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DETERMINING THE RATIONAL HOMOTOPY TYPE OF
THE COMPONENT OF INCLUSION IN THE SPACE OF
CONTINUOUS MAPPINGS FROM $Gr(k, n)$ to $Gr(k, n + r)$

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Submitted in total fulfillment of the requirements for the degree
of Doctor of Philosophy in Pure Mathematics at Strathmore
University.

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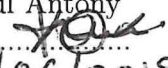
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Abstract

The complex Grassmann manifold $Gr(k, n)$ is the space of k dimensional subspaces of $\mathbb{C}^n$. For $k = 1$, one gets $\mathbb{C}P^{n-1}$, the space of lines in $\mathbb{C}^n$. There is a natural embedding $G(k, n) \hookrightarrow G(k, n + r)$. Moreover, any complex manifold can be embedded in some projective space $\mathbb{C}P^N$. In particular, there is an embedding $Gr(k, n) \hookrightarrow \mathbb{C}P^{N-1}$ where $N = \binom{n}{k}$.

To a simply connected topological space, Sullivan associates in a functional way a commutative differential graded algebra (*cdga*) henceforth $(\wedge V, d)$ which encodes the rational homotopy type of X . This is called a Sullivan model of X . Given that $H^*(\mathbb{C}P^n, \mathbb{Q})$ is the truncated polynomial algebra $\wedge x/(x^{n+1})$, one gets a Sullivan model of the form $(\wedge(x, y), d)$ where $|x| = 2$, $|y| = 2n + 1$ and $dx = 0$, $dy = x^{n+1}$. For $k \geq 1$, one might use the homeomorphism $G(k, n) = U(n)/(U(k) \times U(n - k))$ to find a Sullivan model.

Moreover, if $f : X \rightarrow Y$ is a continuous mapping between *CW*-complexes, then there is a commutative differential graded algebra (*cdga*) morphism $\phi : (\wedge V_Y, d) \rightarrow (\wedge V_X, d)$ between Sullivan models of X and Y . This is called a Sullivan model of f .

In this thesis, we use a Sullivan model of the inclusion $Gr(k, n) \xrightarrow{i} Gr(k, n + r)$ to compute the rational homotopy type of the component of the inclusion in the space of mappings from $Gr(k, n)$ to $Gr(k, n + r)$. Further, we will compute an L_∞ -model of the component of the inclusion i and deduce its Sullivan model, using the generalised cochain Quillen functor.

We seek to define a model both Sullivan and Quillen for the component of the inclusion and from it obtain the cohomology algebra and even attempt to determine whether the space is formal or not.

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To you all I say 'thanks', though a small word, in it lies much appreciation which forever will be etched in my memory.

Indeed, I have come to the conviction after a long meditation, that one of the endlessly alluring aspects of Pure Mathematics is that, its thorniest paradoxes have a way of blooming into beautiful theories (Philip. J. Davis).

Dedication

To mum, Jane Margaret Akinyi and dad, Wilfred Caleb Otieno.

Who paved the path before me and upon whose shoulders I stand.

Publications

The publications presented below are extracts from the thesis. The manuscripts are appended at the end .

1. Paul A. Otieno, J-B. Gatsinzi, Vitalis O. Otieno. (2019). *Rational homotopy of mapping spaces between complex Grassmannians*, to appear in Quaestiones Mathematicae (Accepted).
2. J-B Gatsinzi, V. Onyango-Otieno and P.A Otieno. (2019). *Relative Gottlieb groups of the Plucker embedding of some complex Grassmannians*, to appear in Communications of the Korean Mathematical society (Accepted).
3. Paul A. Otieno, J-B. Gatsinzi, Vitalis O. Otieno. (2019). *Rationalized evaluation subgroups of mapping spaces between complex Grassmannians*, to appear in Afrika Matematika (Under review).
4. Paul A. Otieno, J-B. Gatsinzi, Vitalis O. Otieno. (2019). *Rational cohomology algebra of mapping spaces between complex Grassmannians*, to appear in Iranian Journal of Mathematical Sciences and Informatics (Under review).

Chapter 1

Introduction

1.1 Sullivan and Quillen models

The idea of rational homotopy theory is to assume the torsion part of homotopy groups. Sullivan, using Postnikov towers, builds a *cdga* model, which computes both rational homotopy and cohomology of the space. Quillen, on the other hand builds a differential graded Lie algebra (*dgl*) model which computes the homotopy Lie algebra of the loop space on X and can recover the cohomology [6].

The two distinct approaches by Quillen and Sullivan directly reflect the original two means of computing homology and cohomology at the start of the twentieth century: simplicial homology of a polyhedron, and deRham cohomology of a manifold [6].

