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Measuring and evaluating factors of dynamic term structure models for value-at-risk estimation in the Kenyan market

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Measuring and Evaluating Factors of Dynamic Term Structure Models for Value-at-Risk Estimation in the Kenyan Market

By

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This research project is submitted to the Strathmore Institute of Mathematical Sciences in partial fulfillment of the requirement for the degree of Masters of Science in Mathematical Finance.

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DECLARATION

I declare that this work has not been previously submitted and approved for the award of a degree by this or any other University. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

Name.. **Sign**.. **Date**......................

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This project has been submitted for examination with my approval as the University Supervisor.

Sign.. **Date**..

Dr. Lucy Muthoni

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DEDICATION

This project is dedicated to my dear wife, Mwende Kanda and my precious parents. I am grateful for their love and support. I am also indebted to my siblings and friends, whose support has been priceless. Above all, Soli Deo Gloria; as from Him and through Him and to Him are all things. To Him be glory forever. Amen.

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LIST OF ABBREVIATIONS

1. VaR - Value-at-Risk
2. GARCH - Generalized Autoregressive Conditional Heteroskedasticity
3. CBK - Central Bank of Kenya
4. CMA - Capital Markets Authority
5. NSE - Nairobi Stock Exchange
6. IMF - International Monetary Fund
7. DCC - Dynamic Conditional Correlation
8. GRG - Generalized Reduced Gradient Algorithm
9. LBFGS - Limited Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm
10. AR - Autoregressive Process
11. BIC - Bayesian Information Criterion
12. AIC - Akaike Information Criterion
13. GEV - Generalized Extreme Value Distribution
14. NID - Normally and Independently Distributed

Abstract

The growth of the Kenyan fixed income market and the growing need for investors to diversify the risk in the portfolios has driven the need to quantify the risk associated with the fixed income assets. The measures that are commonly used to estimate bond risk are duration and convexity. However, these measures do not sufficiently assess the risk in fixed incomes. The risk associated with the yield curve illustrates how a portfolio will react to different exposures based on how the yield curve shifts.

In this research, we will seek to model the yield curves for the Kenyan market using the dynamic factor models, namely, Nelson-Siegel and Svensson models. We will estimate the factors for both models and seek to establish the distributions of the estimated factors. We will then seek to use the estimated factors from both models to generate the vector of expected bond yields and the covariance matrix that will be used to measure the Value-at-Risk.

The results of this research will be used to seek a parametric method of measuring risk in a fixed income in an illiquid market and check whether the estimated factors are good fits to be used in the parametric model.

Keywords: **Value at Risk, Nelson-Siegel, Factors, Svensson, Yield Curve, Fixed Income**

Contents

1 Introduction

Kenyan bond market is the third-biggest fixed income market in the Sub-Saharan Africa after those in South Africa and Nigeria. The Kenyan bond market is considerably smaller and illiquid when compared to the markets in South Africa and Nigeria. The trading volume in the Kenyan bond market varies from Kshs.7 billion to Kshs.10 billion daily. The cumulative volume for the 60 issues stands at Kshs.1.48 trillion.(**www.africanbondmarkets.org**). Treasury bonds represent more than 70% of all government securities on offer. Commercial bank share the major investors in treasury bonds at 55% with pension funds at 27.3% ,insurance firms at 11.1%, parastatals at almost 4% and Individual investors at 2%. In its ten-year master plan (2013-2023), the Capital Markets Authority (CMA) cast a vision of revamping Kenya into the core of African Capital Markets. One key purpose of the plan are to achieve a country financing through capital markets of 30% of GDP, an equity market capitalization of 70% and a corporate bond market capitalization of 40%. The corporate bond market capitalization stands at slightly above 2%.(**www.africanbondmarkets.org**)

Deep and liquid markets are essential for a country to sustain development driven by market determined capital allocation. Studies conducted by the International Monetary Fund(IMF) (Yibin et al.2013) show that there is a direct relationship between government securities' market capitalization and interest rate volatility. Moreover, there is an inverse relationship between government securities' market capitalization and the following factors namely; exchange rate volatility, interest rate spreads, fiscal balance and current and capital account openness. The studies further show that corporate bond markets' capitalization has a direct relationship to better institutions, the level of development in the economy, interest rate volatility and economic size. However, corporate bond markets' capitalization is inversely associated with current account openness and higher interest rate spreads.

The most common risk in the bond market is associated with the interest rate movements. This is because of the inverse relationship between bond prices and interest rates; bond prices rise when interest rates drop while the converse is true. Bonds with longer maturities are subject to greater interest rate risk due to the greater probabilities of interest rates rising in the long term.

The most common measure used in bond portfolio risk management is modified duration which measures changes in bond prices on account of changes in interest rates. However, duration is limited because of its assumption of a linear affiliation between interest rates and bond price which is mostly curvilinear. The greater the convexity observed between the interest rates and bond prices, the greater the inaccuracy duration is as a measure. Duration is useful in measuring slight shifts in interest rates because of convexity. However, this decrement in the prices is disproportional for bonds with coupons because of the non-linearity in the relationship. The relationship between the flux in bond value to flux in interest rates is in the shape of a convex curve to origin. Therefore, duration will predict a lesser price than the original. For big changes in interest rates, this margin can be significant. Duration is limited as it can only be applied to measure the approximate change in bond price due to interest changes, only if changes in interest rates do not lead to a change in the shape of the yield curve.

The shape of the yield curve shows the movement of interest rates. Upward sloping yield curve indicate rising interest rates while downward sloping yield curve indicate declining interest rates.

1.1 Background of the study

Government securities are riskless if held to maturity. However, they are exposed to market risk when the interest rates change in the investment horizon. Major investors like banks, insurance companies and pension funds face exposure to risk in their books due to movements in the interest rates.

With increasing government debt in Kenya, the present conundrum lies with how the interest rates will rise to compensate for the perceived risk of the government repaying its debt. Currently, there is a higher demand for short term government securities compared to medium-term or long-term securities resulting in a at yield curve instead of a normal yield curve.

1.2 Problem Statement

The problem this study seeks to address is to estimate and evaluate the parameters from dynamic factor models that will be used to measure the risk exposure with a metric that takes into account the volatility of price and the distribution of the returns. Value at Risk (VaR) measures the worst case loss at a given confidence level and investment horizon. According to Danielsson et al (2005), VaR is proportional to the variance-covariance of portfolio returns because of delta normal and would be subadditive due to the diversification effect even in fat-tail condition.

Several concerns have been raised about measurement of yield curve risk. Financial asset returns are mostly assumed to be normally distributed. However, empirically, most financial returns have been found not to be normally distributed. In the instance where financial returns are not normally distributed, skewness and kurtosis become uncertain parameters resulting in incorrect risk measures.

1.3 Main objectives

- To estimate a time series of the factors for the dynamic factor models.
- To analyze the factors for both models an see to establish characteristics such as normality, multicollinearity and presence of conditional heteroskedasticity.
- Obtain the vector of returns expected and covariance matrix for the bond returns conditioned one step ahead.
- To seek the statistics that will aid in the enumeration of a parametric VaR of a bond portfolio using yield curve models using the vector of bond returns expected and covariance matrix of bond returns based on term structure models namely Nelson-Siegel and Svensson models.
- To analyze price data of bonds in the NSE and establish characteristics such as normality and presence of conditional heteroskedasticity in the term structure of interest rates.

