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Maximal Rank for $\Omega_{\mathbf{P}^n}$

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Abstract. Let $\mathbf{k}$ an algebraically closed field and R the homogeneous coordinate ring of $\mathbf{P}^n$ and $\Omega_{\mathbf{P}^n}$ the cotangent bundle of $\mathbf{P}^n$. In this paper I prove that for a given set S of s general points in $\mathbf{P}^n$ then the evaluation map $H^0\big(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(l)\big) \longrightarrow \bigoplus_{i=1}^{s} \Omega_{\mathbf{P}^n}(l)_{|P_i}$ is of maximal rank. Implying that $a_0 = 0$ or $b_0 = 0$ so that $a_0 b_0 = 0$ as conjectured by Anna Lorenzini [4, 5] see below

$$\cdots \longrightarrow R(-d-2)^{b_1} \bigoplus R(-d-1)^{a_0} \longrightarrow R(-d-1)^{b_0} \bigoplus R(-d)^{\binom{d+n}{n}-s} \longrightarrow I_S \longrightarrow 0$$

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1. INTRODUCTION

For a general set of points $\{P_1, \ldots, P_s\} \in \mathbf{P}^n$, with $s \geq n+1$, then the homogeneous ideal of the sub-scheme of the union of these points, $I_S \subset R = \mathbf{k}[x_0, \ldots, x_n]$, $\mathbf{k}$ an algebraically closed field and R the homogeneous coordinate ring of $\mathbf{P}^n$, has the following expected form:

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I_S \longrightarrow 0,$$

$F_p = R(-d-p)^{a_{p-1}} \bigoplus R(-d-p-1)^{b_p}$,

d being the smallest integer satifying $s \le h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$, with

$a_p = \max\{0, h^0(\mathbf{P}^n, \Omega^{p+1}_{\mathbf{P}^n}(d+p+1)) - \mathrm{rk}(\Omega^{p+1}_{\mathbf{P}^n})s\}$,

$b_p = \max\{0, \mathrm{rk}(\Omega^{p+1}_{\mathbf{P}^n})s - h^0(\mathbf{P}^n, \Omega^{p+1}_{\mathbf{P}^n}(d+p+1))\}$, and

$$\binom{d+n-1}{n} < s \le \binom{d+n}{n}.$$

The problem can be reduced to showing the following; for all $0 \le p \le n-1$ and non-negative integer l then existence of the above resolution is the same as saying the evaluation map below is of maximal rank i.e. it is surjective or injective or both; see [1].

$$H^0\big(\mathbf{P}^n, \Omega^{p+1}_{\mathbf{P}^n}(l)\big) \longrightarrow \bigoplus_{i=1}^{s} \Omega^{p+1}_{\mathbf{P}^n}(l)_{|P_i}.$$

For this consider the exact sequence

$$0 \longrightarrow \Omega_{\mathbf{P}^n}(1) \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}^n} \longrightarrow \mathcal{O}_{\mathbf{P}^n}(1) \longrightarrow 0$$

Here, $W = H^0(\mathcal{O}_{\mathbf{P}^n}(1))$, the set of linear forms and $\mathbf{k}[x_0, x_1, ..., x_n] = \mathrm{Sym}(W)$

Tensoring the sequence above with $\mathrm{T}_S(d)$ gives

$$0 \longrightarrow \mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1) \longrightarrow W \otimes \mathrm{T}_S(d) \longrightarrow \mathrm{T}_S(d+1) \longrightarrow 0$$

Now taking global sections we get;

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) \arrow[r] & W \otimes I_d \arrow[r] & I_{d+1} \arrow[d] \\
 & & & H^1(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) \arrow[d] \\
 & & & 0
\end{tikzcd}$$

Thus $H^1(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) = I_{d+1}/W \cdot I_d$, corresponds to the minimal generators of I_S of degree $d+1$, and its dimension is b_0 i.e. $h^1(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) = b_0$.

Similarly, $H^0(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1))$ is the space of linear relations among the generators of degree d, whose dimension is a_0 i.e. $h^0(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) = a_0$.