1.1.1 Sullivan models

A differential graded algebra (*dga*) is a graded algebra $A = \bigoplus_{n \geq 0} A^n$ together with a derivation d of degree $+1$. This means that for each n , there is a vector space homomorphism $d = d_n : A^n \rightarrow A^{n+1}$ such that $d \circ d = 0$ and $d(a \cdot b) = d(a) \cdot b + (-1)^n a \cdot d(b)$. Then (A, d) is called a cochain algebra. A graded algebra A is commutative if $a \cdot b = (-1)^{\deg a \cdot \deg b} b \cdot a$ for $a, b \in A$ [5, Chap.3].

A cochain algebra of the form $(\wedge V, d)$ in which $V = V^{\geq 0}$ satisfies the Sullivan condition if V is the union of an increasing family of subspaces:

$$V(0) \subset V(1) \subset \cdots \subset V(r) \subset \cdots$$

such that $d = 0$ in $V(0)$ and for $r \geq 1$, $dV(r) \subset \wedge V(r-1)$. A Sullivan algebra is a commutative cochain algebra of the form $(\wedge V, d)$ which satisfies

the Sullivan condition. A Sullivan model for a commutative cochain algebra (A, d) is a quasi isomorphism $m : (\wedge V, d) \rightarrow (A, d)$ from a Sullivan algebra $(\wedge V, d)$. A Sullivan algebra is said to be minimal if the differential is decomposable, that is $Im\,d \subset \wedge^+ V \cdot \wedge^+ V$. Moreover, if $H^0(A) = \mathbb{F}$, then (A, d) has a minimal model which is determined up to isomorphism. A space X is formal if there is a quasi isomorphism $(\wedge V, d) \rightarrow H^*(X, \mathbb{Q})$. Moreover, complex Grassmann manifolds are formal [3]. If X is a nilpotent space, and $A_{PL}(X)$ the commutative differential graded algebra (*cdga*) of piecewise linear forms on X , then a Sullivan model of X is a Sullivan model of $A_{PL}(X)$ [5, Chap.12].

In the following examples of Sullivan models, the subscript on a generator indicates its degree. A model of an odd sphere S^{2n+1} is $\varphi : (\wedge(x_{2n+1}), 0) \rightarrow A_{PL}(S^{2n+1})$, where $\varphi(x)$ is any representative of the unique cohomology generator of degree $2n+1$. On the other hand, a model of an even sphere S^{2n} is $\varphi : (\wedge(y_{2n}, z_{4n-1}), d) \rightarrow A_{PL}(S^{2n})$, where, $dz = y^2$ and $\varphi(y)$ represents the unique cohomology generator of degree $2n$. The complex projective space $(\mathbb{C}P^n)$ has a model given by $\varphi : (\wedge(u_2, x_{2n+1}), d) \rightarrow A_{PL}(\mathbb{C}P^n)$ where, $dx = u^{n+1}$ and $\varphi(u)$ represents the algebra generator of $H^*(\mathbb{C}P^n; \mathbb{Q})$ which is a truncated polynomial algebra on a generator of degree 2 [13].

1.1.2 Quillen model

A differential graded Lie algebra is a vector space (L, d) endowed with a bilinear map $[-, -] : L \otimes L \rightarrow L$ (called bracket) of degree zero such that the following are satisfied:

1. (Graded skew symmetry) $[a, b] = -(-1)^{deg(a)deg(b)}[b, a]$.
2. (Graded Jacobi identity) $[a, [b, c]] = [[a, b], c] + (-1)^{deg(a)deg(b)}[b, [a, c]]$.
3. Graded Leibniz rule $d[a, b] = [da, b] + (-1)^{deg(a)}[a, db]$.

The Leibniz rule implies in particular that the bracket of a differential graded Lie algebra say (L, d) induces a structure of graded Lie algebra on its cohomology $H^*(L) = \bigoplus_i H^i(L)$ [1, Pg.786].

An L_∞ -algebra structure on a graded vector space L often denoted by $(L, \{l_k\}_{k \geq 1})$ is a collection of linear maps called brackets, l_k for $k \geq 1$ of degree $k-2$.

$$l_k = [\cdot \cdots \cdot] : \otimes^k L \rightarrow L$$

such that the following conditions are satisfied.

1. l_k are graded skew symmetric i.e., for any k permutation σ

$$[x_{\sigma(1)}, \dots, x_{\sigma(k)}] = \text{sgn}(\sigma) \epsilon_{\sigma} [x_1, \dots, x_k],$$

where ϵ_{σ} is the sign given by the Koszul convention.

2. The generalised Jacobi identity holds i.e.

$$\sum_{i+j=n+1} \sum_{\sigma \in S(i, n-i)} \text{sgn}(\sigma) \epsilon_{\sigma} (-1)^{i(j-1)} [x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_{\sigma(i+1)}, \dots, x_{\sigma(n)}] = 0,$$

where $S(i, n-i)$ denotes the $(i, n-i)$ shuffles whose elements are permutations σ such that $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(n)$.

L_{∞} structures are in one to one correspondence with codifferentials on the non unital, free commutative co algebra $\Lambda^+ sL$ in which s denotes suspension i.e $(sL)_k = L_{k-1}$ [15].