1.4 Significance of the Study

Treasury and corporate bonds are a major source of capital for government and corporate companies. However, the inverted association between interest rates and bond prices can negatively affect the market price of the bond leading to a loss. Market risk is the risk derived by price fluctuations.

Ultimately, we seek to quantify the market risk by computing (VaR) which is the threshold value such that the probability that the loss on the portfolio over a given time horizon exceeds the value is the given probability level. We will make assumptions we are dealing with normal markets and that there is no trading in the portfolio.

We will observe whether there will be the occurrence of the "fat-tailed problem ". The fat-tailed problem implies that there would be acute price variations in asset prices with a greater probability than predicted by the normal distribution. The assumption of normality for lower tail where there are losses would increasingly be inaccurate, the farther into the tail that one considers the difference. Asset returns that exhibit the fat-tail property often have that losses would tend to be more frequent. The assumption of normality tends to reduce the complexity in equations while non-normality assumptions increases the complexity.

VaR applies to the extreme lower tail of the return distribution, mainly the large losses far way from the mean. VaR computations influence capital requirements for the institutions in compliance to the Basel III guidelines (Burchi et al. 2016). This study seeks to estimate factors from term structure models that will be used to measure the VaR for a bond portfolio using the Yield Curve approach and make comparison with the Duration VaR and Historical-Simulation approaches. If successful, we will then verify each methods credibility using simple backtesting tools.

The VaR as a risk metric has attractive features as highlighted by Carol Alexander (Alexander, 2009):

1. It corresponds to an amount that could be lost with some chosen probability.

2. It measures the risk of the risk factors as well as the risk factor sensitivities.

3. Comparison of VaR measures can be made across different markets and different exposures.

4. VaR is a widely applied measure that can be used for various types of risk.

5. The VaR can be measured at both the individual and portfolio level. Furthermore, a single enterprise-wide VaR measure can be applied to all the risks in the firm as a whole.

6. The VaR can be aggregated and disaggregated to find total VaR of big portfolios and break down to isolate constituent risks corresponding to various risk types. The VaR can be broken down to constituent risks corresponding to various types of risk while considering the dependencies between the assets or portfolios.

2 Literature Review

2.1 Value-at-Risk(VaR): A definition

VaR measures the threshold loss that we are quite convinced will not be surmounted if the current portfolio is held over some period of time. VaR addresses the magnitude of how big or small the loss is. VaR comprises of two basic parameters namely ;

(i) the significance level α ;and

(ii) the risk horizon, denoted h which is the period of time over which the VaR is measured to estimate the potential loss.

Thus, for the continuous distribution, given $(1-\alpha)100\%$ confidence level, the VaR is the $\alpha 100\%$ quantile and can be written in mathematical form as:

$$VaR_{\alpha}(X) = -inf[x|P(X \leq x) > \alpha], \tag{1}$$

where X is a random variable denoting the profit and loss and x is VaR.

2.2 Literature on Fixed Income VaR

Literature on VaR modelling of equity portfolios is vast. The literature on VaR modeling of fixed income securities is not as comprehensive as compared to that on equities.
One explanation given by Caldeira (2015) for the sparse information on VaR modeling

for fixed income securities is the analogous stability and low historical volatility of this class of assets, which deterred the application of more technical approaches to check and to manage the market risk of fixed income securities.

VaR estimation requires knowledge about the nature of the underlying asset or assets one is computing the VaR on. Thus, computing the VaR for bonds requires an extra effort when compared to stocks. For a bond or bond portfolio, additional factors have to be taken into consideration. Specifically, three aspects with regards have to be considered : the ever-changing characteristics of a bond as the maturity approaches, future cash flows and the pull to par effect.

Despite the fact that little or no literature exists on agricultural commodities spread options, the concept of spread option pricing is similar, save for the stylized facts of the different data sets. However, stylized facts like mean reversion are common in both energy and agricultural markets.

Our initial paper of interest is by Sousa et al.(2012). The authors sought to measure the VaR using adjusted historical returns. Bonds historical returns cannot be directly applied to calculate VaR via the historical simulation method since the maturities of the interest rates implied by the historical prices are not the relevant maturities at time VaR is computed.

The authors adjusted the bonds historical returns and directly applied the adjusted returns to calculate VaR by historical simulation. The adjustment was based on the prices, implied by the historical prices, at the times to maturity relevant for the VaR computation. They obtained VaR values that agreed with the common market trend of smaller times to maturity being traded with smaller interest rates, thus, carrying less risk and thereby having a smaller VaR. They observed that the obtained VaR values were greater than the relevant computed maturities-at-time VaR because they corresponded to past times when the bond maturity was further away then it is when VaR is computed. Their main motivation with the keeping the historical simulation was its simplicity.

Another short paper of interest is Barone(Barone et al. 2000). The authors observed that the excess kurtosis will cause more frequent occurrence of losses greater than VaR. The assumption for the normal distribution is because it is well described. The normal distribution is easily understood as it can be defined using only the first two moments, mean and variance. Other distributions could be used, but at a greater computational cost because of complexity.

One interesting paper of note was an investigation on the accuracy of VaR models on Dutch fixed income portfolios by Vlaar (2000). The VaR methods used were historical simulation, variance-covariance and Monte Carlo simulation. He tested the accuracy of the VaR models using the dynamics of the term structure of Dutch interest rates. The best results obtained were from the combined variance-covariance Monte-Carlo method using a term structure model with a normal distribution and GARCH specification while term structure models with a t-distribution or with cointegration performed much worse.

Our paper of interest is that of Caldeira (Caldeira et al.,2015). The authors proposed an approach to measure risk in zero-coupon bond portfolios in terms of VaR. They used closed-form expressions for the vector of expected bond returns and for the covariance matrix of bond returns based on a general class of well established term structure factor models. In order to compute the parametric VaR of a portfolio composed of fixed income securities, they used the dynamic Svensson and Nelson-Siegel models .

The authors highlighted the advantage of flexibility as their approach could adapt alternative specifications to model the yield curve and other specifications to model the conditional heteroskedasticity in bond returns. This approach was empirically applied on a data set composed of constant-maturity future contracts of the liquid Brazilian Inter Bank Deposit Future Contract which is equivalent to a zero-coupon bond. The authors obtained out-of-sample VaR estimates that outperformed benchmark specifications in modeling and forecasting the one-step-ahead VaR at different levels.

The volatility of interest rates usually tend to be high with high interest rates leading to the yield curve exhibiting more curvature. Thus, the presence of conditional heteroskedasticity in the term structure of interest rates has to be considered.

Assumption that the interest rate volatility will be constant ignores the time-varying aspect of interest rate risk. The authors warn of the risk of miscalculation of the confidence intervals for the forecasts obtained from these models in finite samples, in the presence of conditional volatilities. The authors incorporated the effects of time-varying volatility using a multivariate GARCH specification.

