Strathmore UNIVERSITY

Strathmore Institute of Mathematical Sciences (SIMS) Examination for the Degree of Bachelor of Business Science in Financial Economics/Financial Engineering

BSF 4130: Foundations of Asset Pricing

DATE: 22nd July, 2024 **Time: 2 Hours**

Instructions

- **This examination consists of FIVE questions.**
- **Answer Question ONE (COMPULSORY) and any other TWO questions.**

1. (a) Consider the family of Bernoulli utility functions

$$u(w) = \left\{ \begin{array}{ll} \dfrac{w^{1-\sigma}}{1-\sigma}, & \text{if } \sigma \geq 0, \sigma \neq 1 \\ ln(w), & \text{if } \sigma = 1 \end{array} \right\} \tag{1}$$

Throughout the question you can ignore the case where $\sigma = 1$ for simplicity.

(i) Find the coefficient of risk aversion for these utility function. **(4 Marks)**

(ii) Find the coefficient of relative risk aversion for these utility function. **(3 Marks)**

(b) Consider two assets, with percentage returns, $\tilde{R}_1$ and $\tilde{R}_2$ that vary across three states that occur with equal probability as follows:

	State 1 $\pi_1 = 1/3$	State 2 $\pi_2 = 1/3$	State 3 $\pi_3 = 1/3$
Asset 1 retuns, $\tilde{R}_1$	10	0	10
Asset 2 returns, $\tilde{R}_2$	0	10	20

(i) Does one asset exhibit state-by-state dominance over the other? If so, which one? **(2 Marks)**

(ii) Does one asset exhibit mean-variance dominance over the other? If so, which one? **(4 Marks)**

(iii) Which asset has the highest Sharpe ratio? **(2 Marks)**

(c) Provide an intuitive explanation of the following concepts as used in asset pricing literature.

(i) Two-fund theorem (separation theorem) **(4 Marks)**
(ii) Complete Markets **(3 Marks)**
(iii) Roll's critique (of CAPM) **(4 Marks)**
(iv) Equity Premium Puzzle **(4 Marks)**

2. (a) Consider the portfolio allocation problem faced by an investor who has initial wealth $Y_0 = 100$. This investor allocates the amount a to stocks, which provide a return of $r_G = 0.16$ (16 percent) in a good state that occurs with probability $\pi = 1/2$ and a return of $r_B = 0.02$ (2 percent) in a bad state that occurs with probability $1 - \pi = 1/2$. The investor allocates the remaining amount $Y_0 - a$ to a risk-free bond, which provides a return of $r_f = 0.08$ in both states.

   Suppose that the investor's preferences can be described by a von Neumann-Morgenstern expected utility function, with Bernoulli utility function of the form

$$u(Y) = ln(Y),$$

   when ln denotes the natural logarithm, so that his or her portfolio allocation problem can be stated mathematically as

$$\max_a \; \pi ln[(1 + r_f)Y_0 + a(r_G - r_f)] + (1 - \pi)ln[(1 + r_f)Y_0 + a(r_B - r_f)]$$

   Using the specific values for Y_0, r_G, r_B, r_f, and π given above, find the numerical value of the investor's optimal choice a^* **(6 Marks)**

   (b) Consider a risk-averse investor with von Neumann-Morgenstern expected utility and a Bernoulli utility function of the constant relative risk aversion form

$$u(Y) = \frac{Y^{1-\gamma}}{1-\gamma}$$

   with $\gamma = 1/2$. Suppose that this investor has initial income $Y_0 = 100$ and has the chance to acquire a risky asset that pays off $Z^G = 75$ in a good state that occurs with probability $\pi = 0.5$ but pays off only $Z^B = 75$ in a bad state that occurs with probability $1 - \pi = 0.5$.

   (i) Recall that the certainty equivalent $CE(\tilde{Z})$ associated with this risky asset is the maximum riskless payoff that the investor would be willing to give up in order to acquire the risky asset. What is the numerical value for the certainty equivalent $CE(\tilde{Z})$? **(6 Marks)**

   (ii) Recall also that the risk premium $\Psi(\tilde{Z})$ associated with the risky asset is the difference between the expected payoff $E(\tilde{Z})$ from the risky asset and the certainty equivalent $CE(\tilde{Z})$. What is the numerical value for the risk premium $\Psi(\tilde{Z})$? **(4 Marks)**

   (iii) Suppose that instead of having $\gamma = 1/2$, the investor had a coefficient of relative risk aversion equal to $\gamma = 5$. Would the certainty equivalent in this case with $\gamma = 5$ be larger or smaller than it was in part (i) above with $\gamma = 1/2$? Would the risk premium in this case with $\gamma = 5$ be larger or smaller than it was in part (ii) above with $\gamma = 1/2$? Note: to anser these two questions in part (iii), you don't necessarily have to compute the numerical values of the certainty equivalent and risk premium when $\gamma = 5$; all you need to do is to indicate whether each is larger or smaller than in parts (i) and (ii), when $\gamma = 1/2$ instead. **(4 Marks)**

3. (a) Suppose that there are three assets traded. Asset 1 has random return with expected value $\mu_1 = 8$, standard deviation $\sigma_1 = 2$, and variance $\sigma_1^2 = 4$. And asset 2 has random return with with expected value $\mu_2 = 4$, standard deviation $\sigma_2 = 2$, and variance $\sigma_2^2 = 4$. Asset 3, has random return with expected value $\mu_3 = 6$, standard deviation $\sigma_3 = 2$, and variance $\sigma_3^2 = 4$. For simplicityassume that the three asset returns are uncorrelated with $\rho_{12} = \rho_{23} = \rho_{23} = 0$.