Note that if L is a graded vector space of finite type, an L_{∞} structure on L induces a commutative differential graded algebra structure $C^{\infty}(L) = (\wedge (sL)^{\#}, d)$, $d = \sum_{j \geq 1} d_j$, where $d_j : V \rightarrow \wedge^j V$ is defined by $\langle d_j v; sx_1 \wedge \dots \wedge sx_j \rangle = (-1)^{\epsilon} \langle v; sl_j(x_1, \dots, x_j) \rangle$ and $\epsilon = (-1)^{|v| + \sum_{k=1}^{j-1} (j-k)|x_k|}$ [2]. This is a generalisation of the Quillen cochain functor to L_{∞} algebras [5, Chap.23].

An L_{∞} algebra $(L, \{l_k\}_{k \geq 1})$ is minimal if $l_1 = 0$. Further, an L_{∞} algebra is an L_{∞} model of a simply connected complex X , if it is a model of the differential graded vector space $\lambda(X)$, here λ denotes the Quillen functor that associates to any simply connected space X , a differential graded vector space L (free as a Lie algebra) $\lambda(X)$, which determines an equivalence between the homotopy categories of rational simply connected spaces and that of reduced ($L_{\leq 0} = 0$) differential graded vector spaces over the rationals. If X is simply connected and of finite type, and $\mathbb{L}$ denotes the Quillen model of $\lambda(X)$, then $C^{\infty}(\mathbb{L})$ is quasi isomorphic to the Sullivan model of X [2].

1.2 Grassmann manifolds

Let $U(n)$ denote the unitary group, then $Gr(k, n) = U(n) / U(k) \times U(n-k)$ [17]. It is a simply connected complex manifold of dimension $k(n-k)$ [12, Lecture 11, Pg. 138].

The cohomology ring $H^*(Gr(k, n), \mathbb{Q})$ of $Gr(k, n)$ as an algebra is generated

by the Chern classes $c_i \in H^{2i}(Gr(k, n), \mathbb{Q})$ for $1 \leq i \leq k$. Further, the cohomology ring has a presentation:

$$H^*(Gr(k, n), \mathbb{Q}) = \mathbb{Q}[c_1, c_2, \dots, c_k] / (h_{n-k+1}, \dots, h_n)$$

as the quotient of the polynomial ring generated by $c_1, c_2, \dots, c_k$, $|c_i| = 2i$, modulo the ideal generated by the elements h_j , $n - k + 1 \leq j \leq n$, here, h_j is defined as the $2j^{th}$ degree term in the Taylor's expansion of $(1 + c_1 + c_2 + c_3 + \dots + c_k)^{-1}$ where $(1 + c_1 + c_2 + c_3 + \dots + c_k)$ is the total Chern class [18].

The elements h_j , $(n - k + 1) \leq j \leq n$, form a regular sequence in the polynomial algebra $R_k = \mathbb{Q}[c_1, c_2, \dots, c_k]$ for any $n \geq 2n$, i.e., $h_{n-k+1} \neq 0$ and h_{n-k+r} is not a zero divisor in $R_k / \langle h_{n-k+1}, \dots, h_{n-k+r-1} \rangle$, $2 \leq r \leq k$. The natural embeddings $i : Gr(k, n) \hookrightarrow Gr(k, n + 1)$ and $j : Gr(k, n) \hookrightarrow Gr(k + 1, n + 1)$ induce surjections:

$$i^* : H^*(Gr(k, n + 1), \mathbb{Q}) \longrightarrow H^*(Gr(k, n), \mathbb{Q})$$

and

$$j^* : H^*(Gr(k + 1, n + 1), \mathbb{Q}) \longrightarrow H^*(Gr(k, n), \mathbb{Q})$$

where

$$i^*(x_r + \mathcal{I}_{n+1,k}) = x_r + \mathcal{I}_{n,k} = j^*(x_r + \mathcal{I}_{n+1,k+1})$$

for $1 \leq r \leq k$ and $j^*(x_{k+1} + \mathcal{I}_{n+1,k+1}) = 0$, with $\mathcal{I}_{n,k}$ being the ideal $\langle h_j : j > n - k \rangle$, generated by the elements h_j for $n - k + 1 \leq j \leq n$ [21].

We take an example of the minimal model of $Gr(2, 4)$ from its cohomology algebra over the rationals:

$$H^*(Gr(2, 4), \mathbb{Q}) = \mathbb{Q}(c_1, c_2) / (h_3, h_4) = \Lambda(x_2, x_4) / (h_3, h_4),$$

where h_3 and h_4 are the monomials of degree 6 and 8 respectively in the Taylor expansion of $(1 + x_2 + x_4)^{-1}$. This gives $h_3 = 2x_2x_4 - x_2^3$ and $h_4 = x_4^2 - 3x_2^2x_4 + x_2^4$. Hence,

$$H^*(Gr(2, 4); \mathbb{Q}) = \Lambda(x_2, x_4) / (2x_2x_4 - x_2^3, x_4^2 - 3x_2^2x_4 + x_2^4)$$

As $Gr(2, 4)$ is formal, a Sullivan model of $Gr(2, 4)$ is given by $(\Lambda(x_2, x_4, y_5, y_7, d))$, $dx_2 = dx_4 = 0$, $dy_5 = 2x_2x_4 - x_2^3$ and $dy_7 = x_4^2 - 3x_2^2x_4 + x_2^4$.