3 Methodology

This study will ultimately focus on obtaining VaR estimates using the proposed Yield curve approach on a data set of bond prices obtained from the Nairobi stock exchange over a period of 5 years.
The research done by Caldeira et al.(2015) will provide the methodology to be adapted. This is particularly applicable as the VaR model is parametric which can be applied in an illiquid bond market like the Kenyan market. We will seek to apply the proposed parametric VaR model in an illiquid market.

3.1 Specific Objectives

Generation of the closed form expressions for the expected returns and their conditional covariance matrix will be enabled by the factor models for the term structure of interest rates. These moments will guide us in obtaining the distribution of the bond prices and bond returns, that will later be used as an input to calculate the VaR of a bond portfolio.

3.2 Dynamic Term Structure Models

We will consider a set of time series of bond yields with N different maturities $\tau_1, \ldots, \tau_N$. The yield of a bond at time t with maturity τ_i will be denoted by $y_t(\tau_i)$ for $t = 1 \ldots, T$. The $N \times 1$ vector of all yields at time t is given by:

$$y_t(\tau) = (y_t(\tau_1) \ldots y_t(\tau_N)), t = 1 \ \ldots, T.$$

The general specification of the dynamic factor model is given by:

$$y_t = \Lambda(\lambda) f_t + \varepsilon_t, \varepsilon_t \sim \mathcal{NID}(0, \Sigma_t), t = 1 \ldots T. \tag{2}$$

with $\Lambda(\lambda)$ is the $N \times M$ is the matrix of factor loadings, f_t is a M-dimensional stochastic process, ε_t is the $N \times 1$ vector of disturbances and Σ_t is an $N \times N$ conditional covariance matrix of the disturbances. The covariance matrix Σ_t will be restricted to be diagonal so that the covariance between the yields will be explained by the common latent factor f_t alone. The factors f_t will be modeled by the stochastic process below:

$$f_t = \mu + \Upsilon f_{t-1} + \eta_t, \eta_t \sim \mathcal{NID}(0, \Omega_t), t = 1 \ldots T \tag{3}$$

with μ is the $M \times 1$ vector of constants, Υ is the $M \times M$ transition matrix, and Ω_t is the conditional covariance matrix of the disturbance vector η_t, which is independent of the vector of residuals ε_t $\forall t$.

The specification for f_t will be a vector autoregressive process pf lag order 1 for modeling the yield curves. We will consider two factor models for the yield curve namely; the Nelson-Siegel model proposed by Diebold & Li(2006) and the Svensson (1994) model.

3.2.1 Nelson-Siegel Model

The starting point of the original Nelson-Siegel model for the continuously compounded spot rate, with $\tau = T - t$ is

$$y_t(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_3 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right), \tag{4}$$

where $f_t = (\beta_1, \beta_2, \beta_3)$ and λ are parameters. Diebold & Li (2006) interpreted the factors in equation (4) $\beta_1, \beta_2, \beta_3$ are long-term time-varying level, slope and curvature factors. The coefficients that multiply these factors are loadings factor:

$$y_t(\tau) = \beta_{1t} + \beta_{2t} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_{3t} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right), \tag{5}$$

with $y_t(\tau)$ denoting the continuously-compounded zero-coupon yield at maturity τ , and $\beta_{1t}, \beta_{2t}, \beta_{3t}$ and λ_t are time-varying parameters. Thus, equation (4) is a dynamic version that allows the parameters to vary through time. Hence, the Dynamic Nelson-Siegel model.

3.2.2 Svensson model

Svensson(1994) proposed an exponential term as an extension to the Nelson-Siegel model that has an additional decaying parameter. The Svensson model with four factors is written as

$$y_t(\tau) = \beta_{1t} + \beta_{2t} \left(\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau} \right) + \beta_{3t} \left(\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau} \right) + \beta_{4t} \left(\frac{1 - e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau} \right) \tag{6}$$

The fourth factor can be understood as a second curvature. The fourth factor enables the Svensson model to fit term structure curves with more than one maxima or minima along the maturity spectrum.

3.3 Conditional covariance of the factors

To model Ω_t, the conditional covariance matrix of the factors in equation(2), they considered alternative specifications, including not only multivariate GARCH models and multivariate stochastic models. They considered the dynamic conditional correlation model (DCC) that is formulated as;

$$\Omega_t = D_t \Psi_t D_t \tag{7}$$

where D_t is a $M \times M$ diagonal matrix with diagonal elements given by $h_{f_{mt}}$, where $h_{f_{mt}}$ is the conditional variance of the $m-th$ factor, and Ψ_t is a symmetric correlation matrix with elements $\rho_{ij,t}$ where $\rho_{ii,t} = 1$, i,j=1,..., M. In the DCC model, the conditional correlation $\rho_{ij,t}$ is given by:

$$\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}} \tag{8}$$

where $q_{ij,t}$, with $i, j = 1, \ldots M$, are the elements of the $M \times M$ matrix Q_t , which follows a GARCH-type dynamics:

$$Q_t = (1 - \alpha - \beta)\bar{Q} + \alpha z_{t-1} z'_{t-1} + \beta Q_{t-1} \tag{9}$$

where $z_{f_t} = (z_{f_1 t}, \ldots\ldots, z_{f_k t})$ is the standardized vector of returns of the factors, whose elements are $z_{f_{it}} = \frac{f_{it}}{\sqrt{h_{f_i t}}}$, $\bar{Q}$ is the unconditional covariance matrix z_t, α, e, β are non-negative scala parameters satisfying $\alpha + \beta < 1$. The conditional variance of the measurement errors ε_t in equation (1) will be modeled under the assumption that Σ_t is a diagonal matrix with diagonal elements $h_{t\varepsilon_i}$, where $h_{t\varepsilon_i}$ is the conditional variance ε_i . To maintain simplicity, the conditional covariance will depend on one lag of past returns and conditional variances.

3.4 Expected bond returns and the conditional covariance matrix of bond returns

VaR computation requires projections of the expected return of each bond and the covariance matrix of the set of bonds in the portfolio However, the highlighted factor models for the term structure of interest rates cater only for the bond yields. On the basis of the distribution of the expected yields, it is possible for one to get the expected bond return and conditional covariance matrix of bond returns.The following propositions by Caldeira et al.(2013) proposed the following in order to define this distribution.

Proposition 1: Given the system of equations in (1) and (2), the distribution of expected yields $y_{t|t-1}$ is $N(\mu_{y,t}, \Sigma_{y,t})$ with $\mu_{y,t} = \Lambda f_{t|t-1}$ and $\Sigma_{y,t} = \Lambda\Omega_{t|t-1}\Lambda' + \Sigma_{t|t-1}$, where $f_{t|t-1}$ is the one-step-ahead forecast of the factors and $\Sigma_{t|t-1}$ and $\Omega_{t|t-1}$ are the one-step-ahead forecasts of the conditional covariance matrices in (1) and (2) respectively.

Caldeira et al.(2013) showed the vector of bond returns expected is:

$$\mu_t = \mathbb{E}_{t-1}[y_t] = \Lambda(\lambda)\,\mathbb{E}_{t-1}[f_t] = \Lambda(\lambda)f_{t|t-1} \tag{10}$$

where $f_{t|t-1}$ are the one-step-ahead predictions of the factors.