Suppose further that a portfolio manager is assigned the task of forming a portfolio that achieves a six percent target $\overline{\mu} = 6$ for its expected return, while minimizing the variance of the portfolio's random return. If this portfolio manager allocates the share w_1 of his or her total funds to asset 1, share w_2 of total funds to asset 2, and the remaining share $1 - w_1 - w_2$ to asset 3, the expected return on the portfolio will be

$$\mu_p = 8w_1 + 4w_2 + 6(1 - w_1 - w_2)$$

and the variance of the random return on the portfolio will be

$$\sigma_p^2 = 4w_1^2 + 4w_2^2 + 4(1 - w_1 - w_2)^2.$$

Clearly, the portfolio manager can hit the six-percent expected return target

simply by allocating all of the funds to asset 3; he or she would then have to accept the variance $\sigma_3^2 = 4$ of asset 3's return would also become the variance of his or her portfolio's random return. The question is how much better he or she can do by choosing the portfolio weights optimally instead. Recall from class that these optimal weights can be found by maximizing $-\sigma_p^2$, minus one times the variance of the portfolio's ramdom return, subject to the constraint that $\mu_p = \overline{\mu} = 6$. This problem can be stated mathematically as

$$\begin{aligned} \underset{w_1, w_2}{\text{maximize}} \quad & -4w_1^2 - 4w_2^2 - 4(1 - w_1 - w_2)^2 \\ \text{subject to} \quad & 8w_1 + 4w_2 + 6(1 - w_1 - w_2) = 6 \end{aligned} \tag{2}$$

(i) Use the Lagrangian for the portfolio manager's problem to derive the first-order conditions for the optimal choices w_1^* and w_2^* **(6 Marks)**

(ii) Next use your first order conditions from part (i), together with the constraint

$$8w_1 + 4w_2 + 6(1 - w_1 - w_2) = 6$$

to find the numerical values of w_1^* and w_2^* **(7 Marks)**

(iii) Finally, use your results from part (ii) to calculate the minimized variance σ_p^{2*} achieved when the portfolio's weights are chosen optimally. **(3 Marks)**

(b) The prediction test of the CAPM plots estimates of average excess returns $\hat{\mu}_i - r_f$ against beta estimates $\hat{\beta}_i$. Based on these estimates one may estimate the simple linear regression equation

$$\hat{\mu}_i - r_f = \gamma_0 + \gamma_1 \hat{\beta}_i + \epsilon_i, i = 1, \ldots, N \tag{3}$$

if the CAPM were true, what should the estimated values for γ_0 and γ_1 be? what are their interpretations in the context of CAPM? **4 Marks)**

4. (a) State three advantages of the Arrow-Debreu approach to asset pricing compared to Capital Asset Pricing Model (CAPM) **(3 Marks)**

   (b) Consider the following endowments and preferences:

Agents	Endowments			Preferences
	$t=0$	$t=1$		
		θ_1	θ_2	
Agent 1	10	1	2	$\frac{1}{2}c_0^1 + 0.9\left[\frac{1}{3}\ln\left(c_1^1\right) + \frac{2}{3}\ln\left(c_2^1\right)\right]$
Agent 2	5	4	6	$\frac{1}{2}c_0^2 + 0.9\left[\frac{1}{3}\ln\left(c_1^2\right) + \frac{2}{3}\ln\left(c_2^2\right)\right]$

   Find the optimal consumption **(11 Marks)**

   (c) The Arbitage Pricing Theory (APT) is agnostic about what factors should be, and the factor identification has become an empirical issue. The goal is to identify a small number of factors that determine all asset returns. state three approaches used to identify those factors. **(6 Marks)**

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5. Suppose a typical investor solves the following problem

$$\max_{\alpha} U(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})]$$
$$\text{subject to } c_t = e_t - \alpha p_t \text{ and } c_{t+1} = e_{t+1} + \alpha x_{t+1} \tag{4}$$

where c_t and c_{t+1} denotes consumption at date t and $t+1$ respectively. The other parameters in the model are; e_t (investor's endowment at period t and $t+1$ respectively), p_t (asset price at time t), α (the number of units of the asset purchased by the investor), and x_{t+1} (payoff of the asset at period $t+1$), β represents the time preference parameter of the investor ($0 \leq \beta \leq 1$). The general functional form $U(.)$ represents the investor's utility function. An often convenient utility function used in many applications is the power utility $u(c_t) = \dfrac{1}{1-\gamma}c^{1-\gamma}$

   (i) Show that the price of an asset at any given time can be represented as follows: **(3 Marks)**

$$p_t = E_t[\frac{\beta u'(c_{t+1})}{u'(c_t)} x_{t+1}]$$

   (ii) Provide an intuitive interpretation of the term $\dfrac{\beta u'(c_{t+1})}{u'(c_t)}$ and briefly discuss how this term influences the asset price p_t **(3 Marks)**

(iii) Show that a simple manipulation of the price equation in (i) can yield the following equation (ignoring time subscripts)

$$1 = E(mR)$$

where R represents the gross return of the asset, while $m = \frac{\beta u'(c_{t+1})}{u'(c_t)}$ **(2 Marks)**

(iv) Assuning the power utility function provided and the manipulations in parts (i)-(iii) above, one can show that the risk-free rate can be expressed as

$$R^f = \frac{1}{\beta}(\frac{c_{t+1}}{c_t})^\gamma$$

Intuitively discuss three factors that influence interest rates in an economy. **(6 Marks)**

(v) Show that the basic pricing equation ($1 = E(mR)$) can be rewritten as follows;

$$E(R) = R_f - \frac{Cov(u'(c_{t+1}), R)}{E(u'(c_{t+1}))}$$

Interpret this equation. **(6 Marks)**

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