1.3 Statement of the problem

Function spaces are a critical component of homotopy theory whose research dates back to 1930's [23]. The introduction of localization techniques introduced a new depth of study while opening up new problems in the area. There is however no explicit and complete description of the homotopy type of the component of the inclusion $i : Gr(k, n) \hookrightarrow Gr(k, n + r)$ in the space of mappings from $Gr(2, n)$ to $Gr(2, n + r)$, $r \geq 1$. We begin our work with a model of the inclusion.

1.4 Objectives

1.4.1 General objective

To determine the rational homotopy type of the component of inclusion in the space of $i : Gr(k, n) \hookrightarrow Gr(k, n + r)$.

1.4.2 Specific objectives

1. To determine the rational homotopy type of the component of inclusion in the space of continuous functions from $Gr(2, n)$ to $Gr(2, n + r)$ using Sullivan model.
2. To determine the derivation spaces and the rationalized G -sequence of $map\left(Gr(2, n), Gr(2, n + r); i_{n, r}\right)$.
3. To determine the rational homotopy type of the component of inclusion in the space of continuous functions from $Gr(k, n)$ to $Gr(k, n + r)$ using the Sullivan model.

1.5 Significance of the research

Function spaces are at the foundations of homotopy theory and appear in the literature dating back, at least, to Hurewicz's definition of the homotopy groups in the 1930s [23]. Of particular interest to homotopy theorists are function spaces with current research having connections to mathematical Physics in the study of space of loops on the Lie group, closed geodesics problem for a compact Riemann manifold.

A more recent and active area dubbed homotopy type theory for big data

is another frontier opening up where it is possible that homotopy type will be a game changer with regards to automation and computation in the near future.

Chapter 2

Literature Review

2.1 Introduction

Function spaces have been objects of central interest to homotopy theorists, and their study has seen steady activity for over sixty years. Current research in this area is remarkably diverse with connections to other areas ranging from geometry to analysis to robotics. These spaces are at the foundations of homotopy theory and appear in the literature dating back, at least, to Hurewicz's definition of the homotopy groups in the 1930s. Work focusing explicitly on the homotopy theory of a function space first appears in the 1940s. In 1946, Whitehead classified the homotopy types represented by the path-components of a function space focusing on the case $\text{map}(S^2, S^2)$. It was later shown that $\pi_1(\text{map}(S^2, S^2; l_m)) \cong \mathbb{Z}/2|m|$ where l_m is the map of degree m thus distinguishing components of different degree.

Barely a decade later Thom [25] and Federer [4] appeared giving dual methods for computing homotopy groups of components of $\text{map}(X, Y)$. In the same period Gottlieb introduced and studied the evaluation subgroups of Gottlieb groups:

$$G_n(X) = \text{im} \{ \omega : \pi_n(\text{map}(X, X; 1)) \longrightarrow \pi_n(X) \} \subseteq \pi_n(X),$$

initiating a rich literature on the evaluation map [8].

In the 1970s Hansen began a systematic study of the homotopy classification problem for the path components of $\text{map}(X, Y)$ [11]. He completed the classification for $\text{map}(S^n, S^n)$. He and other authors obtained complete results in many special cases involving spheres, suspensions, projective spaces and certain manifolds [7]. The advent of localization techniques introduced new depth to the study of function spaces while opening up a wide range

of fundamental problems and applications. In his seminal paper on rational homotopy theory, D. Sullivan sketched a construction for an algebraic model for components of $\text{map}(X, Y)$ for X and Y simply connected CW -complexes with X finite, as an extension of Thom's ideas [24]. The study of the rational homotopy type of function spaces started with Thom in the case where the codomain is an Eilenberg-MacLane space [25]. Moreover, Sullivan's construction was completed by Haefliger [10].

The first description of a Sullivan model of function spaces is due to Haefliger [10]. Moreover, a model of function spaces between complex projective spaces was given by Møller and Raussen using the Postnikov tower [19].

Since then, the rational homotopy of mapping spaces has been extensively studied in two major settings, the Sullivan and Quillen models approach. Each of these approaches is hinged on replacing the classical homotopy category by that of commutative differential graded algebras, differential graded Lie algebras, respectively. The version of up to homotopy of the latter algebraic structure derives in the concept of L_∞ algebra introduced first in the context of deformation theory of algebraic structure [22].

Lupton and Smith show that the rational homotopy groups of $\text{map}(X, Y; f)$ can be expressed in terms of ϕ -derivations, where, ϕ is a Sullivan(or Quillen) model of f as detailed in Chapter 4.

