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Estimating Heston stochastic volatility model parameters.

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Estimating Heston stochastic volatility model parameters

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Declaration and recommendation

Declaration

I declare that this thesis is my original work and has not been submitted or approved for the award of a degree by Strathmore or any other University. To the best of my knowledge, the thesis contains no material previously published or written by another person except where due reference is made.

Signature .... Date 27^{th} October 2021

Martin Maengo Musya

Recommendation

This proposal has been submitted for assessment with our approval as supervisors according to Strathmore University regulations.

Signature .... Date 12th November 2021

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Abstract

The Black Scholes model is widely favoured for pricing derivatives such as the European put and call options. This model while having the benefit of ease of application has some restrictive assumptions. First there is the assumption that volatility of asset returnsis constant. This assumption is easily violated in the volatility smile that is widely documented in literature as well as observed in option market data, Another assumption is that asset returns are normally distributed. The assumption of normal distribution is reasonable for long term horizons but not for shorter horizons. Market are rarely if ever complete. There always exists informational assymmetry where some investors know more about the market than others. It is also well known that a single asset is insufficient to hedge away risk. The Heston Model improves on previous assumption of constant variance (homoskedasticity) by allowing correlation between volatility and the price of the underlying.
This research sought to estimate the Heston model parameters over various periods using maximum likelihood method. This was to compare the performance of both the Heston model and the Black Scholes model in periods of market uncertainty and in relatively stable periods where markets are performing well. The three periods, also called epochs, in this study are during the 2008 financial crisis, the Covid-19 pandemic and the relatively stable years between the two periods. The data for this thesis was obtained from the S&P 500 volatility index (VIX). Both the option data and the underlying data was available.

Aït-Sahalia (2002) constructed the sequence of approximations to the transition probability for a diffusion process. The first step involved standardizing the diffusion function using the Lamperti transform in order to remove some state dependent term in order to get rid of boundary conditions and correlation structures resulting in simple diffusion terms.This allowed for parameter estimation for the differentiable unit diffusion function. After this, the "pseudo-normalized" increment for random variable of the diffusion function was obtained and maximum likelihood estimation was done on this transformed variable to get rid of the issue of transition probabilities getting peaked when the step size gets small.

The state variables for the Heston diffusion function were those of the underlying process and the diffusion process. The option price and that of the underlying in the data were first organized into a matrix. The determinant of that matrix which was the Jacobian term was then used to estimate the Heston and Black-Scholes parameters. A hypothesis test on the output for the three periods showed that at the 95% significance level the Heston model parameters are significant unlike the Black-Scholes parameters for all periods under study. The Heston model performed better even in times of financial instability such as during the 2008 crisis and the Covid-19 pandemic.

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Dedication

This thesis is dedicated to my dear parents Janet Musya and Daniel Musya in addition to my sister Mercy Musya for their support during the entire period I was working on this project.

Abbreviations

CMA Capital Markets Authority

NSE Nairobi Securities Exchange

Chapter 1

Introduction

1.1 Motivation

The Black Scholes model which is widely utilized in the market when pricing options has significant restrictive assumptions. One such assumption is that volatility is constant. This assumption is violated as many researchers have observed that implied volatility of options varies as the strike price and time to maturity. The pattern has come to be widely known as the volatility smile. Practitioners of finance usually calibrate the Black Scholes by finding the volatility that makes the Black-Scholes option price match the prices of options observed in the market. This implies that the ”true” model for option pricing would require a stochastic process for the volatility. The Heston model is considered an improvement on the Black Scholes model as it allows for stochastic volatility which allows for the volatility smile.

This study first examines the evolution of stochastic volatility models from the classic volatility models, the most widely known being the Black Scholes model. Literature where maximum likelihood estimation of diffusion model parameters are performed are also examined. Those methods are then utilized in a practical and simple framework to estimate Heston model parameters based on option data from the S & P 500 volatility (VIX) index.

1.2 Background to the study

The Heston model is a stochastic volatility model. This is to mean that volatility in this model evolves in a random manner. The Black-Scholes model assumes constant volatility. Empirical observations of asset prices easily shows violation of his assumption. Moreover, Rubinstein (1985) conducted various tests where the assumption of constant volatility was rejected to a

statistically significant level. According to Heston (1993), the Black-Scholes model makes the strong assumption that continuously compounded stock returns have a normal distribution with known mean and variance. Asset returns have also been shown to have fatter tails than those suggested by the Black-Scholes model. Fat tails imply a large probability of having extreme movements in the distribution of returns. An improved model which does not require these assumptions is the Heston model.
All stochastic volatility models do not make the assumption of dependence in the asset price in the diffusion process.The Wiener process for the asset and the asset price volatility are correlated.

1.3 Volatility smile

Implied volatility is the volatility that makes the Black Scholes model yield observed market option prices. The implied volatility of the Black Scholes model varies with strike price and time to maturity. According to Carl Chiarella and Nikitopoulos (2015), implied volatilities observed in market option prices exhibit the mean-reversion property. Thus, this study would require a mean-reverting stochastic process to model volatility. Researchers can reproduce the fat tails and peakedness of observed stock price volatility by modelling volatility with a mean reverting diffusion process.

The wide distribution of simulated sample paths in the case of stochastic volatility implies the distribution is fat tailed. Some authors have argued modelling variance instead of volatility.

1.4 Types of volatility models

Volatility models are broadly categorized into one of three categories. The trilogy of volatilities is implied volatility which is derived from the Black-Scholes equation, local volatility which is the conditional expectation of instantaneous variance & stochastic volatility which varies in a random fashion.

Local volatility is the conditional expectation of instantaneous variance. Gatheral (2011) notes how stochastic volatility models explain in a self-consistent way the existence of the volatility smile. The volatility smile can be observed upon varying the strike price & term to maturity of an option. The effect is an increase in implied volatility.

The stochastic process whose conditional probabilities are given by equation 3.25 is known as Brownian Motion. Stochastic volatility functions arise from Brownian motion ordered by a random clock referred to as the *trading time*. This trading time may be identified with volume of trades or the frequency of trading. Traders who use the Black-Scholes model in practice must continuously change their volatility assumptions in order to match market prices. This is a significant limitation of the Black Scholes model that the Heston model overcomes. *Volatility clustering* is the tendency for large moves in prices to be followed by large moves and vice versa. This implies that volatility is *auto correlated*. This is a consequence of the mean reversion of volatility. *Fat tails* which imply outliers tend to occur when there is a mixture of distributions with different variances.

Implied volatility is the volatility "implied" by the Black-Scholes model. This means that implied volatility allows the option prices generated by the Black Scholes model to match those observed in the market. Implied volatility captures captures the market's view of likely changes in a given assets price. Changes in demand and supply for the asset are thus deduced by investors.

Chiarella and Ziveyi (2013) notes that pricing of options contracts is often done using implied volatility. High implied volatility results in options with higher premiums. Factors such as supply, demand and time value are important for calculating implied volatility. Implied volatility is higher when markets become bearish (pessimistic) and lower when markets become bullish (optimistic).

The implied volatility of the Black-Scholes model was later extended into the *local volatility* model. Local volatility models are special instances of stochastic volatility models. They are thus the simplest *market models*. Practitioners of finance sought a simpler way to price relatively complex (exotic) options than the computationally heavy stochastic volatility models. This way had to also be consistent with the volatility skew.
Breeden and Litzenberger (1978) noted that that density under the risk-neutral numeraire

could be calculated from market prices of European options. Dupire et al. (1994) and Derman and Kani (1994) discovered that under risk-neutral pricing,there was a diffusion process which resulted in these distributions. The diffusion coefficient $\sigma_t(S,t)$ is known as the *local volatility function*.