We will seek to extract the distribution of the anticipated fixed maturity bond process using the results from proposition 1. Considering the price of a bond at time t, $P_t(\tau)$ is the present value at time t of Kshs. 1 receivable τ periods ahead, and letting $y_{t|t-1}$ denoting denote the one-step-ahead forecast of its continuously compounded zero-coupon nominal yield to maturity, we obtain the vector of expected bond prices $P_{t|t-1}$ for all maturities:

$$P_{t|t-1} = \exp\Big(-\tau \otimes y_{t|t-1}\Big) \tag{11}$$

where $\otimes$ is the Hadamard(elementwise) multiplication and τ is the vector of the maturities. Since $y_{t|t-1}$ follows a Normal distribution, $P_{t|t-1}$ has a log-normal distribution.

The log-returns can be written as:

$$r_t = log\left(\frac{P_t}{P_{t-1}}\right) = \log P_t - \log P_{t-1} = -\tau \otimes (y_t - y_{t-1}) \tag{12}$$

Using equation (11), closed form expressions for the vector of expected bond returns and for their conditional matrix can be maintained.

Proposition 2: Given the system of equations in (1) and (2), the vector of expected log returns for the bonds $\mu_{r_{t|t-1}}$, and their conditional covariance matrix $\Sigma_{r_{t|t-1}}$, which is positive-definite $\forall t$ are given by:

$$\mu_{r_{t|t-1}} = -\tau \otimes \mu_{y,t} + \tau \otimes y_{t-1} \tag{13}$$

$$\Sigma_{r_{t|t-1}} = \tau\tau' \otimes \Big[\underbrace{\Lambda\Omega_{t|t-1}\Lambda' + \Sigma_{t|t-1}}_{\Sigma_{y,t}.}\Big] \tag{14}$$

The value can be obtained using the log-return expression:

$$r_t = \log\left(\log\frac{P_t}{P_{t-1}}\right) = \log P_t - \log P_{t-1} = -\tau \otimes (y_t - y_{t-1}) \tag{15}$$

Since $y_{t|t-1} \sim N(\mu_t, \Sigma_{yt})$ where μ_t and Σ_{yt} have been defined in proposition 1 and the expected returns $r_{t|t-1}$ follow $N \sim (\mu_{rt}, \Sigma_{rt})$ where

$$\mu_{r_{t|t-1}} = -\tau \otimes (\mathbb{E}_{t-1}[y_t] - \mathbb{E}_{t-1}[y_{t-1}]) = -\tau \otimes \mu_t + \tau \otimes y_{t-1} \tag{16}$$

$$\Sigma_{r_{t|t-1}} = \tau'\tau \otimes \left[\underbrace{\Lambda\Omega_{t|t-1}\Lambda' + \Sigma_{t|t-1}}_{\Sigma_{y,t}.}\right] \tag{17}$$

The matrix Σ_{rt} is positive. This is because $\Lambda\Omega_t\Lambda'$ is positive-definite since $\Omega_{t|t-1}$ is diagonal and has only positive elements on its diagonal. The term $\Sigma_{t|t-1}$ is also positive. Since τ is positive-definite, then $\tau\tau'$ is also a positive-definite matrix and Schur product theorem ensures that the Hadamard product between Σ_{yt} and $\tau\tau'$ is positive-definite.
Proposition 2 guides in obtaining closed-form expressions for the expected bond log returns and their covariance matrix based on both Nelson-Siegel and Svensson models.
The return on a fixed-maturity zero-coupon bond can be decomposed into two parts; the capitalized deterministic part due to ageing of the bond and secondly, the part linked to changes in the market prices of constant maturity bonds. Thus, the total return is given by the income generated by the variation in ,market prices. The total return between t and $t + h$ on a bond with a fixed maturity τ is given by:

$$R_{t+h}(\tau) = \frac{P_t(\tau)}{P_{t-h}(\tau)} - 1 + \frac{h}{252}y_{t-h}(\tau) = \exp(r_{y,t+h}) - 1 + \frac{h}{252}y_{t-h}(\tau) \tag{18}$$

where h is given on weekdays and $r_{y,t+h}$ is the log-return generated from the changes in yields of fixed maturities from period t to $t + h$.

3.5 VaR Computation

The general solutions for the vector of bond portfolio returns and their covariance matrix will be applied to compute the bond portfolio VaR.
The goal is to finally measure risk associated with increase in bond yields thus leading to lower bond prices and negative returns. An assumption is made for an equally-weighted portfolio.

Let $R_{t+h} = (r_{1,t+h}, \cdots, r_{N,t+h})'$ be the vector of h-period returns between the period t and $t+h$. The bond portfolio return will be given by $r_{p,t+h} = w' R_{t+h}$ where w_t is the vector of portfolio weights at time t.
The portfolio VaR at time t for a given holding period h and confidence level ν will be the ν-quantile of the distribution of the bond portfolio return. Thus, $VaR_t = F^{-1}_{p,t+h}(\nu)$, where $F^{-1}_{p,t+h}$ is the inverse of the cumulative distribution function of $r_{p,t+h}$.
The bond log-returns have a distribution that can be expressed in terms of its two first conditional moments. The return of the portfolio is:

$$r_{p,t+1} = \mu_{p,t+1} + \sigma_{p,t+1} z_{p,t+1} \tag{19}$$

where the standardized unexpected returns $z_{p,t+1}$ are independent and identically distributed with mean equal to 0 and variance equal to 1. $\mu_{p,t+1}$ and $\sigma_{p,t+1}$ are the respective conditional mean and standard deviation of the bond portfolio return, given by:

$$\mu_{p,t+1} = w_t' \mu_{r_{t+1}} \tag{20}$$

and

$$\sigma^2_{p,t+1} = w_t' \Sigma_{r_{t+1}} w_t \tag{21}$$

where $\mu_{r_{t+1}}$ is the $N \times 1$ vector of conditional mean returns for the N individual assets and $\Sigma_{r_{t+1}}$ is their $N \times N$ conditional covariance matrix in equations (12) and (13). The portfolio VaR is thus obtained below

$$VaR_{t+1} = \mu_{p,t+1} + \sigma_{p,t+1} q, \tag{22}$$

where q is the ν quantile of the distribution of $z_{p,t+1}$.

3.6 Estimation and Implementation of Methodology

The parameters of the factor models will first be estimated with the resulting residuals used to estimate the volatility model. This paper will estimate the factors and decaying parameters used in the approach by Diebold and Li (2013) since the the considered alternate yield curve specifications are all nested and can be defined in one general formulation.
A DCC specification will then be used to obtain the conditional covariance matrix of

the factors $\Omega_{t|t-1}$. The computation of the DCC model will be partitioned into two: the volatility part i.e evaluating the univariate volatility models of the factors using a GARCH-type specification and the correlation part, which refers to the estimation of the conditional correlation matrix in equations (7)and (8).

Finally, we will backtest the VaR in order to analyse the VaR violations. The VaR violations may be clustered or sparse. This will aiding analysing the accuracy of the VaR. We will employ the independence, unconditional and conditional coverage tests proposed by Christoffersen [1998].