Now consider the exact sequence

$$0 \longrightarrow \mathrm{T}_S \longrightarrow \mathcal{O}_{\mathbf{P}^n} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

Tensoring it by $\Omega_{\mathbf{P}^n}(d+1)$ gives;

$$0 \longrightarrow \mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1) \longrightarrow \Omega_{\mathbf{P}^n}(d+1) \longrightarrow \Omega_{\mathbf{P}^n}(d+1)_{|S} \longrightarrow 0$$

and now taking global sections yields

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) \arrow[r] & H^0(\Omega_{\mathbf{P}^n}(d+1)) \arrow[r, "\mu"] & H^0(\Omega_{\mathbf{P}^n}(d+1)_{|S}) \arrow[d] \\ & & & H^1(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) \arrow[d] \\ & & & 0 \end{tikzcd}$$

We will prove that μ is of maximal rank for a general set S of s points in $\mathbf{P}^n$.

As result, if μ is injective then its kernel is null i.e. $a_0 = h^0(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) = 0$ and the cokernel is not null that is $b_0 = h^1(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1))$ as expected. On other hand, if μ is surjective then we have the cokernel of μ being null i.e. $b_0 = h^1(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1)) = 0$ and the kernel of μ is not null that is, $a_0 = h^0(\mathrm{T}_S \otimes \Omega_{\mathbf{P}^n}(d+1))$.

2. PRELIMINARIES

We use the statements (the so called *Enonces*) as in [1] by Hirschowitz and Simpson which F Lauze used in [2] to proof maximal rank for $T_{\mathbf{P}^n}$.

Let X a smooth projective variety and X' non-singular divisor of X. Let F be a locally free sheaf on X and

$$0 \longrightarrow \mathrm{F}'' \longrightarrow \mathrm{F}_{|X'} \longrightarrow \mathrm{F}' \longrightarrow 0$$

be a exact sequence of locally free sheaves on X'. The kernel E of $\mathrm{F} \longrightarrow \mathrm{F}'$ is a locally free sheaf on X and we have another exact sequence of locally free sheaves on X'

$$0 \longrightarrow \mathrm{F}'(-X') \longrightarrow \mathrm{E}_{|X'} \longrightarrow \mathrm{F}'' \longrightarrow 0$$

and as well exact sequences of coherent sheaves on X

$$0 \longrightarrow \mathrm{E} \longrightarrow \mathrm{F} \longrightarrow \mathrm{F}' \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{F}(-X) \longrightarrow \mathrm{E} \longrightarrow \mathrm{F}'' \longrightarrow 0.$$

We have the following hypotheses:
$\mathbf{R}(\mathrm{F}, \mathrm{F}', y; a, b, c)$
$\mathbf{RD}(\mathrm{F}, \mathrm{F}', y; a, b, c)$
$\mathbf{RD}(\mathrm{E}, \mathrm{F}'', y'; a', b', c')$

2.1. **Notation.** Set $X = \mathbf{P}^n$, $X' = \mathbf{P}^{n-1}$, $\mathrm{F} = \Omega_{\mathbf{P}^n}$, $\mathrm{F}' = \Omega_{\mathbf{P}^{n-1}}$, $\mathrm{E} = \mathcal{O}_{\mathbf{P}^n}^{\oplus n}(-2)$, $\mathrm{F}'' = \mathcal{O}_{\mathbf{P}^{n-1}}(-1)$.
The exact sequences of the elementary transformations after twisting by $d+1$ are:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_{\mathbf{P}^n}(d) & = & \Omega_{\mathbf{P}^n}(d) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n} & \longrightarrow & \Omega_{\mathbf{P}^n}(d+1) & \longrightarrow & \Omega_{\mathbf{P}^{n-1}}(d+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^{n-1}}(d) & \longrightarrow & \Omega_{\mathbf{P}^n}|_{\mathbf{P}^{n-1}}(d+1) & \longrightarrow & \Omega_{\mathbf{P}^{n-1}}(d+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From which we have the hypotheses:
$\boldsymbol{H}'_{\Omega,n}(d+1; \alpha, \beta, \gamma) = \boldsymbol{H}(\Omega_{\mathbf{P}^n}(d+1), \Omega_{\mathbf{P}^{n-1}}(d+1), \alpha, \beta, \gamma)$ and
$\boldsymbol{H}'_{\mathcal{O},n}(d-1; \rho, \sigma, \tau) = \boldsymbol{H}(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}, \mathcal{O}_{\mathbf{P}^{n-1}}(d); \rho, \sigma, \tau)$ and
$\boldsymbol{H}''_{\mathcal{O},n}(d-1; \rho, \sigma, \tau) = \boldsymbol{H}(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}, \mathcal{O}_{\mathbf{P}^{n-1}}(d); \rho, \sigma, \tau)$.