Local volatility models assume volatility is a deterministic function of the asset price and time. The diffusion process is referred to as local volatility. A deterministic model is one in which the values for the dependent variables of the system are completely determined by the parameters of the model e.g. ordinary differential equations. In contrast, stochastic, or probabilistic, models introduce randomness in such a way that the outcomes of the model can be viewed as probability distributions rather than unique values.
Local volatility models introduce more flexibility into the Black-Scholes model. The local volatility model only introduces stochasticity in the volatility function. Having only one source of stochasticity ensures preservation of the completeness of the Black-Scholes model. This ensures that uniqueness of prices is maintained. Stochastic volatility models introduce new sources of stochasticity (randomness) over the local volatility models. The stochastic volatility cannot be traded, resulting in loss of completeness of the original Black-Scholes model. Local volatility can be calculated from implied volatility.

1.5 Research Objectives

The main objective of this research was to utilize the Heston model in modelling the volatility of option prices obtained using the maximum likelihood method. The specific objectives were to estimate Heston model parameters from option data and to compare the statistical significance of Heston model parameters with those of the Black-Scholes model during the 2008 crash, the Covid pandemic period and the stable period between.

Chapter 2

Literature Review

2.1 Introduction

This literature examines the evolution of stochastic volatility models and methods for estimating volatility model parameters. Section 2.2 begins with an examination of the classic volatility model. Implied volatility is also examined in literature. Jump diffusion processes and literature empirically showing violation of constant volatility is discussed. Section 2.3 then details the evolution of stochastic volatility models such as the Heston model. Numerical methods applied the stochastic volatility models are discussed. In section 2.4, the use of likelihood functions to estimate volatility parameters is discussed.

2.2 Classic Volatility model

The classic volatility model by Black and Scholes assumed constant volatility. They noted that the curve of option price vs strike price is concave upwards for a European style call option Black and Scholes (1973). They also noted that since this curve is below the $45°$ line, the option will have a higher volatility than the stock. Also of note is the fact that a percentage change in the stock price will result in in a larger percent change in the option value. Thus, they deduced that the relative volatility of the option was not constant since it depended on time to maturity and the stock price. A parameter of significant interest in the Black-Scholes formula is the volatility of the stock price. This parameter can be estimated from a history of the stock price. Traders in the market usually work with volatilities implied by option prices. These are known as *implied volatilities*. Implied volatility is an estimate of the future variability for the asset that underlies an options contract such as a put or call option.

Merton observed that new information in the market causes a greater than marginal change in price - a jump Merton (1975). In order to be consistent with the efficient market hypothesis, the jumps have to be martingales. These jumps are unanticipated and modelled using a Poisson process. The arrival of new information into the market is described as an event. These events are independently and identically distributed.

Later on, Cox introduced several jump diffusion processes. First he considers a pure birth process without drift Cox and Ross (1976). This process, unfortunately had no solution. A drift term was subsequently introduced into the model. No closed form solution of this model could be found. Cox introduced several jump diffusion processes Cox and Ross (1976). One of these was the single jump process. The jump is Poisson distributed. The parameter representing this jump is added to the SDE for stock price evolution. Jones derives an option pricing formula when the stock has jumps in its diffusion process Jones (1984).

Rubinstein showed that the assumption of constant volatility is not true Rubinstein (1985). He noted two important assumptions that exist in the Black-Scholes model. First that the instantaneous volatility is non-stochastic and that stock price follows a continuous path through time (no jumps). In order to relax the assumption of constant volatility, he utilized the constant elasticity of variance diffusion model by Cox and Ross, the compound option diffusion model by Geske and the displaced diffusion model by Rubinstein. He performs a hypothesis test with the null hypothesis that option market prices & the Black-Scholes prices are not significantly different. He said that if we reject, it would be because of the form of the Black-Scholes formula. The conclusion drawn after the tests was that no single model captures the observed biases of the Black-Scholes model. He thus concluded that one needs to build a composite model.

The idea behind the Black-Scholes differential equation consists of setting up a riskless portfolio which contains a position in the derivative and a position in the underlying stock. Since the stock price and the derivative price are both affected by the same source of uncertainty, they are perfectly correlated in the short term. A loss or gain in the stock position will always be perfectly correlated with a gain or loss in the derivative position. Thus, the value of the portfolio after the short period is known with certainty.

Louis Scott studied the return and standard deviation parameters of diffusion processes. It was mentioned that the Black-Scholes model calculates implied standard deviations of option prices. He models the asset process either the GBM SDE and the variance process with the OUV process. Since both the GBM and the OUV processes have closed form solutions, calculations should not be a problem. Discretization of both SDEs using Euler-Maruyama method allows us to derive a formula for variance based on the parameters of the SDE Scott (1987). He also noted that changes in volatility are uncorrelated with stock returns. Calculation of option prices was thus done using Monte Carlo simulations.

2.3 Stochastic volatility model

The problem in pricing an option with stochastic volatility was then examined. The option price was then determined for the case in which volatility is correlated with stock price. A significant finding was that the Black-Scholes model overprices options and that the level of overpricing increases with time to maturity Hull and White (1987). Monte Carlo simulation is performed to calculate the option price. He assumes that volatility follows a mean-reverting process. The Monte Carlo simulation is performed by first dividing the time interval into equal sub-intervals. Standard normal distributed random variables are generated and used to calculate variance at points in the time interval. The antithetic technique and the control variates technique can be utilized to reduce variance in simulated estimates. Hull finds that when volatility is uncorrelated with the stock price, the option price is less than the Black-Scholes price estimate.
The Black-Scholes price is lower in deep in and out-of-the-money option prices and higher in at-the-money options when there is no correlation. If correlation is positive, out-of-the-money options are underpriced while in-the-money options are overpriced. When correlation is negative the effect is reversed.

The Heston model was originated by Steven Heston (1993) who was looking for a model to price a European call option on an asset with stochastic volatility. His simulations showed that correlation between volatility and the price of the underlying is important for explaining the volatility skew. He noted how the Black-Scholes model makes strong assumptions such as normal distribution of returns. This model allows for correlation between the volatility and the underlying Heston (1993). This correlation explains the skew in the return density function. The Heston model is a bivariate process which is a process with two random processes. One process is for the asset price and another for the volatility process is a bivariate process.

Dupire describes how implied volatility is not the same across all option prices Dupire et al. (1994). Black-Scholes implied volatility depends on the exercise price (K). Merton (1973) allowed volatility to be time-dependent. However, the dependence of implied volatility on the strike (k) for a given maturity is more complicated. Researchers attempted to enrich the Black-Scholes model to incorporate the ”skew”. Stochastic volatility models were introduced so as to make volatility itself a stochastic process. Duque and Lopes (1999) finds empirical evidence of the volatility skew.

Andersen considers several new algorithms for discretization of the Heston volatility SDE Andersen (2007). Monte Carlo simulation methods are discussed for this discretization. He notes that the use of the maximum with zero function to deal with negative square roots is referred to as the “full truncation”. This helps to keep the process for variance from falling below zero. The Broadie-Kaya scheme is also discussed. It involves sampling from a Poisson distribution followed by an acceptance-rejection sample from a central chi-square distribution. The Broadie-Kaya scheme results in bias free simulation estimates however it is limited by its complexity and lack of speed.

Abe (2008) explains how volatility skew, a key feature of the Heston model is observed in options traded in the market. Zhu (2008) provides a simple and elegant method for simulating the Heston model. The first step involves transforming the Heston stochastic volatility SDE which is a square root Cox Ingersoll Ross (CIR) process into an Ornstein Uhlenbeck Vasicek (OUV) process. He states that the Heston model does not model volatility directly but rather variances. In attempting to estimate the simulated path using the Euler scheme, there is a chance of ending up with negative variance which is a problem in simulation. This is because negative numbers do not have a real square root. A possible solution for this is is to transform the CIR SDE so that it becomes an OUV SDE. The result of this transformation is an OU-process that can easily be simulated.

