STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES

BBS (ACTUARIAL SCIENCE, FINANCE and FINANCIAL ECONOMICS)
MARCH 2019 SPECIAL EXAMINATION
BSA3109/BSF3224/BSA3144 STOCHASTIC MODELS FOR ACTUARIAL APPLICATIONS & FINANCE ANALYSIS

DATE: 4TH MARCH 2019 Time: 2 Hours

Instructions

1. This examination consists of **FIVE** questions.
2. Answer **Question ONE (COMPULSORY)** and any other **TWO** questions.

Question One [30 Marks]

a. Give the Risk-neutral valuation formula explain all the symbols used [4 Marks]

b. State the stochastic differential equation of a Geometric Brownian motion and its solution (No proof required) [4 Marks]

c. Find a solution for the stochastic differential equation: [4 Marks]

$$dX_t = -X_t dt + \sqrt{2} dW_t$$

d. W_t is a standard Brownian motion under the probability measure P and the stochastic process X has the following stochastic differential equation:

$$dX_t = dW_t + \frac{n-1}{X_t} dt$$

Apply Ito's formula to show that $M_t = \frac{1}{X_t{}^{n-2}}$ is a local martingale. [3 Marks]

e. By use of properties of conditional Expectations, show that the discounted share price, $D_t = S_t e^{-rt}$ under the risk-neutral world is is a Martingale. [4Marks]

f. The price of a call option under the Black- Scholes market is given by the following

$$c_t = S_t N(d_1) - Ke^{-r(T-t)} N(d_2)$$

Where:

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T - t}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

i. Delta of an option is how much the value of the option changes with respect to small changes in the underlying price, i.e. $\Delta = \frac{\partial V}{\partial S}$. Show that the delta of a call Δ_c is given by: [7 Marks]

$$\Delta_c = N(d_1)$$

ii. Find the delta at expiration of at-the-money call option

[4 Marks]

Question Two [20 marks]

a. Give the distribution of each of the following quantities: [8]

	Quantity	Distribution
i.	$W_t - W_s$	
ii.	dW_t	
iii.	$f(t)dW_t$	
iv.	$\int_0^t f(s)dW_s$	

Where W_t be a standard Brownian motion.

b. Let W_t be a standard Brownian motion. Consider the following stochastic differential equation (SDE):

$$dX_t = (\alpha - \beta X_t)dt + \sigma dW_t$$

Let α, β *and* σ are constant parameters with $X_0 = x$. The solution is a mean-reverting process called the Ornstein-Uhlenbeck process and has been used to model the stochastic behaviour of interest rates and currency exchange rates.

i. Verify that the solution to the SDE can be expressed in the following form: [8]

$$= e^{-\beta t}\left[x + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma \int_0^t e^{\beta s} dW_s\right]$$

ii. Deduce that: [4]

$$var(X_t) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)$$

Question Three [20 marks]

Assume Black-Scholes framework

i. Describe the assumptions of the Black-Scholes Model. [6]

ii. Show that the value at time t of a European call option with strike price K maturity T is given by the following formula: [8]

$$c_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

Where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$

and

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$

$$\text{Hint: } E\left[e^{\lambda Z} 1_{\{Z>a\}}\right] = e^{\frac{1}{2}\lambda^2} \Phi(\lambda - a)$$

Where a *and* λ are constants and a random variable $Z \sim N(0,1)$.

iii. A stock is currently priced at 400p. The price of a six-month European call option with an exercise price of 420p is 41p. The annual risk-free interest rate (continuously compounded) is 7% and no dividends are payable during the life of the option. Assume the Black-Scholes pricing formula applies.

a. Estimate the implied volatility of the stock to within 1%. [6]

Question Four [20 marks]

a. A put-call parity is an equation giving the relationship between the value at time t of a call option c_t, with maturity T and exercise price K and the value at time t of a corresponding put option p_t with the same exercise price and maturity. Prove the following put-call parity: [10]

$$c_t + Ke^{-r(T-t)} = p_t + S_t$$

Where r *is the risk* $-$ *free rate earned in a cash* $-$ *account* and S_t is the price of the underlying stock at time t.

b. A European chooser option is an option where, at a specified future time u, the holder can choose whether the option is a call or a put, both of which mature at time $T > u > t$ and have the same strike price K. Let the price of this option at time t be denoted h_t. Show that: [10]

$$h_0 = S_0(\Phi(d_1) - \Phi(-d_3)) + Ke^{-rT}\left(\Phi(-d_2) - \Phi(-d_4)\right)$$

Where: $d_1 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ $\qquad d_2 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

$$d_3 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)u}{\sigma\sqrt{u}} \qquad d_4 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)u}{\sigma\sqrt{u}}$$

Question Five [20 marks]

Consider a small investor who starts with an initial capital $X_0 > 0$ and invests in the two assets of the market, stock and money account. Assume that after the initial investment, the investor's portfolio is managed on a self-financing basis. Denoting by π_t the proportion of the portfolio's total value invested in the stock at time t. The dynamics of the assets in this market are:

$$dB_t = rB_t dt \qquad B_t = e^{rt}$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

a. Write down the stochastic differential equation of the portfolio's total value X_t. [5]

b. Derive the solution to the stochastic differential equation in (a). [7]

c. Explain what is meant by a self-financing and replicating strategy for X. [8]