

On the Banach algebra numerical range

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Abstract

Let A denote a unital Banach algebra and SA denote its unit sphere. It was proved by F. F. Bonsall and J. Duncan that when the Banach algebra is unital, the algebra numerical range is identical to a subset of itself. Basing on this fact, we have established that the union over all elements of SA of the sets of support functionals for the unit ball at each $x \in SA$ i.e. $S_x \in SD(A, x)$ is equal to the set of normalized states i.e. $D(A, e)$ when the Banach algebra A is unital (e is the unit element). The implication is that even when the algebra is smooth, the set of normalized states i.e. $D(A, e)$ is not a singleton. Hence algebra numerical range is not singleton except when the element is a scalar multiple of the unit. Also, consider the statements $P1$ and $P2$; ($P1$) The union over all elements of SA of the sets of support functionals for the unit ball at each $x \in SA$ i.e. $S_x \in SD(A, x)$ is equal to the set of normalized states i.e. $D(A, e)$ -component algebra numerical ($P2$) The algebra numerical range i.e. $V(A, a)$ is equal to the e -range i.e. $V(A, a, e)$. We have proved that the statements $P1$ and $P2$ are equivalent under a suitable condition. Further, F.F. Bonsall and J. Duncan proved that for a unital Banach algebra, the numerical radius is an equivalent algebra norm. In that proof, the inequalities $k > v(a)$ and $k \geq 1 \cdot a$ were used to conclude that $1 \leq v(a)$ (e is the irrational number 2.718...). However, these two inequalities lead to the undesired inequality $1 \geq v(a)$. In this dissertation we improve on this proof by merging the property of the complex roots of unity used by Bonsall and Duncan together with the geometric series argument and making a different choice of the arbitrary element $b \in A$ of norm less than one to derive the desired inequality $1 \leq v(a)$. and $D(A, e) = \{f \in SA : f(e) = 1\}$. Elements of $D(A, x)$ are called support functionals for the unit ball at $x \in SA$ while elements of $D(A, e)$ are called normalized states. For each $a \in A$ and $x \in SA$ define $V(A, a, x) = \{f(ax) : f \in D(A, x)\}$