Monte Carlo simulation of the Heston model is done & subsequently tested by Bech Rasmussen (2009). The Euler-Maruyama method for simulating the two Heston processes is described, and the pricing ability of the method is compared to the solution of the Heston model. It is shown that increasing the number of simulations & decreasing the discrete time steps in each simulation converges the simulated price to the solution of the Heston model.

It is noted by Gauthier and Possamaï (2010) that numerical simulation may be challenging due to the slow convergence speed of the Euler scheme. Suggestions made to solve this problem include adding degrees of freedom by including additional parameters.

Gatheral studies local and implied volatilities Gatheral (2011) . He discusses how the implied volatility (given by the Black Scholes) in practice depends on both the exercise (strike) price and the time to maturity. This is what is referred to as the *volatility skew*. The collection of all such implied volatilities is referred to as the volatility surface.

Grzelak describes how the variance process is always positive and cannot reach zero - Grzelak and Oosterlee (2011). Thus we are required to have a variance process greater than zero so as to have a well defined value for the square root of the process. As mentioned before, this is what is referred to as the *feller condition*. Chiarella provides a good background for the Heston model. They explain that one of the motivations for modelling variance with the square root CIR process is its tendency to keep variance away from zero Chiarella and Ziveyi (2013). This is referred to as the feller condition.

Rouah (2013) describes the issues that arise when simulating bivariate processes such as the Heston model. The possibility of negative square roots in simulation of the variance process is problematic. One possible way he suggested to deal with this is to use the *full truncation scheme* where you use maximum of the variance and zero. The other way is to replace negative variances with their positive equals. This is referred to as the *reflection scheme*.

2.4 Likelihood function

Atiya and Wall (2009) utilized the Heston model to approximate the volatility likelihood function. The model analyzes the joint probability of the asset and the volatility processes. Historical volatility variables are integrated out. The likelihood simplifies to a product of variables (T) which are the length of the history considered. The integration based approach (INT) using Simsons method of integration and the particle filtering (PF) approach are compared using the mean absolute percentage error (MAPE). The optimal estimate for the new method, integration method and the particle filtering approach showed that computational cost is the same across all three methods.

Cacace et al. (2019) investigated the problem of estimating volatility using the Heston model. A polynomial filtering method was developed to estimate volatility. This method was compared with the analytical likelihood method (AWLIKE) by Atiya and Wall. The polynomial filtering method was found to be a significant improvement due to analytic approximation of the likelihood function for the Heston stochastic volatility function.

Redroban and Cifuentes (2021) compared performance of the Black-Scholes model before, during and after the subprime crisis of 2008, the Covid-19 pandemic. The subprime crisis begun in particular segments of the capital markets. The impact of this crisis was mostly borne in some sectors of the economy. The crisis due to the Covid-19 pandemic resulted from an external shock. The findings were consistent with previous studies that the accuracy of the Black-Scholes model is very poor. The significant errors in pricing of options have been found to be significant in all the scenarios analyzed.

The reviewed literature demonstrates the evolution of stochastic volatility models and the numerical techniques that have been applied when applying the Heston model to option data. The utilization of the maximum likelihood method to obtain Heston model parameters has also been discussed.

2.5 Thesis Structure

The thesis covers stochastic volatility models in chapter 2. Local volatility models are discussed and how this leads to stochastic volatility models. In chapter 3, we discussed a methodology for estimation of the Heston model SDE parameters using the maximum likelihood method. A hypothesis test with a 95% level of significance was utilized to determine the significance of parameters in the Heston model and the Black-Scholes model for three epochs, the 2008 crisis, the Covid-19 pandemic and the period between them. In all epochs, the significance of parameter estimates were determined with p-values. The Heston parameters had greater statistical significance for all epochs than the Black-Scholes estimates.

Chapter 3

Stochastic Volatility (SV) Models

A stochastic volatility model is a model for which the volatility evolves in a random process. These kinds of models explain in a self consistent way the volatility skew. Stochastic volatilities give a more realistic dynamics of the underlying than the local volatility models. Stochastic volatility models have volatility that follows a diffusion process. The Black-Scholes model assumes volatility of returns of an underlying asset is constant. The distribution of returns shows fat tails and peaking around the mean. Observation of implied volatility shows violation of the assumption of constant volatility. Implied volatility exhibits the property of mean reversion. In order to show fat tails and peakedness, consider the process

$$\frac{dS}{S} = \mu dt + \sigma dz_s, \tag{3.1}$$

with initial condition $S(0) = S_0$ where volatility (σ) follows the diffusion process

$$d\sigma = k(\bar{\sigma} - \sigma)dt + \delta dz_\sigma, \tag{3.2}$$

with initial condition $\sigma(0) = \sigma_0$ and $\mathbb{E}(dz_s dz_\sigma) = \rho dt$. The long term variance ($\bar{\sigma}$) is defined in equation 3.13. Volatility is modelled using

$$dv = a(\sigma, t)dt + b(\sigma, t)dz_\sigma. \tag{3.3}$$

The drift and diffusion coefficients depend only on volatility and time. No stochastic volatility models to date include asset price S in the model parameter definition. The parameters z_s and z_σ are sources of uncertainty in the asset price and the volatility. The correlation between these two parameters is defined as

$$\mathbb{E}(dz_s, dz_\sigma) = \rho dt. \tag{3.4}$$

In order to work with independent Wiener processes, we can define the parameters as

$$\begin{aligned} dz_s &= dw_s, \\ dz_\sigma &= \rho dw_s + \sqrt{1-\rho^2} dw_\sigma. \end{aligned} \tag{3.5}$$

Thus, the stochastic system defined above can be written as

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \sigma dw_s, \\ d\sigma &= a(\sigma, t)dt + \rho b(\sigma, t) dw_s + \sqrt{1-\rho^2} b(\sigma, t) dw_\sigma. \end{aligned} \tag{3.6}$$

Most stochastic volatility models are defined by a specific selection of the functions a, b and the correlation coefficient ρ. Some authors suggest modelling variance $v \equiv \sigma^2$ instead of volatility. Taking this approach using Ito's lemma, results in

$$dv = (2\sqrt{v}a + b^2)dt + 2\rho\sqrt{v} b dw_s + 2\sqrt{1-\rho^2}\sqrt{v} b dw_\sigma \tag{3.7}$$

with initial condition $v(0) = v_0$. The parameter a is usually selected to define a mean reverting process hence it takes the form $k(\bar{\sigma} - \sigma)$. Thus equation the volatility equation in 3.6 takes the form

$$d\sigma = k(\bar{\sigma} - \sigma)dt + \rho b(\sigma, t) dw_s + \sqrt{1-\rho^2} b(\sigma, t) dw_\sigma \tag{3.8}$$

with initial condition $\sigma(0) = \sigma_0$. Thus, we can get volatility at any point in time by performing the integrals in the equation

$$d\sigma = \bar{\sigma} + k(\sigma_0 - \bar{\sigma})e^{-kt} + \rho \int_0^t e^{-k(t-\tau)} b(\sigma,\tau) dw_s(\tau) + \sqrt{1-\rho^2} \int_0^t e^{-k(t-\tau)} b(\sigma,\tau) dw_\sigma(\tau). \tag{3.9}$$

with initial condition $\sigma(0) = \sigma_0$. If $b = 0$ then volatility is not stochastic but rather a time varying deterministic function. stochastic Volatility can be viewed as having three components. The first is a deterministic function of time, the second is integral over all shocks in the asset price and the third is an integral over all shocks in the volatility. Both integrals are weighted with an exponentially declining term as well as a function for past volatilities b. The path dependence of volatility is transmitted to the diffusion process for the asset price. Thus, the stochastic process for the asset (S) is non-Markovian. Before we discretize equation 3.9 above, we need to perform a change of variable ($t - \tau = u$) resulting in

$$\int_0^t e^{-k(t-\tau)} b(\sigma,\tau) dw(\tau) = -\int_t^0 e^{-ku} b(\sigma, t-u) dw(t-u). \tag{3.10}$$

Discretizing the above equation and defining $n\Delta\tau = t$ and $\alpha_j = e^{-kj\Delta\tau}$ gives us

$$\int_0^t e^{-k(t-\tau)} b(\sigma,\tau) dw(\tau) \approx -\sum_{j=1}^{n} \alpha_j b(\sigma_{t-j}, (n-j)\Delta\tau)\sqrt{\Delta} e_{t-j}, \tag{3.11}$$

where $e_{t-j} \sim N(0,1)$.

