)

(d) Consider now that the definition of age x for those who died is: age x next birthday on previous 1 January.

(i) State the type and range of the rate interval under this definition of age for deaths.

(ii) Assume that we obtain new estimates for μ_{51} and q_{51} based on this definition of age for deaths. State the exact ages at which these estimates apply, justifying your answers. Any assumptions made should be clearly stated.

(3)

END