Moreover, Buijs, Felix and Murillo give an explicit L_∞ model of $\text{map}(X, Y; f)$ We follow [2] for the following.

Let $\phi : (A, d_A) \rightarrow (B, d_B)$ be a morphism of cochain algebras. A ϕ -derivation θ of degree n is a linear map $A^* \rightarrow A^{*-n}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{n|a|}\phi(a)\theta(b)$. We denote by $\text{Der}_n(A, B, \phi)$ the vector space of all ϕ -derivations of degree n . Define a differential of chain complexes $\delta : \text{Der}_n(A, B, \phi) \rightarrow \text{Der}_{n-1}(A, B, \phi)$, by $\delta\theta = d_B\theta - (-1)^{|\theta|}\theta d_A$.

If $\phi : (\wedge V, d) \rightarrow (B, d)$ is a morphism of commutative differential graded algebras, then there is an isomorphism of vector spaces

$$\text{Der}(\wedge V, B, \phi) \xrightarrow{\cong} \text{Hom}(V, B).$$

Moreover, if $\{v_1, \dots, v_n\}$ is a basis for V we will denote by (v_i, b_i) the ϕ -derivation θ such that:

$$\begin{cases} \theta(v_i) = b_i, \\ \theta(v_j) = 0 \quad i \neq j, b_i \in B. \end{cases}$$

Consider positive derivations defined by:

$$\widetilde{Der}_i(\wedge V, B, \phi) = \begin{cases} Der_i(\wedge V, B, \phi) & \text{if } i \geq 2 \\ Z\left(Der_i(\wedge V, B, \phi)\right) & \text{if } i = 1. \end{cases}$$

If $(A, d) = (\wedge V, d)$ is a Sullivan algebra, then given $j \geq 2$ and $\varphi_1, \dots, \varphi_j \in \widetilde{Der}(\wedge V, B, \phi)$, of degrees $p_1, \dots, p_j$, we define their bracket of length j , $[\varphi_1, \dots, \varphi_j] \in \widetilde{Der}(\wedge V, B, \phi)$ by

$$[\varphi_1, \dots, \varphi_j](v) = (-1)^{p_1 + \dots + p_j - 1} \sum_{i_1, \dots, i_j} \left(\sum \epsilon \phi(v_1 \dots \widehat{v}_{i_1} \dots \widehat{v}_{i_j} \dots v_k) \varphi(v_{i_1}) \dots \varphi_j(v_{i_j}) \right),$$

where $dv = \sum v_1 \dots v_k$ and ϵ is the sign given by the Koszul convention. These operations may be *desuspended* to define a set of linear maps $\{l_j\}_{j \geq 1}$ each of degree $j - 2$ on $s^{-1}\widetilde{Der}(\wedge V, B, \phi)$ as follows :

For $j = 1$

$$l_1 : s^{-1}\widetilde{Der}(\wedge V, B, \phi) \longrightarrow s^{-1}\widetilde{Der}(\wedge V, B, \phi), \quad l_1(s^{-1}\theta) = -s^{-1}\delta\theta.$$

For $j = 2$ define

$$l_j : s^{-1}\widetilde{Der}(\wedge V, B, \phi) \otimes \dots \otimes s^{-1}\widetilde{Der}(\wedge V, B, \phi) \longrightarrow s^{-1}\widetilde{Der}(\wedge V, B, \phi)$$

by

$$l_j(s^{-1}\varphi_1, \dots, s^{-1}\varphi_j) = (-1)^\alpha s^{-1}[\varphi_1, \dots, \varphi_j]$$

and $\alpha = \sum_{n=1}^{j-1} (j - n) + |\varphi_n|$.

Further, let $f : X \longrightarrow Y$ be a continuous function between 1-connected spaces and $map(X, Y; f)$ the component of f in the space of continuous maps from X to Y . Let $\phi : (\wedge V, d) \longrightarrow B$ be a commutative differential graded algebra model of f . *Lemma 3.3* of [2] proves that $\left(s^{-1}\widetilde{Der}(\wedge V, B, \phi), \{l_j\}_{j \geq 1}\right)$ is an L_∞ model of $map(X, Y; f)$.

Chapter 3

Rational Homotopy Type of the component of Inclusions

3.1 Rational homotopy type of $Gr(2, 4) \hookrightarrow Gr(2, 5)$

3.1.1 The inclusion and its model

Recall that

$$H^*(Gr(2, 4), \mathbb{Q}) = \wedge(y_2, y_4) / (2y_2y_4 - y_2^3, y_2^4 - 3y_2^2y_4 + y_4^2).$$

Moreover,

$$H^*(Gr(2, 5), \mathbb{Q}) = \wedge(x_2, x_4) / (x_2^4 - 3x_2^2x_4 + x_4^2, 4x_2^3x_4 - 3x_2x_4^2 - x_2^5).$$

Hence the minimal Sullivan model of $Gr(2, 5)$ is given by $(\wedge(x_2, x_4, x_7, x_9), d)$ where $dx_2 = dx_4 = 0$, $dx_7 = x_2^4 - 3x_2^2x_4 + x_4^2$ and $dx_9 = 4x_2^3x_4 - 3x_2x_4^2 - x_2^5$.