3.1 Implied Volatility model

Implied volatility is the volatility that makes the Black-Scholes model match option prices observed in the market. The Black-Scholes model assumes that the stock price (asset) satisfies the SDE

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dz_s. \tag{3.12}$$

The initial condition for this process is $S(0) = S_0$. The Black-Scholes model holds when σ becomes a function of time $\sigma(t)$. Thus, we replace σ by

$$\bar{\sigma} = \left[\frac{1}{(T-t)} \int_t^T \sigma^2(s) ds \right]^{1/2}. \tag{3.13}$$

In the market, the Black-Scholes model is usually calibrated by finding the value of σ that makes the theoretical model match option prices observed in the market to obtain the implied volatility $\hat{\sigma}$. It is commonly observed across many underlying assets that $\hat{\sigma}$ varies across both exercise price & time-to-maturity. This pattern, when observed is known as the volatility skew. This is incompatible with the assumption of constant volatility (homoskedasticity) in the Black-Scholes model.

It is known that local variance is a conditional expectation of instantaneous variance. Estimation of local volatilities given by a given stochastic volatility model can be done. Implied volatility can thus be calculated. Finally approximation of the shape of the volatility surface can be done. This would allow researchers to deduce some characteristics of the underlying process.

Upon observing evolution of volatilities in options being traded in the market, it becomes clear that different options on the same underlying asset are governed by different volatilities Kamp (2009). The constant volatility which is called the *implied volatility* is utilized in the Black-Scholes equation. This implied volatility seems to depend on the term to maturity (T) and the the strike price (K) of the option. This dependence is referred to as the volatility skew.

In order to understand the volatility skew, we need to first understand the *implied volatility* (σ_{imp}). Volatility skew is the ratio of σ_{imp} to the strike level K.

Another issue that arises when observing log asset returns of prices observed in the market is *volatility clustering*. This is the tendency of small moves follow small moves and large moves follow large moves in log returns. This can be demonstrated in the figure below.

3.2 Local Volatility

The *local volatility* is the volatility (σ) that is used in the Black-Scholes model to calculate Put and Call option prices. This was studied by Dupire et al. (1994) & Derman and Kani (1994).

Dupire begins with stock price following GBM

$$\frac{dS}{S} = \mu_t dt + \sigma(S_t, t) dZ.$$

The boundary condition for the process is $S(0) = S_0$. The payoff for a European call option under the risk-neutral numeraire (**Q**) denoted as $C(S_0, K, T)$ is given by

$$C(S_0, K, T) = \int_K^\infty dS_{T\varphi}(S_T, K; S_0)(S_T - K)$$

where $\varphi(S_T, K; S_0)$ is the quasi-probability density for the spot price at maturity (T). The *definition* of the local volatility function is

$$\sigma^2(K, T, S_0) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}. \tag{3.14}$$

The Heston SDE takes the form

$$\frac{dS}{S} = \mu dt + \sqrt{v} dz_s, \tag{3.15}$$

$$dv = k_v (\bar{v} - v) dt + \sigma_v \sqrt{v} dz_v, \tag{3.16}$$

where $\mathbb{E}(dz_s, dz_v) = \rho$. Transforming the paired SDE above to independent Wiener processes results in

$$\frac{dS}{S} = \mu dt + \sqrt{v} dz_s, \tag{3.17}$$

$$dv = k_v (\bar{v} - v) dt + \sigma_v \sqrt{v} \rho dz_s + \sigma_v \sqrt{v} \sqrt{1 - \rho^2} dz_v, \tag{3.18}$$

Notation	Meaning
x_t	transformed asset price
v_t	local volatility
ρ	correlation between local volatility & asset price

Table 3.1: Local volatility model parameters (Heston)

Chiarella and Ziveyi (2013) aimed to calculate local variances in the Heston model and integrate local variance to approximate the Black-Scholes implied variance. To do this they took the steps outlined below. First they consider unconditional expectation of instantaneous variance at time s as

$$\tilde{v}_s = (v_0 - \bar{v}) e^{-\lambda s} + \bar{v}.$$

The Total variance to time *t* is defined as

$$\hat{w}_t := \int_0^t \tilde{v}_s ds = (v_0 - \bar{v}) \left\{ \frac{1 - e^{-\lambda t}}{\lambda} \right\} + \bar{v} t.$$

Finally, they let $u_t = \mathbb{E}[v_t | x_T]$ be expectation of variance at time t given x_T.

3.3 Stochastic volatility models

One stochastic volatility model by Hull and White (1987) was a model where the variance process takes the form

$$dv = k_v v(t) dt + \sigma_v v(t) dz_v. \tag{3.19}$$

The correlation between the underlying process and the variance process is set to $\rho = 0$. Another model was also proposed where the volatility is modelled after an Ornstein-Uhlenbeck process. An important property of this model is *mean-reversion*. This version of the model is

$$dv = (\bar{v} - k_v v)\, dt + \sigma_v \sqrt{v} dz_v. \tag{3.20}$$

The correlation between the asset process & the volatility process in the model is set to $\rho = 0$. A general problem with these models is the lack of analytical expressions in cases where the correlation parameter ρ is not zero. i.e. $\rho \neq 0$ The solution is that one has to either ignore the correlation parameter ($\rho = 0$) or use cumbersome numeric solution methods that are computationally heavy.

3.4 Heston model

The model by Heston was originated to derive a closed form solution for the price of a European call option on an asset with stochastic volatility. The asset price under the probability measure $\mathbb{P}$ follows the process

$$dS(t) = \mu S(t) dt + \sqrt{v(t)} S(t) dz_s^{\mathbb{P}}(t). \tag{3.21}$$

The variance process for the underlying asset is assumed to be stochastic and follows the process

$$dv(t) = \kappa \left(\theta - v(t)\right) dt + \sigma \sqrt{v(t)} dz_v^{\mathbb{P}}(t). \tag{3.22}$$

The variance follows the square root Cox Ingersoll Ross (CIR) stochastic process. This used to model variance because it ensures that the variance process remains away from zero. The Heston model makes it possible to calculate the characteristic function of the joint distribution of S and v. A table summarizing the parameters of of interest in the Heston model is

Notation	Meaning
μ	the drift of the process for the stock
κ>0	the mean reversion speed for the variance
θ>0	the mean reversion level for the variance
σ>0	the volatility of the variance
v_0>0	the initial (time zero) level of the variance

Table 3.2: Model Parameters

The correlation between the two Wiener processes above is

$$\left(dz_s^{\mathbb{P}}(t), dz_v^{\mathbb{P}}(t)\right) = \rho dt.$$

As with other pricing measures, we wish to perform a *change of numeraire (measure)* to the risk neutral probability measure $\mathbb{Q}$ also known as its equivalent martingale target. The risk-neutral process is the one used for further pricing.

Two assumptions for this thesis are that there is no dividend payment and the interest rate is a constant, hence μ is a fixed value. A risk-averse investor expects to achieve a risk-premium. The risk premium λ is deducted from the operation in the variance process. The model under $\mathbb{Q}$ measure can be summarized as

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dz_s^{\mathbb{Q}}, \tag{3.23}$$

$$dv(t) = \kappa\left(\theta - v(t)\right)dt + \sigma\sqrt{v(t)}dz_v^{\mathbb{Q}}. \tag{3.24}$$

Kurtosis is a measure of the steepness of the distribution. The normal distribution has a kurtosis of 3. A kurtosis greater than 3 implies a steeper distribution than the normal distribution and vice versa.

