STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES

MASTER OF SCIENCE MATHEMATICAL FINANCE & RISK ANALYTICS

MASTER OF SCIENCE IN STATISTICAL SCIENCE

END OF SEMESTER EXAMINATION

MFI 8102: PROBABILITY AND MEASURE THEORY/STA 8101: PROBABILITY THEORY

Date: 11th December ,2023 Duration: 3 Hours

Attempt Question ONE and any other two questions:

Question ONE (30 marks)

a. Show $f(x) = x^2$ is Reimann integrable over E= (0,1). (7 marks)

b. If E_1 and E_2 are measurable sets, show that $E_1 \cup E_2$ is measurable. Hence, if $E_1 \cap E_2 = \emptyset$ then $m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$. (8 marks)

c. Show that that the mapping $A \longmapsto P(A|B)$ is countably additive on, σ-algebra, $\mathcal{F}_B$. (6 marks)

d. Let X_1, X_2 ... be i.i.d with mean μ and variance σ.^{2} Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Prove that

$$Z = \frac{n^{\frac{1}{2}}(\bar{X}_n - \mu)}{\sigma} \to N(0,1)$$

in distribution as $n \to \infty$. (9 marks)

Question TWO (15 marks)

a. If $f, g: X \to \mathbb{R}$ are real valued measurable functions and $k \in \mathbb{R}$ show that

$$kf, \quad f+g, \quad fg, \ f/g$$

are measurable functions, where we assume that $g \neq 0$ in the case of f/g. (9 marks)

b. Show that the outer measure is translation invariant, that is,
$m^*(A+t) = m^*(A)$ for any set $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$
where $A + t = \{a + t | a \in \mathbb{R}, t \in \mathbb{R}\}$ (6 marks)

Question THREE (15 marks)

a. Find the Lebesgue integral of simple function

i. $\varphi(x) = Int(x)$ over $E = (0,10)$, (3 marks)

ii. $\varphi(x) = Int(x^2)$ over $E = (0,2)$, (5 marks)

where $Int(w)$ return the integer part of w.

b. Prove that convergence in probability implies convergence in distribution

(7 marks)

Question FOUR (15 marks)

a) For random variables X and Y from the probability space $(\Omega, \mathfrak{F}, P)$, prove that $E(E(Y|X))$ =E(Y). (7 marks)

a) Let $(\Omega, \mathfrak{F}, P)$ be a probability space. Show that for all sets $A_1, A_2, \ldots \in \mathfrak{F}$ we have that:

$$P\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} P[A_i]$$

(8 marks)