For the plane divisorial, with $H \subseteq \mathbf{P}^n$ a hyperplane isomorphic to $\mathbf{P}^{n-1}$ we shall utilize the sequence;

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(d-2)^{\oplus n} \longrightarrow \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n} \longrightarrow \mathcal{O}_H(d-1)^{\oplus n} \longrightarrow 0..$$

Hypothesis 2.1. $\boldsymbol{H}'_{\Omega,n}(d+1; \alpha, \beta, \gamma)$
The hypothesis $\boldsymbol{H}'_{\Omega,n}(d+1; \alpha, \beta, \gamma)$ *asserts that for non-negative integers* α, β, γ *and* ε *satisfying the conditions:*

$0 \leq \gamma \leq 1$, *and* $1 \leq \varepsilon \leq n-2$,
$n\alpha + n - 1\beta + \varepsilon\gamma = h^0(\Omega_{\mathbf{P}^n}(d+1))$, *and*

$(n-1)\beta + \varepsilon\gamma \leq h^0(\Omega_{\mathbf{P}^{n-1}}(d+1))$ *having for* $\gamma = 1$ *a quotient* Γ' *then the map*

$$\eta : H^0\big(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{\alpha} \Omega_{\mathbf{P}^n}(d+1)_{|A_i} \oplus \bigoplus_{j=1}^{\beta} \Omega_{\mathbf{P}^{n-1}}(d+1)_{|B_j} \oplus \Gamma'_{|C}$$

is bijective with $h^0(\Omega_{\mathbf{P}^n}(d+1)) = d\binom{d+n}{d+1}$ *and for* α *general points* $A_1 \ldots A_\alpha \in \mathbf{P}^n$, $\beta+1$ *general points* $B_1 \ldots B_\beta, C \in \mathbf{P}^{n-1}$.

Hypothesis 2.2. $\boldsymbol{H}_{\Omega,n}(d+1)$
The hypothesis $\boldsymbol{H}_{\Omega,n}(d+1)$ *asserts that* $\boldsymbol{H}'_{\Omega,n}(d+1;\alpha,\beta,\gamma)$ *is true for all* α, β *and* γ *satisfying the conditions above.*

Hypothesis 2.3. $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$
The hypothesis $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$ *asserts that for non-negative integers* ρ, σ, τ *and* θ *satisfying the conditions:*

$0 \leq \tau \leq 1$ *and* $2 \leq \theta \leq n-1$,
$n\rho + \sigma + \theta\tau = h^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n})$, *and*
$\sigma + \theta\tau \leq h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d))$ *having for* $\tau = 1$ *a quotient* Γ *then the map*

$$\phi : H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}\big) \longrightarrow \bigoplus_{i=1}^{\rho} \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}_{|R_i} \oplus \bigoplus_{j=1}^{\sigma} \mathcal{O}_{\mathbf{P}^{n-1}}(d)_{|S_j} \oplus \Gamma(S)_{|T}$$

is bijective with $h^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) = n\binom{d+n-1}{d-1}$ *and for* ρ *general points* $R_1 \ldots R_\rho \in \mathbf{P}^n$, $\sigma+1$ *general points* $S_1 \ldots S_\sigma, T \in \mathbf{P}^{n-1}$.