The *Black-scholes model* has a log-normal distribution with the density function

$$\mathbb{Q}(z) = \frac{1}{\sigma\sqrt{2\pi(T-t)}}\text{exp}\left[-\frac{\left\{ln(S_T/S) - \left(r - \sigma^2/2\right)(T-t)\right\}^2}{2\sigma^2(T-t)}\right]. \tag{3.25}$$

The variance of the Heston model follows a CIR process whose pdf follows the χ^2 distribution. The average variance is given by

$$\frac{1}{\tau}\mathbb{E}\int_t^T v_s ds = \theta + (v_t - \theta)\frac{1-e^{-\kappa\tau}}{\kappa\tau}. \tag{3.26}$$

Note that the long-term level of variance θ is equal to the initial variance v_t.

The Heston model has various advantages namely the existence of a Semi-closed form solution, it allows for non-lognormal probability distribution (high peak, fat tails), it fits the implied volatility surface of option prices when the maturity is not too small, has mean reverting volatility & into account leverage effect (negative correlation of equity returns and implied volatility). The disadvantages of the Heston model are: the parameter values in the Heston model are not easily estimated, prices produced by the model are parameter sensitive, fails to produce decent results for short maturities.

3.5 Valuation framework

Under the assumption that markets are complete and arbitrage free, options can be calculated as the present value of thie expected value under the risk neutral numeraire

$$V_0 = e^{-rT} E_Q\left[H(S_t)\right] \tag{3.27}$$

where V_0 is the option value at time $t = 0$, r is the risk free rate, T is the time to maturity and $H(S_t)$ is the option payoff. When the option is a European Call, the payoff $H(S_t) = (S_t - K)^+$. We work with the logarithm of asset prices to obtain simpler characteristic functions.

Characteristic functions have a one-to-one relationship with density functions. The characteristic function of a given stochastic process X is the Fourier transform of its probability density function

$$\psi(w) = E[e^{iwX}] = \int_{-\infty}^{\infty} e^{iwX} f(x) dx.$$

By applying the Fourier Inversion theorem, the density function of the process X can be recovered in terms of its characteristic function

$$f(x) = \frac{1}{2\pi} = \int_{-\infty}^{\infty} e^{-iwX} E[e^{iwX}] dw.$$

Thus, all probability evaluations required to calculate option values can also be computed using characteristic functions.

3.6 Valuing European Call through Characteristic Functions

The European call option is valued using the formula

$$C_0 = S_0 P_1 - e^{-rT} K P_2 \tag{3.28}$$

where P_1 is the option delta and P_2 is the probability that the option will be exercised. Instead of using density functions, these probabilities can be derived from the characteristic functions

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re \left[\frac{e^{i\psi \ln K} f_j}{i\phi} \right] d\phi \tag{3.29}$$

for $j = 1\&2$. The function f_j is the characteristic function of the log-price.

3.7 Application of the Black-Scholes Model

The Black-Scholes SDE under the risk neutral measure takes the form

$$dS_t = rS_t + \sigma S_t dW_t. \tag{3.30}$$

The closed form solution to this equation takes the form

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z} \tag{3.31}$$

where Z is the standard normal distribution. The distribution of S_t is lognormal and $\ln S_t$ is normally distributed. Thus $\ln S_t \sim N(ln(S_0) + (r - 0.5\sigma^2)t, \sigma^2 t)$

3.8 Black and Scholes Characteristic Function

The characteristic function for a normally distributed random variable is given by

$$\psi(w) = e^{iw(mean) - \dfrac{1}{2}w^2(variance)}. \tag{3.32}$$

The characteristic function for $\ln(S_t)$ can be calculated using the function

$$\psi^{BSM}_{ln(S_t)}(w) = e^{iw\left[ln(S_0) + (r - 0.5\sigma^2)t\right] - 0.5w^2\sigma^2 t}. \tag{3.33}$$

Once we have the characteristic function, the next step is to estimate P_1 and P_2. These two probabilities can be computed with numerical integration or with Euler's formula ($e^{ix} = \cos x + i \sin x$).

3.9 Heston characteristic function

The characteristic function proposed by Gatheral (2006) takes the form

$$
\begin{aligned}
\psi^{Heston}(w) &= e^{\left[C(t,w)\bar{V}+D(t,w)V_0+iw\ln S_0 e^{rt}\right]} \\
C(t,w) &= a\left[rt - \frac{2}{\eta^2}\ln\left(\frac{1-ge^{-ht}}{1-g}\right)\right] \\
D(t,w) &= r\frac{1-e^{-ht}}{1-ge^{-ht}} \\
r_{\pm} &= \frac{\beta \pm h}{\eta^2}; h = \sqrt{\beta^2 - 4\alpha\gamma} \\
g &= \frac{r_-}{r_+} \\
\alpha &= -\frac{w^2}{2} - \frac{iw}{2}; \beta = a - \rho\eta iw; \gamma = \frac{\eta^2}{2}.
\end{aligned}
$$

Christostomo (2014) utilizes the above approach which avoids having two distinct functions for P_1 and P_2 utilized by Gatheral (2006).

3.10 Significance of the study

The implied volatility of put and call options traded in the market demonstrate that the constant volatility assumption of the Black Scholes model is far from being realized. The mean reversion property in a stochastic volatility process seems to be far more representative of the real evolution of implied volatility in the market. The Heston model is a widely favoured stochastic volatility model due to the existence of a closed form solution.

This research adds to the existing body of knowledge and provides a framework for estimating the Heston stochastic volatility model parameters by using the maximum likelihood method. The significance of these parameters is considered for three epochs during the 2008 financial crash, the Covid-19 pandemic and the relatively stable period between them. The Heston parameters had greater statistical significance than the Black-Scholes parameters. None of the Black-Scholes parameters were statistically significant.

Chapter 4

Research Methodology

4.1 Introduction

This chapter provides the logical framework that was adopted in conducting this research. It involved the choice of data for this study as well as the analysis of the analysis method used to obtain the parameter estimates and test the significance of the parameter estimates.

4.2 Description of data

This data was derived from option prices traded in the S&P 500 volatility (VIX) index (link). The VIX index measures the 30 day *expected volatility* of the S&P 500 index by aggregating the weighted prices of multiple S&P 500 puts and calls over a wide range of strike prices. In percentage times it measures a 1% standard deviation in market returns. The S&P 500 index has an inverse relationship with the VIX index. The data obtained was for the period from July 1, 2005 to April 1,2021.

Chapter 5

Data Analysis and Results

5.1 Introduction

This chapter provides a discussion of the data analysis including the steps taken to standardize the diffusion functions of the Heston model, define the input variables from the option data,replacement of the instantaneous volatility using the Jacobian term which is the determinant of the data matrix. This Jacoian term converts the state variable in the data (X) to a form that can be an input for the likelihood function which generates the Heston stochastic parameters.

5.2 Descriptive Statistics

Descriptive statistics based on datly S&P 500 values were considered in this research and the results are shown in the table below.

Observations	1826
Min	-0.407
Max	0.308
Mean	-0.004
Std Dev	0.040
Skew	0.460
Kurtosis	-0.1125
Jarque-Bera test	X squared = 3.2135 p-value = 0.22
Augmented Dicky-Fuller test	Dickey Fuller = -11.257 p-value = 0.01

Table 5.1: Descriptive Statistics of daily S&P 500 index returns

From table 5.1 above, the mean of the data is -0.004 and its kurtosis is -0.1125 which is less than 3 indicating a light tailed distribution. Skewness of 0.460 is within -0.5 to 0.5 range indicating the data is symmetrical. A further test for normality of the data was done using the Jarque-Berra test whose null hypothesis is that the data is normally distributed. The p-value (0.22) obtained was higher than the critical level (0.05) hence the null hypothesis could not be rejected at the 5% significance level.