The inclusion $i : Gr(2, 4) \hookrightarrow Gr(2, 5)$ has a model given by

$$\phi : \wedge((x_2, x_4, x_7, x_9), d) \longrightarrow \wedge(y_2, y_4) / (2y_2y_4 - y_2^3, y_2^4 - 3y_2^2y_4 + y_4^2),$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_7) = 0$ and $\phi(x_9) = 0$. Let $(\wedge V, d) = (\wedge(x_2, x_4, x_7, x_9), d)$ and $B = \wedge(y_2, y_4) / (2y_2y_4 - y_2^3, y_2^4 - 3y_2^2y_4 + y_4^2)$.

The vector space $Der(\wedge V, B; \phi)$ is generated by:

$$\begin{aligned} \alpha_9 &= (x_9, 1), \alpha_7 = (x_7, y_2), \alpha_5 = (x_9, y_4), \alpha'_5 = (x_9, y_2^2), \alpha_3 = (x_9, y_2y_4), \\ \alpha_1 &= (x_9, y_2^4 - 2y_2^2y_4), \beta_7 = (x_7, 1), \beta_5 = (x_7, y_2), \beta_3 = (x_7, y_4), \beta'_3 = \\ &= (x_7, y_2^2), \beta_1 = (x_7, y_2y_4), \gamma_4 = (x_4, 1), \gamma_2 = (x_4, y_2), \theta_2 = (x_2, 1). \end{aligned}$$

Computing the Lie brackets we get:

$$\begin{aligned}
[\theta_2, \theta_2] &= -96\alpha_3 + 24\beta_3' - 6\beta_3, \\
[\theta_2, \gamma_2] &= 18\alpha_3 - 6\beta_3', \\
[\theta_2, \gamma_4] &= 24\alpha_5' - 6\alpha_5 - 6\beta_5, \\
[\gamma_2, \gamma_2] &= -6\alpha_3 + 2\beta_3', \\
[\gamma_2, \gamma_4] &= -6\alpha_5' + 2\beta_5, \\
[\gamma_4, \gamma_4] &= 2\beta_7, \\
[\theta_2, \theta_2, \theta_2] &= 24\alpha_5 - 120\alpha_5' + 24\beta_5, \\
[\theta_2, \theta_2, \gamma_2] &= 24\alpha_5' - 6\beta_5, \\
[\theta_2, \theta_2, \gamma_4] &= 24\alpha_7 - 6\beta_7, \\
[\theta_2, \gamma_2, \gamma_2] &= -6\alpha_5', \\
[\theta_2, \gamma_4, \gamma_4] &= -6\alpha_9, \\
[\theta_2, \theta_2, \theta_2, \theta_2] &= -120\alpha_7 + 24\beta_7, \\
[\theta_2, \theta_2, \theta_2, \gamma_2] &= 24\alpha_7, \\
[\theta_2, \theta_2, \theta_2, \gamma_4] &= 24\alpha_9, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= -120\alpha_9.
\end{aligned}$$

and other brackets are zero.

We further obtain $\delta\theta_2 = -4\alpha_1 + 2\beta_1$, $\delta\gamma_2 = -\alpha_1 - 4\beta_1$ and $\delta\gamma_4 = -\alpha_3 + 4\beta_3'$ with $\delta\alpha_i = \delta\beta_j = 0$. If $L = s^{-1}Der(\wedge V, B, \phi)$, then letting $x_i = \alpha_i^\sharp$, $y_j = \beta_j^\sharp$, $w_j = \gamma_j^\sharp$ and $z_2 = \theta_2^\sharp$, we obtain

$$C^\infty(L) = \left(\wedge (x_1, y_1, w_2, z_2, x_3, y_3, y_3', w_4, x_5, x_5', y_5, x_7, y_7, x_9), d \right),$$

where

$$\begin{aligned}
d(x_1) &= -\frac{1}{4}z_2 - w_2, \\
d(y_1) &= \frac{1}{2}z_2 - \frac{1}{4}w_2, \\
d(x_3) &= w_4 - \frac{1}{96}z_2^2 + \frac{1}{18}w_2z_2 - \frac{1}{6}w_2^2, \\
d(y_3) &= \frac{1}{4}w_4 - \frac{1}{6}z_2^2, \\
d(y_3') &= \frac{1}{24}z_2^2 - \frac{1}{6}z_2w_2 + \frac{1}{2}w_2^2, \\
d(x_5) &= \frac{1}{24}z_2^3 - \frac{1}{6}z_2w_4, \\
d(x_5') &= -\frac{1}{120}z_2^3 + \frac{1}{24}z_2^2w_2 + \frac{1}{24}z_2w_4 - \frac{1}{6}z_2w_2^2 - \frac{1}{6}w_2w_4,
\end{aligned}$$

