STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES

MASTER OF SCIENCE IN STATISTICAL SCIENCE

END OF SEMESTER EXAMINATION

STA 8101 PROBABILITY THEORY/MFI 8102 MEASURE AND PROBABILITY THEORY

Date: 14th December, 2021 **Duration:** 3 Hours

Attempt Question ONE and any other two questions:

Question ONE (30 marks)

a. Prove the following properties of Lebesgue outer measure on (0,1):

i. $0 \leq m^*(E) \leq 1$ for any $E \subset (0,1)$. (3 marks)

ii. $m^*(E) \leq m^*(F)$ if $E \subset F$. (2 marks)

iii. $m^*(\emptyset) = 0$ (2 marks)

b. Find the Lebesgue integral of simple function

i. $\varphi(x) = Int(x)$ over $E = (0,10)$, (3 marks)

ii. $\varphi(x) = Int(x^2)$ over $E = (0,2)$, (4 marks)

where $Int(w)$ return the integer part of w.

c. Show that that the mapping $A \longmapsto P(A|B)$ is countably additive on , σ-algebra , $\mathcal{F}_B$.

(6 marks)

d. Let X_1, X_2 ... be i.i.d with mean μ and variance $\sigma.^2$ Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Prove that

$$Z = \frac{n^{\frac{1}{2}}(\bar{X}_n - \mu)}{\sigma} \to N(0,1)$$

in distribution as $n \to \infty$. (10 marks)

Question TWO (15 marks)

a) Prove that if $X_n \overset{L^2}{\rightarrow} X$ then $X_n \overset{p}{\rightarrow} X$. (5 marks)

b) The number of students X who are going to fail in the exam is a Poisson variable with mean 75,i.e, X~Poi(75). We admit that the exam was too hard if more than 90 student fail. What is the probability for it to happen? (4 marks)

c) Consider random variables X and Y from the probability space $(\Omega, \mathcal{F}, P)$, prove that

$E(E(Y|X))$ =E(Y). (6 marks)

Question FOUR (15 marks)

a. State and prove the Markov's inequality. Hence or otherwise, use your results to prove Chebyshev's inequality. (8 marks)

b. Let $(\Omega, \mathcal{F}, P)$ *be a probability space*. Show that for all sets $A_1, A_2, \ldots \in \mathcal{F}$ we have that:

$$P\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} P[A_i]$$

(7 marks)

Question FIVE (15 marks)

e. State and prove the Weak Law of Large Numbers (6 marks)

a. Find the Riemann and Lebesgue integral of a step function $f(x)$ defined as:

$$f(x) = \begin{cases} 1 \text{ if} -1 < x < 2 \\ 2 \text{ if} \quad 2 \leq x < 4 \\ 3 \text{ if} \quad 4 \leq x < 8 \\ 0 \quad \text{otherwise.} \end{cases}$$

(9 marks)