Augmented Dicky-Fuller test (ADF) was conducted on daily S&P 500 volatility index (VIX) data to determine the stationarity of the data. The null hypothesis is that the data is non-stationary. The p-value (0.01) obtained was smaller than the critical level (0.05) indicating that null hypothesis was rejected at the 5% significance level. Thus the stationary data did not contain a unit root.

5.3 Standardization of diffusion function

The first step involved standardizing the diffusion function of the underlying and volatility processes using the Lamperti transform

$$Y \equiv \gamma(S, v; 0) = \int^{S} \frac{du}{u; \theta}. \tag{5.1}$$

It was impractical to expand p_Y, the conditional probability of the transformed variable Y due to the fact that p_Y gets peaked around the conditional value y_0 when Δ gets small. Thus, a further transformation was performed on Y leading to a "pseudo-normalized" increment of Y defined as

$$Z \equiv \Delta^{-\frac{1}{2}}(Y - y_0). \tag{5.2}$$

The transition density function for Z is defined as p_z. For any random process X that is bounded as X varies in D_X ($X \in D_X$) with maximum class concentration C_X, the maximum likelihood function of the likelihood function used to approximate Heston model parameters proposed by Aït-Sahalia (2002) takes the form

$$
\begin{aligned}
l_X^{(J)}(\Delta, x \mid x_0; \theta) = & -\frac{m}{2} \ln(2\pi\delta) - D_v(x, \theta) + \frac{C_X^{(-1)}(x \mid x_0; \theta)}{\Delta} \\
& + \sum_{k=0}^{J} C_X^{(k)}(x \mid x_0; \theta) \frac{\Delta^k}{k!}.
\end{aligned}
\tag{5.3}
$$

5.4 Definition of Input Variables

Wang et al. (2016) outlined the steps utilized in the estimation of Heston model parameters given option data. The first step involved defining the input variables for the likelihood function from the data.

The spatial dimension of input variables are limited before input. The duration $T - t$ is limited to a fixed dimension. Also, call options refer to options which are considered are in the money options with $\frac{S}{K}$ greater than 1, at the money options with $\frac{S}{K}$ equal to 1 and out of the money options with $\frac{S}{K}$ less than 1. The price of a European call option can be expressed as $[E(S_T) - K]^+ e^{-\int r(t)dt}$. A substitution of $e^{-\int r(t)dt} = \alpha$ which results in

$$
\begin{aligned}
C(T, \alpha S_T, \alpha K) = (\alpha S_T - \alpha K)^+ = \alpha(S_T - K)^+ \\
= \alpha C(T, S_T, K).
\end{aligned}
\tag{5.4}
$$

In order to ensure homogeneity of option prices from these model parameters, additional limitations are included. This ensures that the gamma of the option price (second derivative of option price with respect to spot price) is always non-negative. The additional limitations affect the correlation structure of instantaneous variance estimates. They are,

$$
\begin{aligned}
\sigma_1(X_t)\sigma_1^{'}(X_t) &= \varphi_{11}(Y_t)S_t^2, \\
\sigma_1(X_t)\sigma_i^{'}(X_t) &= \varphi_{1i}(Y_t)S_t, = \varphi_{1i}(Y_t)S_t, i > 1 \\
\sigma_1(X_t)\sigma_i^{'}(X_t) &= \varphi_{ij}(Y_t)S_t, = \varphi_{1i}(Y_t), i > 1, j > 1 \\
\sigma_1^Q(X_t) &= \phi_i(Y_t).
\end{aligned}
\tag{5.5}
$$

This reduces three dimensions of input variables.

5.5 Replacement of instantaneous volatility

Since instantaneous volatility is an unobservable variable, we can estimate it in one of two ways. The first way is to modify input variables from observable to unobservable variables. We convert the unobservable X_t variable into an observable variable Y_t. In order to estimate parameters, the likelihood function is adjusted around X_t with respect to S_t and C_t. These two variables are put in a matrix and the determinant of that matrix is the Jacobian term defined as

$$
\begin{aligned}
J_t &= \det \begin{bmatrix} \frac{\partial S_t}{\partial S_t} & \frac{\partial S_t}{\partial Y_t(1)} & \cdots & \frac{\partial S_t}{\partial Y_t(N)} \\ \frac{\partial C_t(1)}{\partial S_t} & \frac{\partial C_t(1)}{\partial Y_t(1)} & \cdots & \frac{\partial C_t(1)}{\partial Y_t(N)} \\ \cdots & \cdots & \ddots & \cdots \\ \frac{\partial C_t(N)}{\partial S_t} & \frac{\partial C_t(N)}{\partial Y_t(1)} & \cdots & \frac{\partial C_t(N)}{\partial Y_t(N)} \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial C_t(1)}{\partial S_t} & \frac{\partial C_t(1)}{\partial Y_t(1)} & \cdots & \frac{\partial C_t(1)}{\partial Y_t(N)} \\ \cdots & \cdots & \ddots & \cdots \\ \frac{\partial C_t(N)}{\partial S_t} & \frac{\partial C_t(N)}{\partial Y_t(1)} & \cdots & \frac{\partial C_t(N)}{\partial Y_t(N)} \end{bmatrix} \\
&= \det \begin{bmatrix} \frac{\partial C_t(1)}{\partial Y_t(1)} & \cdots & \frac{\partial C_t(1)}{\partial Y_t(N)} \\ \cdots & \ddots & \cdots \\ \frac{\partial C_t(N)}{\partial Y_t(1)} & \cdots & \frac{\partial C_t(N)}{\partial Y_t(N)} \end{bmatrix}
\end{aligned} \tag{5.6}
$$

Now the state variable X_t is converted into the state variable G_t through the Jacobian term.Supposing the transition density function of X_t is $p_X(\Delta, x \mid x_0; \theta)$ and the transition density for G_t is $p_G(\Delta, g \mid g_0; \theta)$. The observable asset price vector G_t as a function of the state variable X is given by

$$G_{t+\Delta} = f\left(X_{t+\Delta};\Theta\right). \tag{5.7}$$

The inverse function that expresses the state variable as a function of asset price is given by

$$X_{t+\Delta} = f^{-1}\left(X_{t+\Delta};\Theta\right). \tag{5.8}$$

Given than $G_t = g_0$, the conditional probability density of G is

$$\begin{aligned} p_G(\Delta, g \mid g_0;\theta) &= det\left(\frac{\partial f\left(f^{-1}\left(g;\theta\right)\right)}{\partial x}\right)^{-1} \cdot p_X\left(\Delta, f^{-1}\left(g;\theta\right) \mid f^{-1}\left(g;\theta\right);\theta\right) \\ &= J_t(\Delta, g \mid g_0;\theta)^{-1} \cdot p_X\left(\Delta, f^{-1}\left(g;\theta\right) \mid f^{-1}\left(g_0;\theta\right);\theta\right) \end{aligned} \tag{5.9}$$

where the multiplier $J_t\left(\Delta, g \mid g_0;\theta\right)$ will be calculated using the equation (5.6).
Asset prices which can be observed in the market prices exhibit the Markov property and can be obtained using the Bayes Rule. The likelihood function takes the form

$$\ell_n(\theta) = n^{-1}\sum_{i=1}^{n} l_G\left(t_i - t_{i-1}, g_{ti} \mid g_{ti-1};\theta\right), \tag{5.10}$$

where

$$\begin{aligned} l_G\left(\Delta, g \mid g_0;\theta\right) &= \ln p_G\left(\Delta, g \mid g_0;\theta\right) \\ &= \ln J_t\left(\Delta, g \mid g_0;\theta\right) + l_X\left(\Delta, f^{-1}(g,\theta) \mid f^{-1}\left(g_0,\theta\right);\theta\right). \end{aligned} \tag{5.11}$$