Hypothesis 2.4. $\boldsymbol{H}_{\mathcal{O},n}(d-1)$
The hypothesis $\boldsymbol{H}_{\mathcal{O},n}(d-1)$ *asserts that* $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$ *is true for any* ρ, σ, *and* τ *satisfying the conditions above.*

Hypothesis 2.5. $\boldsymbol{H}''_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$
A variant version of the hypothesis $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$ *with* Γ *independent of* Γ' *takes the form* $\boldsymbol{H}''_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$ *and it makes the same assertion as the hypothesis* $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$ *the only difference being quotient dependency.*

3. THE METHODS OF HORACE

Méthode d'Horace simple[3] lemme 1

Lemma 3.1. *Suppose we have a bijective morphism of vector spaces* $\gamma : H^0(X', \mathrm{F}') \longrightarrow L$ *and that we have* $H^1(X, \mathrm{E}) = 0$. *Let* $\mu : H^0(X, \mathrm{F}) \longrightarrow L$ *be a morphism of vector spaces. Then for* $H^0(X, \mathrm{F}) \longrightarrow M \oplus L$ *to be of maximal rank it suffices that* $H^0(X, \mathrm{E}) \longrightarrow M$ *is of maximal rank.*

Differential méthode d'Horace([1] lemme 1)

Lemma 3.2. *Suppose we are given a surjective morphism of vector spaces,* $\lambda : H^0(\mathbf{P}^{n-1}, \Omega_{\mathbf{P}^{n-1}}(d+1)) \longrightarrow L$ *and suppose there exists a point* $Z' \in \mathbf{P}^{n-1}$ *such that* $H^0(\mathbf{P}^{n-1}, \Omega_{\mathbf{P}^{n-1}}(d+1)) \hookrightarrow L \oplus \Omega_{\mathbf{P}^{n-1}}(d+1)_{|Z'}$ *and suppose* $H^1(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) = 0$*. Then there exists a quotient* $\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}_{|Z'} \longrightarrow D(\lambda)$ *with kernel contained in* $\Omega_{\mathbf{P}^{n-1}}(d)_{|Z'}$ *of dimension* $\dim(D(\lambda)) = \mathrm{rk}(\Omega_{\mathbf{P}^n}(d+1)) - \dim(\ker \lambda)$ *having the following property. Let* $\mu : H^0(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(d+1)) \longrightarrow M$ *be a morphism of vector spaces then there exists* $Z \in \mathbf{P}^{n-1}$ *such that if* $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) \longrightarrow M \bigoplus D(\lambda)$ *is of maximal rank then* $H^0(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(d+1)) \longrightarrow M \oplus L \bigoplus \Omega_{\mathbf{P}^n}(d+1)|_Z$ *is also of maximal rank.*

The sequences for the quotient are as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\dim n-1 & \Omega_{\mathbf{P}^{n-1}}(d)_{|Z} \arrow[r, two heads] \arrow[d, hook] & D'_{|Z} \arrow[d, hook] & \dim n-3\ (n-2) \\
\dim n & \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}_{|Z} \arrow[r, two heads] \arrow[d, two heads] & D_{|Z} \cong \mathcal{O}_{\mathbf{P}^{n-1}}|_Z \oplus D'_{|Z} \arrow[d, two heads] & \dim n-2\ (n-1) \\
\dim 1 & \mathcal{O}_{\mathbf{P}^{n-1}}(d)_{|Z} \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbf{P}^3}(d)_{|Z} \arrow[d] & \dim 1 \\
 & 0 & 0 &
\end{tikzcd}$$

3.1. **The Vectorial Methods.**

Lemma 3.3. *Vectorial Method* 1
Let α, β, γ, d *and* ε *be non-negative integers satisfying the conditions of Hypothesis* 2.1 *and* ρ, σ, τ *and* θ *non-negative integers satisfying the conditions of Hypothesis* 2.3 *then the Hypothesis* $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$ *implies* $\boldsymbol{H}'_{\Omega,n}(d+1;\alpha,\beta,\gamma)$.