Empirical tests will be carried out using R-version computer software and Microsoft Excel-Addins (XLSTAT and Solver).

4 Findings

4.1 Bond Yields Data and Data Analysis

The data set consists of bond yields on the Nairobi Stock Exchange. The sample period consists of 436 bond yields observed monthly from December 2008 to December 2017. The bond yields are annual for the bonds with tenors of 2 years, 5 years, 10 years and 15 years.

Several optimization methods that are non-linear in nature can be used to solve for the parameters in the Nelson-Siegel and Svensson models. Muthoni(2015) noted that the grid search method initially proposed by Nelson and Siegel could result in erratic behaviour over a period of time. We use the Generalized Reduced Gradient (GRG) algorithm to estimate the parameters $\beta_1, \beta_2, \beta_3, \lambda$ for the Nelson-Siegel and the parameters $\beta_1, \beta_2, \beta_3, \beta_4, \lambda_1, \lambda_2$ for the Svensson model while fixing the maturity of the bond τ be constant. The GRG method is similar to the L-BFGS method used by Muthoni as it maintains a dense BFGS approximation of the Hessian of the function with respect to the variables.

The table below shows the descriptive statistics for the bond yields on the Nairobi Stock Exchange.

Table 1: **Descriptive Statistics of the Bond Yields**

Statistic	**2-Year Yield**	**5-Year Yield**	**10-Year Yield**	**15-Year Yield**
Minimum	0.031	0.042	0.058	0.065
Maximum	0.180	0.183	0.187	0.228
$1st$ Quartile	0.100	0.109	0.116	0.124
Median	0.112	0.118	0.124	0.131
$3rd$ Quartile	0.121	0.130	0.134	0.139
Mean	0.106	0.115	0.123	0.132
Variance	0.001	0.001	0.001	0.001
Standard Deviation	0.027	0.025	0.023	0.024

4.1.1 Nelson Siegel

The descriptive statistics for the estimated parameter values of the time series of the $\beta_1, \beta_2, \beta_3, \lambda$ factors are presented in the table below.

Table 2: **Descriptive Statistics of the Nelson-Siegel Parameters**

Statistic	β_1	β_2	β_3	λ
Minimum	0.005	-1579.917	-0.484	0.000
Maximum	0.267	0.034	448.977	12,909.174
1st Quartile	0.118	-0.028	-0.146	0.053
Median	0.125	-0.002	0.024	0.071
3rd Quartile	0.142	0.000	0.045	3.640
Mean	0.130	-29.060	6.019	198.120
Variance	0.002	43,305.047	2,177.795	2,102,425.751
Standard Deviation	0.039	208.099	46.667	1,449.974

The estimated values for β_2 and β_3 are mostly negative. This indicates that the yield curves are mostly positively upward-sloping with visible humps at times. The estimated values for β_1 are always positive.

Diebold and Li (2006) treated their parameter λ as fixed for simplicity in the estimation procedure for the remaining parameters. However, we treated the λ as an unknown so that we could fit its distribution.

Using the maximum likelihood method to fit the distribution at 5% significance level, the table below shows the distributions that best fit the parameters.

The estimated parameters were observed to have varied distributions. The tables below show the parameters with the corresponding best-fit distribution.

The Arcsine distribution best fits the β_1 using the goodness of fit test. The following distributions had zero p-values from the goodness of fit tests: Chi-square, Exponential, Fisher-Tippett (1), Gumbel, Standard Normal, Student-t and Weibull (1).

Table 3: **Distribution Fitting for β_1**

Distribution	**p-value**
Arcsine	1.00000000
Beta4	0.00404498
Logistic	0.06225391
Normal	0.00360232

Table 4 below shows the estimated parameter for the β_1 for its' arcsine distribution.

Table 4: **Estimated Parameter for β_1**

Parameter	**Value**	**Standard Error**
alpha	0.322	0.026

The Log-likelihood statistics for β_1 are shown below;

Table 5: **Log-Likelihood Statistics for β_1**

Log-likelihood(LL)	17.691
BIC(LL)	-30.691
AIC(LL)	-33.383

The table below shows the fitted distribution for the estimated parameter for the β_2. There was no distribution that best fit the β_2 for the goodness of fit test. The maximum likelihood estimation algorithm has reached an undefined value. The displayed parameters below have been estimated using the moments method.

Table 6: **Distribution Fitting for β_2 Using Moments Method**

Distribution	**p-value**
Beta4	0.000
GEV	0.000
Normal	0.000
Normal (Standard)	0.000

The table below shows the estimated parameter for the β_2 though the distribution is unknown.

Table 7: **Estimated Parameters for β_2**

Parameter	**Value**	**Standard Error**
	9.891	0.000
	1,647.707	0.000
	-1,595.874	0.000

The GEV distribution best fits the β_3 for the goodness of fit test. The following distributions had zero p-values from the goodness of fit tests: Fisher-Tippett (1), Fisher-Tippett (2), Gumbel, Logistic and Normal.

Table 8: **Estimated Parameters for β_2**

Distribution	**p-value**
GEV	0.00000000000585332
Normal (Standard)	0.00000000000063216
Student-t	0.00000000000003908

The table below shows the estimated parameter for the β_3.

Table 9: **Estimated Parameters for β_3**

Parameter	**Value**	**Standard Error**
k	1.752	0.001
beta	0.745	0.000
μ	-0.071	0.000

The Log-likelihood statistics for β_3 are shown below;

Table 10: **Estimated Parameters for β_3**

Log-likelihood(LL)	-117.360
BIC(LL)	248.794
AIC(LL)	240.720

The GEV distribution best fits the λ_t using the goodness of fit test. The following distributions had zero p-values from the goodness of fit tests: Fisher-Tippett (1), Fisher-Tippett (2), Gumbel, Logistic and Normal.

Table 11: **Estimated Parameters for λ_t**

Distribution	**p-value**
Gamma (1)	0.0000002
Log-normal	0.0003432
Weibull (2)	0.0000225

The table below shows the estimated parameter for the λ_t

Table 12: **Estimated Parameters for λ_t**

Parameter	**Value**	**Standard Error**
μ	-1.341	0.242
$sigma$	2.701	0.171
μ	-0.071	0.000

The Log-likelihood statistics for λ_t are shown below;

Table 13: **Estimated Parameters for λ_t**

Log-likelihood(LL)	-116.830
BIC(LL)	243.044
AIC(LL)	237.661

The Kolmogorov-Smirnov test is used to check the distributional accuracy. The β_1 reflects the best accuracy while the β_2, β_3 and λ_t have poor accuracies. The computed p-value

for β_1 is greater than the significance level hence the β_1 follows the Arcsine distribution more accurately. The other estimated distributions for β_2, β_3 and λ_t have lower computed p-values than the significance level hence are poor distribution fittings. The tables below show the p-values for the Kolmogorov-Smirnov tests for β_1 , β_2, β_3 and λ_t.
The Kolmogorov-Smirnov test for β_1 yields

Table 14: **Estimated Parameters for β_3**

Parameter	**Value**
D	0
p-value (Two-tailed)	1.000
alpha	0.05

The Kolmogorov-Smirnov test for β_2 below the computed p-value is lower than the significance level $alpha = 0$. This shows that the fitted distribution for the β_2 is inaccurate.