Wang et al. (2016) made an important note that the implied volatility for short term at-the money options to replace momentary volatility since they converge to the momentary volatility and because they are less impacted by market micro-structure. Applying the maximum likelihood method to the function above results in

$$\frac{\partial l}{\partial g} = 0 \tag{5.12}$$

The optimal parameter vector ($\hat{g}$) that satisfies the above equation is the vector of Heston model parameters. i.e. $\hat{g} = (\hat{r}, \hat{\sigma}, \hat{\theta}, \hat{\kappa}, \hat{\rho}, \hat{\sigma}_v)$

5.6 Results of Parameter Estimates

The data used to obtain these results is the call option data for the S & P 500 volatility (VIX) index obtained on a daily basis from 2005 to 2020. The optimal values of the parameters give a vector of Heston model parameters $\hat{g} = (\hat{r}, \hat{\sigma}, \hat{\theta}, \hat{\kappa}, \hat{\rho}, \hat{\sigma}_v)$. These parameters are $\hat{\mu}$ which is the estimate of long term drift for the underlying, $\hat{\sigma}$ is the estimate for the volatility of the underlying process, $\hat{\theta}$ is the mean reversion level for the variance, $\hat{\kappa}$ is the estimate for the mean reversion speed for variance, $\hat{\rho}$ is the estimate of the correlation between the underlying and volatility process and $\hat{\sigma}_v$ is the estimate for the volatility of the variance. The parameter estimates and their 95% confidence bounds during the 2008 financial crash epoch are summarized in the table below.

Parameter	Estimate	Lower 95% CI	Upper 95% CI	P-Value
r	-0.277	-0.624	0.069	1.17×10^{-1}
σ	1.442	0.917	1.968	1.26×10^{-7}
θ	0.060	-0.104	0.223	4.81×10^{-1}
κ	2.287	-4.291	8.865	5.06×10^{-1}
ρ	0.045	0.026	0.064	4.58×10^{-6}
σ_v	0.269	-0.236	0.773	3.00×10^{-1}

The p-value for the Black-Scholes estimate for the parameter is 0.29 which is *not* statistically significant. The parameter estimates and their 95% confidence bounds for the relatively stable period after the 2008 financial crisis but before the Covid-19 pandemic are summarized in the table below.

Parameter	Estimate	Lower 95% CI	Upper 95% CI	P-Value
r	0.132	0.059	0.204	3.87×10^{-4}
σ	0.435	0.367	0.502	1.74×10^{-33}
θ	0.164	0.067	0.261	9.64×10^{-4}
κ	2.396	0.984	3.808	9.24×10^{-4}
ρ	0.072	0.059	0.086	1.01×10^{-23}
σ_v	-0.046	-0.265	0.173	6.94×10^{-1}

The p-value for the Black-Scholes estimate for the parameter is 0.32 which is *not* statistically significant. The parameter estimates and their 95% confidence bounds during the Covid-19 epoch are summarized in the table below.
The p-value for the Black-Scholes estimate for the parameter is 0.42 which is *not* statistically significant. The p-values for the parameters estimates indicate that they are all significant.

Parameter	**Estimate**	**Lower 95% CI**	**Upper 95% CI**	**P-Value**
r	0.422	-0.421	1.265	3.32×10^{-1}
σ	2.666	1.859	3.473	2.56×10^{-10}
θ	0.182	-0.058	0.421	1.366×10^{-1}
κ	4.063	-3.188	11.314	2.75×10^{-1}
ρ	0.349	0.233	0.464	6.59×10^{-9}
σ_v	-0.129	-0.547	0.289	5.57×10^{-1}

Hence we reject the null hypothesis that the parameters are zero. except for the volatility of volatility parameter. The price of a call option using the Heston model is calculated the function

$$C_{Heston}(S_T, v_T, T) = SP_1 e^{-r(T-t)} - Ke^{-r(T-t)} P_2. \tag{5.13}$$

The price of a call option using the Black-Scholes model is calculated using the function

$$C_{BS}(S_T, v_T, T) = SN(d_1)e^{-r(T-t)} - KN(d_2)e^{-r(T-t)} \tag{5.14}$$

where

$$d_1 = \frac{\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{5.15}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{5.16}$$

$N(d_1)$ is the delta of the option which represents the change in the option price as a result of a change in the price of the underlying and $N(d_2)$ is the probability that the option is in the money at expiry.

Chapter 6

Conclusion and Discussion

6.1 Summary of Findings

The Black-Scholes pricing model has limitations such as the assumption of constant volatility. This is not consistent with volatilities of options that are traded in the market. As an example, the volatility smile is a key feature that violates this assumption of homoskedasticity. Three (3) types of volatility models are investigated namely *classic volatility models*, *local volatility models* & *stochastic volatility models*. The features of each of these volatility models are investigated.

The data used is a time series of daily option data on the S&P 500 index and the underlying S&P 500 index value over the same time. First the state variables of the option price and the underlying are organized into a matrix. Then the determinant of that matrix which is the Jacobian term is used to estimate the Heston and Black-Scholes parameters. The significance of the Heston and Black-Scholes parameter estimates were calculated for three epochs, the 2008 crisis, the Covid-19 pandemic and the period between them. In all epochs, the Heston parameters were found to be significant unike the Black-Scholes parameters. The results are summarized in the table below.

Parameter	2008 crash epoch	Stable epoch	Covid-19 epoch
r	1.17×10^{-1}	3.87×10^{-4}	3.32×10^{-1}
σ	1.26×10^{-7}	1.74×10^{-33}	2.56×10^{-10}
θ	4.81×10^{-1}	9.64×10^{-4}	1.366×10^{-1}
κ	5.06×10^{-1}	9.24×10^{-4}	2.75×10^{-1}
ρ	4.58×10^{-6}	1.01×10^{-23}	6.59×10^{-9}
σ_v	3.00×10^{-1}	6.94×10^{-1}	5.57×10^{-1}
σ_{bs}	0.29	0.32	0.42

6.2 Directions for Future Research

Future studies extensions of this thesis could involve studying the distribution and other statistical properties of the volatility and underlying processes in other stochastic volatility models. Solving for European and American option prices using the Heston model could be another extension. A deeper comparative analysis of the Heston model against other volatility models could also offer other insights.

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Appendix A

Appendices

The R code to implement the Heston model SDEs can be seen below:

A.1 Appendix I: Estimation function

Listing A.1: **R Code: Packages & Data declaration**

```
# Package ———————————————
library(devtools)
library(Quandl)
library(DiffusionRgqd)

# Data ———————————————
# Source data for the S&P500 index (SPX).
    qdata1 <- Quandl("MULTPL/SP500_REAL_PRICE_MONTH")

# Source data for the volatility   index (VIX).
    qdata_vol <- Quandl("CHRIS/CBOE_VX1")
```

Listing A.2: **R Code: Data epoch segmentation**

```
# Epochs ---------------------------------------------------
## (1) during 2008 "2007-07-01" -> "2009-07-01"
## (2) b4 covid "2009-07-01" -> "2020-02-01"
## (3) during covid "2020-02-01" -> "2021-07-01"

qdata_08<-qdata1[which(qdata1$Date>"2007-07-01" &
                        qdata1$Date<"2009-07-01"),]
qdata_vol_08<-qdata_vol[which(qdata_vol$`Trade Date`>"2007-07-01" &
                              qdata_vol$`Trade Date`<"2009-07-01"),]

qdata_mid<-qdata1[which(qdata1$Date>"2009-07-01" &
                        qdata1$Date<"2020-02-01"),]
qdata_vol_mid<-qdata_vol[which(qdata_vol$`Trade Date`>"2009-07-01" &
                              qdata_vol$`Trade Date`<"2020-02-01"),]

qdata_covid<-qdata1[which(qdata1$Date>"2020-02-01" &
                        qdata1$Date<"2021-07-01"),]
qdata_vol_covid<-qdata_vol[which(qdata_vol$`Trade Date`>"2020-02-01" &
                              qdata_vol$`Trade Date`<"2021-07-01"),]
```