$$d(y_5) = \frac{1}{24}z_2^3 - \frac{1}{6}z_2^2w_2 - \frac{1}{6}z_2w_4 + \frac{1}{2}w_2w_4,$$

$$d(x_7) = \frac{1}{24}z_2^2w_4 + \frac{1}{24}z_2^3w_2 + \frac{1}{120}z_2^4,$$

$$d(y_7) = \frac{1}{24}z_2^4 - \frac{1}{6}z_2^2w_4 + \frac{1}{24}w_4^2,$$

$$d(x_9) = -\frac{1}{120}z_2^5 - \frac{1}{24}z_2^3w_4 - z_2w_4^2.$$

Note that the ideal generated by $\langle x_1, y_1, w_2, z_2 \rangle$ is acyclic, hence $C^\infty(L)$ is quasi-isomorphic to

$$\left(\wedge (x_3, y_3', x_5, x_5', y_5, x_7, y_7, x_9), 0 \right).$$

Hence, $\text{map}(Gr(2, 4), Gr(2, 5); i)$ has the rational homotopy type of $S^3 \times S^3 \times S^5 \times S^5 \times S^5 \times S^7 \times S^7 \times S^9$. We can hence generalise the result as follows:

Theorem 1. *Let $i_n : Gr(2, n) \hookrightarrow Gr(2, n+1)$ be the inclusion. The mapping space $\left(Gr(2, n), Gr(2, n+1); i_n \right)$ has the rational homotopy type of a product of odd spheres.*

Proof. Since $H^*(Gr(k, n), \mathbb{Q}) = \wedge(c_1, c_2, \dots, c_k) / (h_{n-k+1}, \dots, h_n)$, $|c_i| = 2i$, then it follows that:

$$H^*(Gr(2, n), \mathbb{Q}) = \wedge(x_2, x_4) / (h_{n-1}, h_n) \text{ and}$$

$$H^*(Gr(2, n+1), \mathbb{Q}) = \wedge(x_2, x_4) / (h_n, h_{n+1}).$$

Hence a model of $Gr(2, n+1)$ is given by $\left(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d \right)$ with $dx_2 = dx_4 = 0$, $dx_{2n-1} = h_n$ and $dx_{2n+1} = h_{n+1}$. Assume $n \geq 4$. The inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ hence has a model,

$$\phi : \wedge(x_2, x_4, x_{2n-1}, x_{2n+1}) \longrightarrow \wedge(y_2, y_4) / (h_{n-1}, h_n).$$

Moreover, $Gr(2, n)$ is a complex manifold of dimension $2(n-2)$, and a real dimension of $2(2(n-2)) = 4n-8$.

The Grassmann manifold $Gr(2, n)$ being a symplectic manifold has a symplectic class represented by $y_2 \in H^2(Gr(2, n), \mathbb{Q})$. Further, y_2^i is a non zero cohomology class for $1 \leq i \leq (2n-4)$.

Let

$$L = \text{Der} \left(\left(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d \right), \wedge(y_2, y_4) / (h_{n-1}, h_n); \phi \right).$$

The only even degree ϕ -derivations are $\theta_2 = (x_2, 1)$, $\gamma_2 = (x_4, y_2)$ and $\gamma_4 = (x_4, 1)$. Let $\alpha_1 = (x_{2n-1}, y_2^{n-1})$ and $\beta_1 = (x_{2n+1}, y_2^n)$. Note that $L_{\text{even}} = \langle \theta_2, \gamma_2, \gamma_4 \rangle$. Let

$$dx_{2n-1} = a_n x_2^n + a_{n-2} x_2^{n-2} x_4 + \cdots, a_n \neq 0 \quad \text{and}$$

$$dx_{2n+1} = b_n x_2^{n+1} + b_{n-2} x_2^{n-1} x_4 + \cdots, b_n \neq 0.$$

Then,

$$\delta\theta_2(x_{2n-1}) = a_n \cdot n \cdot y_2^{n-1} + b_{n-2} \cdot (n-2) \cdot y_4 y_2^{n-3} + \cdots$$

$$\delta\theta_2(x_{2n+1}) = b_n \cdot n \cdot y_2^n + b_{n-2} \cdot (n-1) \cdot y_4 y_2^{n-2} + \cdots$$

$$\delta\gamma_2(x_{2n-1}) = a_{n-2} \cdot (n-2) \cdot y_2 y_4^{n-3} + \cdots$$

$$\delta\gamma_2(x_{2n+1}) = b_{n-2} \cdot (n-1) \cdot y_2^{n-1} + \cdots$$

$$\delta\gamma_4(x_{2n-1}) = a_{n-2} \cdot (n-2) \cdot y_2^{n-3} + \cdots$$

$$\delta\gamma_4(x_{2n+1}) = b_{n-2} \cdot (n-1) \cdot y_2^{n-2} + \cdots$$

Hence,

$$\delta\theta_2 = n \cdot a_n \cdot \alpha_1 + n \cdot b_n \cdot \beta_1 + \cdots$$

$$\delta\gamma_2 = b_{n-2}(n-1)\alpha_1 + 0\beta_1 + \cdots$$

We deduce that $\delta : L_2 \rightarrow L_1$ is injective. Therefore, $H_2(L, \delta) = 0$ as $\delta\theta_2$ and $\delta\gamma_2$ are linearly independent. In the same way $H_4(L, \delta) = 0$. Moreover, all brackets involving odd dimension ϕ -derivations are zero. We conclude that $H_*(L)$ is concentrated in odd degrees with all brackets being zero.