Proof. Consider the exact sequence;

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n} \longrightarrow \Omega_{\mathbf{P}^n}(d+1) \longrightarrow \Omega_{\mathbf{P}^{n-1}}(d+1) \longrightarrow 0$$

and let B and C be general subsets of $\mathbf{P}^{n-1}$. We specialize A to $R \cup S \cup T$ with R a general set of ρ points in $\mathbf{P}^n$ and S and T sets of σ and τ general points in $\mathbf{P}^{n-1}$. To run points to $\mathbf{P}^{n-1}$, consider the map, $\gamma : H^0(\Omega_{\mathbf{P}^{n-1}}(d+1)) \longrightarrow H^0(\Omega_{\mathbf{P}^{n-1}}(d+1)_{|B}) \oplus \Gamma'_{|C}$, if the number of points we have satisfy $h^0(\Omega_{\mathbf{P}^{n-1}}(d+1))$ then γ is bijective, if not then we specialize as many more points as we need to $\mathbf{P}^{n-1}$ in order for γ to become bijective.

Taking global sections for the exact sequence above and evaluating we construct;

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^0(\Omega_{\mathbf{P}^{n-1}}(d+1)) & \xrightarrow[\cong]{\gamma} & H^0(\Omega_{\mathbf{P}^{n-1}}(d+1)_{|\{B\cup S\}}) \oplus \Gamma'_{|C} \oplus \Gamma_{|T} \\
\uparrow & & \uparrow \\
H^0(\Omega_{\mathbf{P}^n}(d+1)) & \xrightarrow{\beta} & H^0(\Omega_{\mathbf{P}^n}(d+1)_{|R\cup S\cup T=A}) \oplus H^0(\Omega_{\mathbf{P}^{n-1}}(d+1)_{|B}) \oplus \Gamma'_{|C} \\
\uparrow & & \uparrow \\
H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) & \xrightarrow{\alpha} & H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}_{|R}) \oplus H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)_{|S}) \oplus \Gamma_{|T} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

From the above diagram of exact sequences, by Inductive hypothesis on $\mathbf{P}^{n-1}$ and Lemma 3.2 the map γ is bijective and hence if α is bijective then β is bijective as well and this gives $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$ implies $\boldsymbol{H}'_{\Omega,n}(d+1;\alpha,\beta,\gamma)$ □

Lemma 3.4. *Vectorial Method* 2
Let ρ, σ, τ and θ non-negative integers satisfying the conditions of Hypothesis 2.3 *and $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\varepsilon}$ be non-negative integers satisfying conditions similar to those of Hypothesis* 2.1 *with the Hypothesis $\boldsymbol{H}'_{\Omega,n}(d;\overline{\alpha},\overline{\beta},\overline{\gamma})$ being the same as Hypothesis* 2.1 *but twisted by 1, then the Hypothesis $\boldsymbol{H}'_{\Omega,n}(d;\overline{\alpha},\overline{\beta},\overline{\gamma})$ implies $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$.*

Proof. Consider the exact sequence;

$$0 \longrightarrow \Omega_{\mathbf{P}^n}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n} \longrightarrow \mathcal{O}_{\mathbf{P}^{n-1}}(d) \longrightarrow 0$$

and let S and T general sets of σ and τ points in $\mathbf{P}^{n-1}$, specialize R to $A \cup B$, where A is a general set of $\overline{\alpha}$ points in $\mathbf{P}^n$ and B is a general set of $\overline{\beta}$ points in $\mathbf{P}^{n-1}$ with $C = T$.

Now consider the evaluation map, $\gamma : H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)_{|S\cup T})$, if the number of points we have are enough to satisfy $h^0(\mathcal{O}_{\mathbf{P}^n}(d))$ then $\overline{\gamma}$ bijective, if not then we specialize as many more points, $\overline{\beta}$, in this case, to $\mathbf{P}^{n-1}$ in order for $\overline{\gamma}$ to become bijective.