A.2 Appendix II: Option Pricing & Errors

Listing A.3: **R Code: SDE parameter functions**

```
# R_t coefficients:
    a00 <- function(t){theta[1]}
    a01 <- function(t){-0.5*theta[2]*theta[2]}
    c01 <- function(t){theta[2]*theta[2]}
    d01 <- function(t){theta[2]*theta[5]*theta[6]}
# V_t coefficients:
    b00 <- function(t){theta[3]}
    b01 <- function(t){-theta[4]}
    e01 <- function(t){theta[2]*theta[5]*theta[6]}
    f01 <- function(t){theta[5]*theta[5]}
```

A.3 Appendix III: Hypothesis test

Listing A.4: **R Code: MLE estimates function (1)**

```
Heston_Param_fn<-function(qdata_under, qdata_vol){
    # Package & data -----
        library(DiffusionRgqd)

        Vt <- rev(qdata_vol[,names(qdata_vol)=='Close'])
        time2 <- rev(qdata_vol[,names(qdata_vol)=='Trade Date'])

    # Plots -----------
        qdata_under = qdata_under[which(qdata_under$Date %in%
                                        qdata_vol$`Trade Date`),]

        St <- rev(qdata_under[,names(qdata_under)=='Value'])
        time1 <-rev(qdata_under[,names(qdata_under)=='Date'])


        plot(St~time1, type= 'l', col ='steelblue',
            main = 'S&P 500 (SPX)', ylab  = 'S&P 500', xlab = 'Date')

        plot(Vt~time2, type= 'l', col ='steelblue',
            main = 'Volatility (VIX)', ylab  = 'Volatility',
            xlab = 'Date')
```

Listing A.5: **R Code: MLE estimates function (2)**

```
# Heston calib ———————————————————
    Vt=Vt[1:length(St)]



# Create data matrix and numerical time vector :
      X <- cbind(log(St),(Vt/100)^2)
      time <- cumsum(c(0, diff(as.Date(time1))*(1/365)))

# Some starting parameters for the optimization routine:————————
      theta.start <- c(0, 1, 1, 0.5, 1, 0)

# Calculate MLEs of the parameter vector:——————————
      model_1 <- BiGQD.mle(X, time, mesh = 10, theta = theta.start)

# Retreve parameter estimates and appr. 95% CIs:
      Heston_Param <- GQD.estimates(model_1)

return(Heston_Param)
}
```

Listing A.6: **R Code: Calling function for epochs**

```
# Calling Heston function for epochs ----------------
    Heston_Param_08<-Heston_Param_fn(qdata_08,qdata_vol_08)
    Heston_Param_mid<-Heston_Param_fn(qdata_mid,qdata_vol_mid)
    Heston_Param_covid<-Heston_Param_fn(qdata_covid,qdata_vol_covid)



# Black Scholes Calib ----------
BlackSch_Param_fn<-function(qdata_under){
  St<-qdata_under$Value
  Rt<-log(St[-1]/St[-length(St)])

  mu<-mean(Rt)
  sig_s<-sqrt(var(Rt))

  p_val<-(mu-0)/sig_s

  # BS Param -----------
  BS_Param <- list(mu = mu, sig_s = sig_s, p_val = p_val)
  BS_Param<-as.data.frame(BS_Param)
  return(BS_Param)
}
```

A.4 Appendix IV: Simulation

Listing A.7: **R Code: p-values for parameters**

```
# Calling Heston function for epochs ----------------
    BlackSch_Param_08<-BlackSch_Param_fn(qdata_08)
    BlackSch_Param_mid<-BlackSch_Param_fn(qdata_mid)
    BlackSch_Param_covid<-BlackSch_Param_fn(qdata_covid)

# P values ------------------
p_value_fn<-function(Heston_Param){
  Heston_Param$Std_Error<-(Heston_Param$Upper_95-Heston_Param$Lower_95)/
                                (2*1.96)
  Heston_Param$z<-abs(Heston_Param$Estimate)/Heston_Param$Std_Error
  Heston_Param$P<-exp(-0.717*Heston_Param$z -0.416*(Heston_Param$z)^2)
  return(Heston_Param)
}

Heston_Param_08<-p_value_fn(Heston_Param_08)
Heston_Param_mid<-p_value_fn(Heston_Param_mid)
Heston_Param_covid<-p_value_fn(Heston_Param_covid)
```

A.5 Appendix V: Descriptive statistics

Listing A.8: **R Code: Descriptive statistics function**

```
# Descriptive statistics ----------------
Desc_Stats_fn<-function(qdata1,qdata_vol){
        # Define output list
                Desc_lst<-list()
        # Get returns
                St<-qdata1$Value
                Rt<-log(St[-1]/St[-length(St)])

        summ_Rt<-summary(Rt)[-c(2,5)]
        sd_Rt<-sd(Rt)
        adf_Rt<-adf.test(Rt)
        skew_Rt<-skewness(Rt)
        kurt_Rt<-kurtosis(Rt)
        JB_Rt<-jarque.bera.test(Rt)
        len_Rt<-length(Rt)

        Desc_lst<-list(summ_Rt,adf_Rt,skew_Rt,kurt_Rt,JB_Rt,len_Rt)
        return(Desc_lst)
}
```

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Estimating Heston Stochastic Volatility model parameters Martin Musya Thesis presented in fulfillment of the academic requirement for the degree of Masters in Mathematical Finance of Strathmore University August 2021

Declaration and recommendation Declaration This proposal is my original work and has not been submitted or presented for assessment in any institution Signature .... Date Martin Maengo Musya Recommendation This proposal has been submitted for assessment with our approval as supervisors according to Strathmore University regulations. Signature .... Date Dr. Samuel Chege Maina Strathmore University Signature .... Date Meleah Oleche Strathmore University v

Abstract In this thesis, we begin by investigating volatility in the Black Scholes model. This model; while having the benefit of ease of application has some restrictive assumptions. These in- clude constant volatility of asset returns, normality assumptions for returns and the assump- tion that markets are complete. The assumption of normal distribution is reasonable for long term (LT) horizons but not for shorter horizons. Market are rarely if ever complete. There always exists informational as- symetry where some investors know more about the market than others. It is also well known that a single asset is insufficient to hedge away risk. Modern pricing models overcome these assumptions and give more realistic prices. One such pricing model is the Heston Model. It overcomes the assumption of constant variance (homoskedasticity). We will utilize begin by estimating the Heston model parameters over three periods. These are during the 2008 financial crisis, the Covid-19 pandemic and the relatively stable years be- tween the two periods. The maximum likelihood method will be utilized to estimate these pa- rameters. The state variables of the option price and that of the underlying are organized into a matrix. The determinant of that matrix which is the Jacobian term is then used to estimate the Heston and Black-Scholes parameters. These estimates are then used to price call option based on the data. A hypothesis test on the output for the three periods shows that at the 95% significance level the Heston model parameters are significant unlike the Black-Scholes pa- rameters for all periods under study. The Heston model performs better even in times of finan- cial instability such as during the 2008 crisis and the Covid-19 pandemic. vi

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Dedication This thesis is dedicated to my dear parents Janet Musya and Daniel Musya in addition to my sister Mercy Musya for their support during the entire period I was working on this project. xii

Abbreviations CMA Capital Markets Authority NSE Nairobi Securities Exchange xiii