Table 15: **Estimated Parameters for β_3**

Parameter	**Value**
p-value (Two-tailed)	< 0.0001
alpha	0

The Kolmogorov-Smirnov test for β_3 below the computed p-value is lower than the significance level $alpha = 0$. This shows that the fitted distribution for the β_3 is inaccurate. Thus β_3 does not follow a GEV distribution.

Table 16: **Estimated Parameters for** β_3

Parameter	Value
D	0.345
p-value (Two-tailed)	< 0.0001
alpha	0.05

The Kolmogorov-Smirnov test for λ_t below the computed p-value is lower than the significance level $alpha = 0$. This shows that the fitted distribution for the λ_t is inaccurate. Thus λ_t does not follow a Log-normal distribution.

Table 17: **Estimated Parameters for** β_3

Parameter	Value
D	0.197
p-value (Two-tailed)	0.000
alpha	0.05

4.1.2 Svensson

The mean values of the time series of the β_1, β_2, β_3 , β_4 , λ_1 and λ_2 factors are presented in table 18 below.

Table 18: **Descriptive Statistics for the Svensson Factors**

Statistic	β_1	β_2	β_3	β_4	λ_1	λ_2
Minimum	0.034	-0.128	-0.460	0.074	0.160	0.006
Maximum	0.146	0.360	-0.028	1.513	6.914	0.209
1st Quartile	0.098	0.043	-0.090	0.098	2.489	0.038
Median	0.102	0.077	-0.062	0.109	2.651	0.052
3rd Quartile	0.107	0.102	-0.054	0.132	2.896	0.106
Mean	0.101	0.070	-0.079	0.146	2.728	0.072
Variance	0.000	0.004	0.003	0.026	0.797	0.002
Standard Deviation	0.019	0.064	0.057	0.160	0.893	0.044

The estimated values for β_3 are negative which means that the yield curves are mostly positively upward-sloping with visible humps at times. The estimated values for β_1 are always positive.

The parameters were observed to have varied distributions similarly to the Nelson-Siegel parameters. The table below shows the parameters with the corresponding best-fit distribution.

Table 19: **Distribution Fitting for β_1**

Distribution	**p-value**
Arcsine	1.000000
Normal	0.000006
Weibull (2)	0.000710

The Arcsine distribution best fits β_1 for the goodness of fit test . The following distributions had zero p-values from the goodness of fit tests: Chi-square, Erlang, Exponential, Fisher-Tippett (1), Fisher-Tippett (2), Gamma (2), GEV, Log-normal, Gumbel, Standard Normal, Student-t and Weibull (1).
The table below shows the estimated parameter for the β_1

Table 20: **Parameter Estimation for β_1**

Parameter	**Value**	**Standard Error**
alpha	0.305	0.025

The Log-likelihood statistics for β_1 are shown below;

Table 21: **Log-likelihood Statstics for β_1**

Log-likelihood(LL)	32.217
BIC(LL)	-59.742
AIC(LL)	-62.434

The normal distribution best fits β_2 for the goodness of fit test. The following distributions had zero p-values from the goodness of fit tests: Fisher-Tippett (1), Gumbel, Standard Normal and Student-t.

Table 22: **Distribution Fitting for β_2**

Distribution	**p-value**
Beta4	0.07344
Fisher-Tippett (2)	0.00227
GEV	0.00015
Normal	0.07588

The table below shows the estimated parameter for the β_2

Table 23: **Parameter Estimation for β_2**

Parameter	**Value**	**Standard Error**
μ	0.070	0.006
sigma	0.064	0.004

The Log-likelihood statistics for β_2 are shown below;

Table 24: **Likelihood Statistics for β_2**

Log-likelihood(LL)	145.616
BIC(LL)	-281.850
AIC(LL)	-287.233

The normal distribution best fits the β_3 for the goodness of fit test. The following distributions had zero p-values from the goodness of fit tests: Fisher-Tippett (1), Fisher-Tippett

(2), GEV, Gumbel, Standard Normal and Student-t.

Table 25: **Distribution Fitting for β_3**

Distribution	**p-value**
Normal	0.00006639

The table below shows the estimated parameter for the β_3

Table 26: **Parameter Estimation for β_3**

Parameter	**Value**	**Standard Error**
μ	-0.079	0.005
sigma	0.057	0.004

The Log-likelihood statistics for β_3 are shown below;

Table 27: **Likelihood Statistics for β_3**

Log-likelihood(LL)	158.392
BIC(LL)	-307.401
AIC(LL)	-312.783

The GEV distribution best fits β_4 for the goodness of fit test. The following distributions had zero p-values from the goodness of fit tests: Beta4, Chi-square, Exponential, Fisher-Tippett (1), Gamma (1), Gamma (2), Gumbel, Log-normal, Logistic, Normal, Normal (Standard), Student, Weibull (1) and Weibull (2).

Table 28: **Distribution Fitting for** β_4

Distribution	**p-value**
GEV	0.01318

The table below shows the estimated parameter for the β_4

Table 29: **Parameter Estimation for** β_4

Parameter	**Value**	**Standard Error**
k	0.448	0.047
beta	0.023	0.001
μ	0.103	0.001

The Log-likelihood statistics for β_4 are shown below;

Table 30: **Log-Likelihood Statistics for** β_4

Log-likelihood(LL)	210.712
BIC(LL)	-407.349
AIC(LL)	-415.424

The logistic distribution best fits λ_1 for the goodness of fit test. The following distributions had zero p-values from the goodness of fit tests: Chi-square, Erlang, Exponential, Fisher-Tippett (1), Gamma (1), Gamma (2), Gumbel, Log-normal, Logistic, Normal, Normal (Standard), Student, Weibull (1) and Weibull (2).

Table 31: **Distribution Fitting for λ_1**

Distribution	p-value
Beta4	0.000023
Logistic	0.001379
Normal	0.000042
GEV	0.000004

The table below shows the estimated parameter for the λ_1

Table 32: **Parameter Estimation for λ_1**

Parameter	Value	Standard Error
μ	2.688	0.061
s	0.380	0.033

The Log-likelihood statistics for λ_1 are shown below;

Table 33: **Log-Likelihood Statistics for λ_1**

Log-likelihood(LL)	-123.093
BIC(LL)	255.569
AIC(LL)	250.186

The Arcsine distribution best fits λ_2 for the goodness of fit test. The following distributions had zero p-values from the goodness of fit tests: Chi-square, Erlang, Exponential, Fisher-Tippett (1), Gamma (1), GEV, Gumbel, Logistic, Normal (Standard), Student and Weibull (1).