Taking global sections for the exact sequence above and evaluating at corresponding points we construct a diagram of exact sequences as follows;

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)) & \xrightarrow[\overline{\gamma}]{\hspace{3cm}} & H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)_{|\{S\cup T\cup B\}}) \\
\uparrow & & \uparrow \\
H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}_{|R}) \oplus H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)_{|S}) \oplus \Gamma_T \\
\uparrow & & \uparrow \\
H^0(\Omega_{\mathbf{P}^n}(d)) & \longrightarrow & H^0(\Omega_{\mathbf{P}^n}(d+1)_{|A}) \oplus H^0(\Omega_{\mathbf{P}^{n-1}}(d+1)_{|B} \oplus \overline{\Gamma}_C \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
$$

The map $\overline{\gamma}$ is bijective giving the Hypothesis $\boldsymbol{H}'_{\Omega,n}(d;\overline{\alpha},\overline{\beta},\overline{\gamma})$ implies $\boldsymbol{H}'_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$. When the number of points we have in $\mathbf{P}^{n-1}$ are few relative to d we use the plane divisorial method in preference to this method. $\square$

Lemma 3.5. *Plane Divisorial*
Let ρ, σ, τ and θ non-negative integers satisfying the conditions of Hypothesis 2.3 *and set $\rho' = \rho - h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d-1))$. If $\rho' \geq 0$ and $\sigma + \tau \leq h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d-1))$ then the Hypothesis $\boldsymbol{H}_{\mathcal{O},n}(d-2;\rho',\sigma,\tau)$ implies $\boldsymbol{H}_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$.*

Proof. Let R be a general set of ρ points in $\mathbf{P}^n$, S and T be general sets of σ and τ points in $\mathbf{P}^{n-1}$ such that they are fewer relative to d (i.e. when Vectorial Method 2 fails). We choose a hyperplane $H \subset \mathbf{P}^n$ disjoint from S and T with $H \cong \mathbf{P}^{n-1}$ and specialize ρ' points from $\mathbf{P}^n$ to H (i.e. R' is the set we have after specializing from R in $\mathbf{P}^n$) so that $H^0(H, \mathcal{O}_H(d-1)^{\oplus n}) \longrightarrow H^0(\mathcal{O}_H(d-1)^{\oplus n}_{|R'})$ is bijective that is set $\rho - \rho' = h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d-1))$ and so taking global sections for the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(d-2)^{\oplus n} \longrightarrow \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n} \longrightarrow \mathcal{O}_H(d-1)^{\oplus n} \longrightarrow 0$$

we construct a diagram of exact sequences:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^0(H, \mathcal{O}_H(d-1)^{\oplus n}) & \xrightarrow[\cong]{\alpha} & H^0(\mathcal{O}_H(d-1)^{\oplus n}_{|R'}) \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) & \xrightarrow{\beta} & H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}_{|R} \oplus H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d)_{|S}) \oplus \Gamma_{|T} \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d-2)^{\oplus n}) & \xrightarrow{\gamma} & H^0(\mathcal{O}_{\mathbf{P}^n}(d-2)^{\oplus n}_{|R\setminus R'} \oplus H^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d-1)_{|S} \oplus \Gamma_{|T} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

Since α is bijective then γ bijective implies β is also bijective and this gives the Hypothesis $\boldsymbol{H}_{\mathcal{O},n}(d-2;\rho',\sigma,\tau)$ implies $\boldsymbol{H}_{\mathcal{O},n}(d-1;\rho,\sigma,\tau)$. □

3.2. **Hypercritical mèthode d'Horace.**

Lemma 3.6. *Consider $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,0)$ with $d \geq 1, s_1,$ and s_2 being non-negative integers that satisfy: $ns_1+s_2 = h^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n})$ and $s_2 \leq h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d))$. Now suppose that the $H^0(\Omega_{\mathbf{P}^n}(d)) \longrightarrow H^0(\Omega_{\mathbf{P}^n}(d)|_{S_1})$ is injective and $H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}|_{S_1})$ is surjective with a general $S_1 \subseteq \mathbf{P}^n$ then the Hypothesis $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,0)$ is true.*

This Lemma is for when we have no quotient.

