Strathmore
UNIVERSITY

INSTITUTE OF MATHEMATICAL SCIENCES
MASTER OF SCIENCE IN STATISTICAL SCIENCES
END OF SEMESTER EXAMINATION
STA 8101 STATISTICAL INFERENCE

DATE: Wednesday, April 14, 2021 Time: 3hrs

Instructions

1. This examination consists of **FOUR** questions.
2. Answer **Question ONE (COMPULSORY)** and any other **TWO** questions.

Question 1 (**20** Marks)

a) Suppose that $X_1,\ldots,X_n$ be a random sample from a population with the probability density function $f(x;\theta)$ and that $T_n = t(X_1,\ldots,X_n)$ is an estimator of θ. When would the estimator T_n be

i) An Unbiased estimator for θ; and

ii) A uniformly minimum variance unbiased estimator for θ.

(4 Marks)

b) Let $X_1,\ldots,X_n$ be a random sample a $f(x,\theta)$ population, with θ unknown. If $T(X_n,\ldots,X_n)$ is an unbiased estimator of θ, prove (assuming regularity conditions) that

$$Var(T(X_1,\ldots,X_n) \geq \frac{1}{I(\theta)}$$

where $I(\theta)$ is Fisher's Information based on $X_1,\ldots,X_n$.

(5 Marks)

c) Let $X_1,\ldots,X_n$ be a random sample a $N(\theta,\sigma^2)$ population, with θ and σ^2 are unknown. Find the Cramer-Rao lower bound for θ. Is this lower bound achieved for some statistic?

(5 Marks)

d) Suppose that $X_1, \ldots, X_n$ is a random sample of size n for a $N(\mu, \sigma^2)$ population with μ and σ^2 is unknown. Show that the size α generalized likelihood ratio test of the hypothesis $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$ rejects H_0 if and only if

$$\left|\frac{\bar{X} - \mu_0}{S/\sqrt{n}}\right| \geq t_{\alpha/2}(n-1)$$

(6 Marks)

Question 2 (**20** Marks)

a) Let $X_1, \ldots, X_n$ be a random sample from a population with the probability density function $f(x;\theta)$.

i) Letting $y = \frac{\partial}{\partial\theta} ln[f(x;\theta)]$, show that

$$E(y) = 0 \text{ and } Var(y) = -E\left[\frac{\partial^2}{\partial\theta^2} ln[f(x;\theta)]\right] = i(\theta), \text{ the information number.}$$

(6 Marks)

ii) Hence, by the central limit theorem, explain how the

$$S(\theta; \boldsymbol{x}) = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial\theta} ln[f(x_i;\theta)]}{f(x_i;\theta)} \sim N(0, \Im(\theta)),$$

where $\Im(\theta) = ni(\theta)$, the expected Fisher's information

(6 Marks)

b) Let $X_1, \ldots, X_9$ be an iid sample from Exponential(θ), $f(x;\theta) = \theta\exp - \theta x, x > 0$. Suppose we observe $\bar{X}$= 1, use the score test to test the following hypothesis $H_0: \theta = 0.5$ versus $H_0: \theta \neq 0.5$. Use a 5% level of significance.

(8 Marks)

Question 3 (**8** Marks)

a) Let $X_1, \ldots, X_n$ be a random sample from a Exponential(θ)distribution. Using the asymptotic normality property of maximum likelihood estimators, derive an expression for a $100(1-\alpha)\%$ Wald confidence interval for θ.

(6 Marks)

b) Based on part (a), construct a 90% confidence interval for θ for the following random sample:

X: 1,2, 3, 2, 3, 5, 3, 3, 4, 2

(3 Marks)

c)

i) Let $X_1,\ldots,X_n$ be a random sample from a Poisson(θ)distribution. Using the asymptotic normality property of maximum likelihood estimators, show that an approximate $100(1-\alpha)\%$ confidence interval for θ is:

$$\left[\overline{X}+\frac{Z_{\alpha/2}^2}{2n}-\sqrt{\frac{\overline{X}Z_{\alpha/2}^2}{n}+\frac{Z_{\alpha/2}^4}{4n^2}},\overline{X}+\frac{Z_{\alpha/2}^2}{2n}+\sqrt{\frac{\overline{X}Z_{\alpha/2}^2}{n}+\frac{Z_{\alpha/2}^4}{4n^2}}\right].$$

(7 Marks)

ii) Based on part (i), construct a 90% confidence interval for θ for the following random sample:

X: 1,2, 3, 2, 3, 5, 3, 3, 4, 2

(4 Marks)

Question 4 (**20** Marks)

a) State and prove Neyman-Pearson's Lemma

(12 Marks)

b) Suppose that $X_1, \ldots, X_n$ is a random sample of size n for a $N(\mu, \sigma^2)$ population. Show that the most critical test of size α of the hypothesis $H_0: \sigma^2 = \sigma_0^2$ against $H_1: \sigma^2 = \sigma_1^2$ $(\sigma_1^2 < \sigma_0^2)$ is

$$R = \left\{x: \sum_{i=1}^{n} (x_i - \mu)^2 \leq \sigma_0^2 \chi_\alpha^2(n)\right\}$$

(8 Marks)