Table 34: **Distribution Fitting for λ_2**

Distribution	p-value
Arcsine	1.0000
Beta	0.010
Gamma (2)	0.011
Log-normal	0.056
Logistic	0.009

The table below shows the estimated parameter for the λ_2

Table 35: **Parameter Estimation for λ_2**

Parameter	Value	Standard Error
alpha	0.271	0.023

The Log-likelihood statistics for λ_2 are shown below;

Table 36: **Log-Likelihood Statistics for λ_2**

Log-likelihood(LL)	70.509
BIC(LL)	-136.326
AIC(LL)	-139.017

The Kolmogorov-Smirnov test will also be used to check the distributional accuracy of the factors. The β_1, β_2 and λ_2 reflect the best accuracy in while the β_3, β_4 and λ_1 have poor accuracy.

The Kolmogorov-Smirnov test for β_1 below the computed p-value is greater than the significance level $\alpha = 0.05$. This shows that the fitted distribution for the β_1 is accurate. Thus β_1 does follow an Arcsine distribution.

Table 37: **Kolmogorov-Smirnov Test for** β_1

Parameter	**Value**
D	0
p-value (Two-tailed)	1.000
alpha	0.05

The Kolmogorov-Smirnov test for β_2 below the computed p-value is greater than the significance level $alpha = 0.05$. This shows that the fitted distribution for the β_2 is accurate. Thus β_2 does follow a Normal distribution.

Table 38: **Kolmogorov-Smirnov Test for** β_2

Parameter	**Value**
D	0.121
p-value (Two-tailed)	0.076
alpha	0.05

The Kolmogorov-Smirnov test for β_3 below the computed p-value is lower than the significance level $alpha = 0.05$. This shows that the fitted distribution for the β_3 is inaccurate. Thus β_3 does not follow a Normal distribution.

Table 39: **Kolmogorov-Smirnov Test for β_3**

Parameter	**Value**
D	0.215
p-value (Two-tailed)	¡ 0.0001
alpha	0.05

The Kolmogorov-Smirnov test for β_4 below the computed p-value is lower than the significance level $alpha = 0.05$. This shows that the fitted distribution for the β_4 is inaccurate. Thus β_4 does not follow a GEV distribution.

Table 40: **Kolmogorov-Smirnov Test for β_4**

Parameter	**Value**
D	0.150
p-value (Two-tailed)	0.013
alpha	0.05

The Kolmogorov-Smirnov test for λ_1 below the computed p-value is lower than the significance level $\alpha = 0.05$. This shows that the fitted distribution for the λ_1 is inaccurate. Thus λ_1 does not follow a logistic distribution.

Table 41: **Kolmogorov-Smirnov Test for λ_1**

Parameter	Value
D	0.181
p-value (Two-tailed)	0.001
alpha	0.05

The Kolmogorov-Smirnov test for λ_2 below the computed p-value is greater than the significance level $\alpha = 0.05$. This shows that the fitted distribution for the λ_2 is accurate. Thus λ_2 does follow an arcsine distribution.

Table 42: **Kolmogorov-Smirnov Test for λ_2**

Parameter	Value
D	0
p-value (Two-tailed)	1.000
alpha	0.05

4.2 Test Statistics

4.2.1 The Coefficient of Determination (R^2)

We will us the coefficient of determination to measure the accuracy of the curve. This statistic will aid us in checking how well the model fits into the data. A high value of the statistic R^2 is usually associated with the degree of accuracy while the converse is true.

Table 43: R^2 **for the Nelson-Siegel and Svensson Models**

Model	**2 Year**	**5 Year**	**10 Year**	**15 Year**
Nelson-Siegel	0.70957	0.68991	0.67295	0.63108
Svensson	0.87869	0.88625	0.90316	0.83778

The R^2 shows that the Svensson model is a better fit for the data than the Nelson-Siegel.

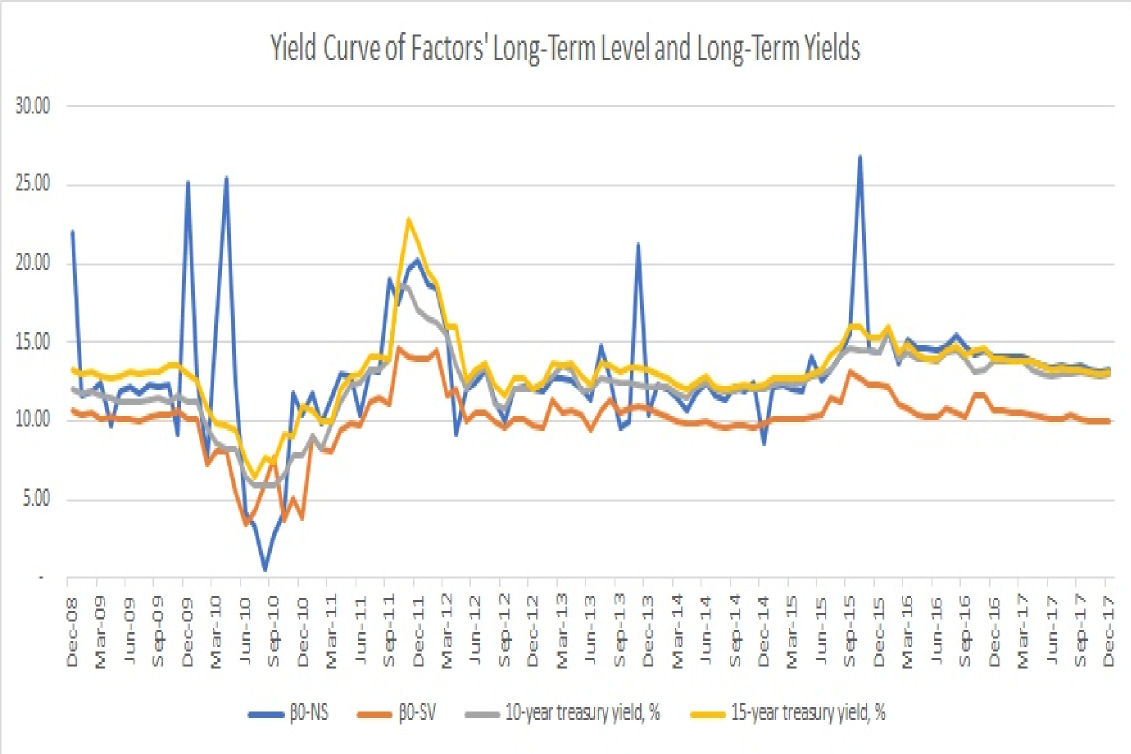
4.2.2 Frechet Distance

Frechet distance is a measure of the similarity between curves that takes into account the location of the ordering of points along the curves.
The Frechet distance for the parametric models' estimates, the 10-year treasury yields and the 15-year treasury yields were generated using R-programming. The table below surprisingly shows us that the Nelson-Siegel yield curves is closer in similarity to the long-term treasury yields than the Svensson yield curves.

Table 44: **Frechet Distances Between Yields and the Long-Term Levels**

Period	**N-S/10Year**	**N-S/15Year**	**SV/10Year**	**SV/15Year**
2017-2017	1.4668	0.5989	10.2977	11.1418
2016-2017	2.8946	1.5858	15.4092	16.8560
2015-2017	12.6987	11.1571	17.2673	19.6404

The plot below shows the evolution of the yields and long-term levels for both models for the time horizon December 2008 to December 2017.



