Strathmore
UNIVERSITY

STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES (SIMS)
BBS (ACTUARIAL SCIENCE, FINANCE and FINANCIAL ECONOMICS)
END OF SEMESTER EXAMINATION
BSA3109/BSF3224/BSA3144 STOCHASTIC MODELS FOR ACTUARIAL APPLICATIONS & FINANCE ANALYSIS

DATE: 19TH JULY 2017 Time: 2 Hours

Instructions
1. This examination consists of **FIVE** questions.
2. Answer **Question ONE (COMPULSORY)** and any other **TWO** questions.

SECTION A
Question One

a. Let $c(0)$ *and* $p(0)$ denote the prices of call and put options at time 0 with payoffs given by:

$$(S_T - K)^+ \text{ and } (K - S_T)^+ \text{ respectively}$$

Show that $c(0) - p(0) = S_0 - Ke^{-rT}$ [4]

b. The price of a European call option with strike price K maturity T is given by the following formula:

$$c_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

Where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

and

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Derive using the put-call parity the formula for a corresponding European call option with strike price K maturity T [4]

c. A one-year European Call option on a non-dividend paying stock in Company ABC has a strike price of \$150. The continuously compounded risk-free rate is 2% p.a. The current stock price is \$117.98. Assume that the market follows the assumptions of a Black Scholes model. An institutional investor holds a number of call options on the stock. Each of the call options has a price of 2.36. Calculate the implied volatility for the underlying. [4]

d. Find the hedging strategy associated with the derivative that has a payoff of $(\ln S_T)^2$ [5]

e. Distinguish between the Girsanov and Martingale representation measure. Explain their significance of each in valuation of derivatives. [6]

f. Consider a non dividend paying equity of price St and a risk-free savings account of value B_t, whose evolution at time t follows these processes:

$$dS_t = \mu S_t dt + \sigma S_t dw_t$$

and

$$dB_t = rB_t dt$$

where w_t is a standard Brownian motion, and μ, σ and r are constants.

where W_t is a standard Brownian motion, and μ, σ and r are constants. Find the value of μ that makes $Z_t = \frac{S_t}{B_t}$ a martingale under a suitable probability risk- neutral measure. [4]

v. Explain what is meant by self-financing in the context of continuous-time derivative pricing, defining all notation used. [3]

SECTION B

<u>Question Two</u>

a. Given constant a and λ and a random variable $Z \sim N\ (0,1)$. Show that: [3]

$$\mathrm{E}\left[\mathrm{e}^{\lambda Z} 1_{\{Z>a\}}\right] = \mathrm{e}^{\frac{1}{2}\lambda^2} \Phi(\lambda - a)$$

Where Φ is a standard normal distribution

b. A European cash or nothing call option has the following payoff at maturity T >0:

$$K 1_{\{S_T > K\}}$$

Given that the price of the stock at time 0 is S_0 , Exercise process K and volatility σ. Derive an explicit formula of the price C_{cash} of the European cash or nothing call option. [8]

c. A European asset or nothing put option has the following payoff at maturity T >0:

$$S_T 1_{\{S_T < K\}}$$

Given that the price of the stock at time 0 is S_0 , Exercise process K and volatility σ. Derive an explicit formula of the price p_{asset} of the European asset or nothing put option. [8]

Question Three

Consider a small investor who starts with an initial capital $X_0 > 0$ and invests in the two assets of the market, stock and money account. Assume that after the initial investment, the investor's portfolio is managed on a self-financing basis. Denoting by π_t the proportion of the portfolio's total value invested in the stock at time t. The dynamics of the assets in this market are:

$$dB_t = rB_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

a. Write down the stochastic differential equation of the portfolio's total value X_t. [4]

b. Derive the solution to the stochastic differential equation in (a). [6]

c. Derive the Risk-neutral valuation of the European contingent claims stated below: [10]

$$V_t = E_Q[V_T|F_t]e^{-r(T-t)}$$

Question Four

a.

i. The general stochastic differential equation of an Ornstein-Uhlenbeck process has the following solution:

$$X_t = X_0 e^{at} + b \int_0^t e^{a(t-s)} dW_t$$

Verify that X_t satisfies the following stochastic differential equation: [4]

$$dX_t = aX_t dt + b dW_t$$

ii. Show that $X_t \sim N\left(X_0 e^{at}, \frac{b^2}{2a}(e^{2at} - 1)\right)$ [3]

iii. What is the distribution of the random variable $\int_0^t X_t \, dt$ [6]

b. Critique Vasicek as a model for interest rate. [7]

Question Five

A stock price under the Risk-neutral world follows a geometric Brownian motion with the following stochastic differential equation, SDE. Where ${W_t}^{\theta}$ is the standard Brownian motion under the risk-neutral world.

$$dS_t = rS_t dt + \sigma S_t d{W_t}^{\theta}.$$

a. Find the solution of S_T [4]

b. Find the stochastic differential equation of ${S_t}^p$: [3]

c. Deduce the distribution of ${S_T}^p$ as well as its solution [3]

d. A power option gives at maturity, a payoff of $({S_T}^p - K)^+$ where S_T is the stock price at maturity. Where $p, K > 0$. Find the price $C\ (0)$ of this power option. [10]