STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES
MASTER OF SCIENCE IN STATISTICAL SCIENCE
MASTER OF SCIENCE IN MATHEMATICAL FINANCE & RISK ANALYTICS
END OF SEMESTER EXAMINATION
STA 8101: PROBABILITY THEORY/MFI 8102: MEASURE & PROBABILITY THEORY

Date: 13th January, 2023 Duration: 3 Hours

Attempt Question ONE and any other TWO questions:

Question ONE (30 marks)

a. Show $f(x) = x^2$ is Reimann integrable over E= (0,1). (8 marks)

b. If E_1 and E_2 are measurable sets, show that $E_1 \cup E_2$ is measurable. Hence, if $E_1 \cap E_2 = \emptyset$ then $m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$. (8 marks)

c. Find the Lebesgue integral of simple function

i. $\varphi(x) = Int(x)$ over $E = (0,10)$, (4 marks)

ii. $\varphi(x) = Int(x^2)$ over $E = (0,2)$, (5 marks)

where $Int(w)$ return the integer part of w.

d. Prove that E[(Y-E(Y|X))h(X)) = 0 . (5 marks)

Question TWO (15 marks)

a. Show that that the mapping $A \longmapsto P(A|B)$ is countably additive on σ-algebra, $\mathcal{F}_B$.

(8 marks)

b. Consider random variables X and Y from the probability space $(\Omega, \mathfrak{F}, P)$, prove that $E(E(Y|X))$ =E(Y). (7 marks)

Question THREE (15 marks)

a. State and prove the Markov's inequality. Hence or otherwise, use your results to prove Chebyshev's inequality. (8 marks)

b. Let $(\Omega, \mathcal{F}, P)$ *be a probability space*. Show that for all sets $A_1, A_2, \ldots \in \mathcal{F}$ we have that:

$$P\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} P[A_i]$$

(7 marks)

Question FOUR (15 marks)

a. Suppose that X ~ N(0,1) and let Y= X^2|. find E[Y|X] and E[X|Y] (6 marks)

b. Find the Riemann and Lebesgue integral of a step function $f(x)$ defined as:

$$f(x) = \begin{cases} 1 \text{ if} -1 < x < 2 \\ 2 \text{ if } \quad 2 \leq x < 4 \\ 3 \text{ if } \quad 4 \leq x < 8 \\ 0 \quad \text{otherwise.} \end{cases}$$

(9 marks)