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4.3 Modelling the Dynamic Factors

Dibold and Li[2006] modeled the dynamics of the factors by fitting either the VAR(1) model and estimating separate AR(1) models. Their main goal was to model the factors so that they could be able to provide forecast for the yield curve models.

For the Nelson-Siegel model, the AR(1) model is a good fit for β_1 while it poorly fits for both β_2 and β_3.

For the Svensson model, the AR(1) model is a good fit for β_1 and β_3 while it poorly fits for both β_2 and β_4.

The mixed results for the AR(1) models indicate that the models cannot be relied upon to accurately describe the conditional means of the level, scope and curvature factors. The inaccurately fitted AR(1) models would increase the difficulty of accurately forecasting the yield curve using the forecasted models.

4.4 Modeling the Covariance Matrix of the Bond Yields

The conditional covariance matrix of the factors, $\Omega_{t|t-1}$ was to be obtained using the DCC model. However, subsequent tests for heteroskedasticity on the factors revealed no ARCH-type effect across the factors. The β_2 factor for the Nelson-Siegel would not generate the necessary test-statistics to evaluate it.

5 Limitation of the Study

We have estimated the factors for both the Nelson-Siegel and Svensson models. While we were able to generate the vectors of bond expected returns and the correlation matrix and volatility vector for the factors, we were not able to generate the covariance matrix of the bond returns based on both Nelson-Siegel and Svensson models. Without the covariance matrix from the DCC model, the computation of the parametric VaR is hampered.

The practical estimation of the DCC models uses the univariate GARCH modes to obtain the volatilities of the time series. However, testing the time series of the factors for both Nelson-Siegel and Svensson models reveals that there is no time-varying volatility present in the data. This can be explained by the frequency of the bond yields that are monthly by nature.

However, despite the forecasting failure of this approach, we were able to find the distribution of the decay parameters for both models. Diebold and Li imposed a priori structure motivated by parsimony, simplicity and theory that aided them in making the decay parameter constant. This is obvious after fitting the distribution of the decay parameters. The arc-sine distribution creates a lot of parsimony and is rarely used in bond markets.

The different distributions of the factors for both Nelson-Siegel and Svensson models shows the difficulty in modeling the factors using a multivariate distribution.

6 Discussion, Conclusions and Recommendations

The parameters λ_t for the Nelson-Siegel model and the λ_1 and λ_2 govern the exponential decay rates. Small values of the decay parameters produce slow decay that generate better fits for the curve at long maturities. Large values of the decay parameters produce fast decay and better fit the curve at short maturities. Finding the optimum value for the decay parameters while obtaining a priori structure that improves accuracy is a challenge

that needs to be tested using various algorithms.
As noted by Caldeira, obtaining accurate risk measures is an important issue in risk management. This challenge is increased further in modelling the risk measures in an illiquid market using parametric models.
The long-term levels β_1 for both the Nelson-Siegel and Svensson models were observed to follow an arcsine distribution. The dependence of the dynamic model factors can be investigated using copulas.
We suggest the use of copulas to improve the descriptions of the dependencies among the factors. This will provide a method to create distributions that will model the correlated multivariate data from the factors obtained.
We recommend for a multivariate distribution to be constructed by specifying the marginal univariate distributions of the factors and choosing a copula to provide a correlation structure between the factors. However, having observed the varied distributions fitted to the estimated factors, some of the standard multivariate distributions can only model limited type of dependencies.
Furthermore, the number of parameters for both Nelson-Siegel and Svensson models limit the usage of one-parameter family of copulas such as the Archimedean copula families namely, Clayton copulas, Frank copulas and Gumbel copulas.

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List of Appendices

Appendix A: R Code for Frechet Distance Calculation

Using package SimilaritiyMeasures in R

```
Beta0Nelson1<-c(14.08406,14.13419,14.10588,13.81419,13.56129,......)
Beta0Nelson2<-c(15.67598,13.68775,15.17946,14.63481,14.58888,......)
Beta0Nelson3<-c(12.21001,12.27883,11.98281,11.89632,14.05857,......)

Beta0Svensson1<-c(10.6111,10.4795,10.504,10.4117,10.2344,...... )
Beta0Svensson2<-c(12.1726,11.04,10.7696,10.454,10.2478,...... )
Beta0Svensson3<-c(10.0928,10.0997,10.1629,10.1747,10.2579,...... )

TenYear1<-c(13.85,13.8,13.8,13.25,12.95,12.8,12.95,12.966,......)
FifteenYear1<-c(13.9,13.9,13.9,13.8,13.5,13.3,13.4,13.3,.......)
TenYear2<-c(15.8,14,14.4,14,13.9,13.8,14.3,14.5,.......)
FifteenYear2<-c(16,14.3,14.9,14.2,14.0,14.0,14.5,14.8,.......)
TenYear3<-c(12.3,12.4,12.4,12.4,12.9,13.0,13.2,14.2,.......)
FifteenYear3<-c(12.8,12.8,12.8,12.8,13.0,13.2,14.2,14.8,......)

Beta0Nelson1
Beta0Nelson2
Beta0Nelson3
Beta0Svensson1
Beta0Svensson2
Beta0Svensson3

TenYear1
TenYear2
TenYear3
FifteenYear1
FifteenYear2
FifteenYear3
```

```
data11<-matrix(Beta0Nelson1,1)
data12<-matrix(Beta0Nelson2,1)
data13<-matrix(Beta0Nelson3,1)
data21<-matrix(Beta0Svensson1,1)
data22<-matrix(Beta0Svensson2,1)
data23<-matrix(Beta0Svensson3,1)

data31<-matrix(TenYear1,1)
data41<-matrix(FifteenYear1,1)
data32<-matrix(TenYear2,1)
data42<-matrix(FifteenYear2,1)
data33<-matrix(TenYear3,1)
data43<-matrix(FifteenYear3,1)

data11
data12
data13
data21
data22
data23
data31
data32
data33
data41
data42
data43

dNelsonSiegelTenYear1<-Frechet(data11,data31)
dNelsonSiegelFifteenYear1<-Frechet(data11,data41)
dNelsonSiegelTenYear2<-Frechet(data12,data32)
dNelsonSiegelFifteenYear2<-Frechet(data12,data42)
```

```
dNelsonSiegelTenYear3<-Frechet(data13,data33)
dNelsonSiegelFifteenYear3<-Frechet(data13,data43)

dSvenssonTenYear1<-Frechet(data21,data31)
dSvenssonFifteenYear1<-Frechet(data21,data41)
dSvenssonTenYear2<-Frechet(data22,data32)
dSvenssonFifteenYear2<-Frechet(data22,data42)
dSvenssonTenYear3<-Frechet(data23,data33)
dSvenssonFifteenYear3<-Frechet(data23,data43)

dNelsonSiegelTenYear1
dNelsonSiegelFifteenYear1
dNelsonSiegelTenYear2
dNelsonSiegelFifteenYear2
dNelsonSiegelTenYear3
dNelsonSiegelFifteenYear3

dSvenssonTenYear1
dSvenssonFifteenYear1
dSvenssonTenYear2
dSvenssonFifteenYear2
dSvenssonTenYear3
dSvenssonFifteenYear3
```