Proof. See [6] Lemma 1.11. □

Lemma 3.7. *Consider $\boldsymbol{H}'_{\mathcal{O},n}(d-1;s_1,s_2,1)$ where $d \geq 1, s_1,$ s_2 and $2 \leq \theta \leq n-1$ are non-negative integers such that, $ns_1+s_2+\theta = h^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n})$ and $s_2+\theta \leq h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(d))$. Under the same Hypotheses as Lemma 2.1 i.e. $H^0(\Omega_{\mathbf{P}^n}(d)) \longrightarrow H^0(\Omega_{\mathbf{P}^n}(d)|_{S_1})$ is injective and $H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^n}(d-1)^{\oplus n}|_{S_1})$ is surjective then the Hypothesis $\boldsymbol{H}''_{\mathcal{O},n}(d-1;s_1,s_2,1)$ is true.*

Proof. See [6] Lemma 1.12. □

3.3. **The Main Theorem.**

Theorem 3.8. *Suppose $\boldsymbol{H}_{\Omega,n}(d+1)$ is true. Then for any non-negative integer m, there exists a set, $M = \{P_1, P_2, \ldots, P_m\}$ of m points in $\mathbf{P}^n$ such that the evaluation map, μ, is of maximal rank.*

$$\mu : H^0\big(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^n}(d+1)_{|P_i}$$

Proof. (a) If $h^0(\Omega_{\mathbf{P}^n}(d+1)) \equiv 0 \pmod{n}$ then r is the critical number of points needed for bijectivity i.e. the map $H^0\big(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^n|P_i}$ is bijective. Set $\pi = \lfloor \frac{1}{n} h^0(\Omega_{\mathbf{P}^n}(d+1)) \rfloor$
we now have the following cases:

(i) if $m = r$ then our map is bijective since we have the same number of points as the critical number i.e. the map α is bijective and γ an identity map and so μ is bijective see below:

$$\begin{tikzcd}
H^0(\mathbf{P}^n, \Omega_{\mathbf{P}^n}(d+1)) \arrow[r, "\mu"] \arrow[dr, "\alpha"'] & \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^n|P_i} \\
 & \bigoplus_{i=1}^{n} \Omega_{\mathbf{P}^n|P_i} \oplus \bigoplus_{i=n+1}^{r} \Omega_{\mathbf{P}^n|P_i} \arrow[u, "\gamma"']
\end{tikzcd}$$

(ii) if $m > r$ i.e. we have more points than the critical number and our map is injective i.e. since α is bijective and γ surjective then our map μ has to inject see below:

$$\begin{tikzcd}
H^0(\mathbf{P}^n, \Omega_{\mathbf{P}^n}) \arrow[r, "\cong", "\alpha"'] \arrow[dr, hook, "\mu"'] & \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^n|P_i} \\
 & \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^4|P_i} \oplus \bigoplus_{i=r+1}^{m} \Omega_{\mathbf{P}^4|P_i} \arrow[u, two heads, "\gamma"']
\end{tikzcd}$$

(iii) if $m < r$ then we have the less points than the critical number thus our map surjects i.e. since α is bijective and γ surjective then our map μ is surjective. □

References

[1] A Hirschowitz and C Simpson: *La résolution minimale de l'idéal d'un arrangement général d'un grand nombre de points dans* $\mathbf{P}^n$. Invent. Math. **126**, 467–503 (1996).

[2] F Lauze: *Rang maximal pour* $T_{\mathbf{P}^n}$. Manuscripta Math. **92**, 525–543 (1997).

[3] F Lauze: *Sur la résolution des arrangements génèriques de points dans les espaces projectifs* Thèse, Université de Nice-Sophia Antipolis, Septembre 1994.

[4] A Lorenzini: *The Minimal Resolution Conjecture*. J Algebra **156**, 5–35 (1993).

[5] A Lorenzini: *On the Betti numbers of points in projective space*. Ph.D. thesis. Queen's University, Kingston, Ontario, 1987. J Algebra **156**, 5–35 (1993).

[6] D M Maingi: *On the Application of the Method of Horace to get the number of generators for an ideal of s general points in* $\mathbf{P}^4$ International Journal of Algebra, Vol. 4, **10**, 477 – 500 (2010), to appear.

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