Hence, $C^\infty(s^{-1}L)$ has a minimal model of the form $(\wedge X, 0)$, where $X = X^{\text{odd}}$. Therefore, $\text{map}(Gr(2, n), Gr(2, n+1); i_n)$ has the rational homotopy type of a product of odd spheres.

□

3.2 Rational homotopy type of $Gr(2, 4) \hookrightarrow Gr(2, 7)$

3.2.1 Introduction

Having established that for $r = 1$ and $\text{map}(Gr(2, n), Gr(2, n+r); i_{n,r})$ has the rational homotopy type of a product of odd spheres, we now set to vary r in the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+r)$, $r \geq 2$.

3.2.2 The inclusion $Gr(2, 4) \hookrightarrow Gr(2, 7)$ and its model

A model of the inclusion is:

$$\phi : \left(\wedge (x_2, x_4, x_{11}, x_{13}), d \right) \longrightarrow \wedge (y_2, y_4) / (h_3, h_4),$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$ and $\phi(x_{11}) = \phi(x_{13}) = 0$. Here, $dx_2 = dx_4 = 0$,

$$dx_{11} = -x_4^3 + 6x_2^2x_4^2 - 5x_2^4x_4 - x_2^6, \quad dx_{13} = 4x_2x_4^3 - 10x_2^3x_4^2 - x_2^7 + 6x_2^5x_4,$$

with $h_3 = 2y_2y_4 - y_2^3$ and $h_4 = y_4^2 - 3y_2^2y_4 + y_2^4$. The vector space $Der(\wedge V, B, \phi)$ is spanned by:

$$\alpha_{13} = (x_{13}, 1), \alpha_{11} = (x_{13}, y_2), \alpha_9 = (x_{13}, y_4), \alpha'_9 = (x_{13}, y_2^2), \alpha_7 = (x_{13}, y_2^3 = 2y_2y_4), \alpha_5 = (x_{13}, y_4^2 = 2y_2^2y_4 = y_2^4),$$

$$\beta_{11} = (x_{11}, 1), \beta_9 = (x_{11}, y_2), \beta_7 = (x_{11}, y_4), \beta'_7 = (x_{11}, y_2^2), \beta_5 = (x_{11}, y_2^3 = 2y_2y_4), \beta_3 = (x_{11}, y_4^2 = 2y_2^2y_4 = y_2^4),$$

$$\gamma_2 = (x_4, y_2), \gamma_4 = (x_4, 1),$$

$$\theta_2 = (x_2, 1)$$

Computing the brackets we get;

$$\begin{aligned} [\theta_2, \theta_2] &= 12\beta_3, \\ [\gamma_4, \gamma_4] &= -6\beta_7, \\ [\gamma_2, \gamma_2] &= 26\beta_3, \\ [\theta_2, \gamma_2] &= 12\beta_3, \\ [\theta_2, \gamma_4] &= 12\beta_5 - 2076\alpha_5, \end{aligned}$$

$$\begin{aligned} [\gamma_2, \gamma_4, \gamma_4] &= -6\beta_9, \\ [\gamma_4, \gamma_4, \gamma_4] &= -6\beta_{11}, \\ [\theta_2, \gamma_2, \gamma_2] &= 24\beta_5 + 168\alpha_5, \\ [\gamma_2, \gamma_2, \gamma_4] &= -2\beta'_7, \\ [\gamma_2, \gamma_2, \gamma_2] &= -6\beta_5, \\ [\theta_2, \theta_2, \gamma_2] &= -2076\alpha_5 - 132\beta_5, \\ [\theta_2, \theta_2, \gamma_4] &= 24\beta_7 + 24\beta'_7 - 2076\alpha_7, \\ [\theta_2, \theta_2, \theta_2] &= -72\beta_5 - 6030\alpha_5, \\ [\theta_2, \gamma_4, \gamma_4] &= 168\alpha_9 + 24\beta_9, \end{aligned}$$

$$\begin{aligned} [\theta_2, \theta_2, \theta_2, \gamma_2] &= -24\beta'_7 - 2076\alpha_7, \\ [\theta_2, \theta_2, \theta_2, \gamma_4] &= 168\alpha_9 - 2160\alpha'_9 - 144\beta_9, \\ [\theta_2, \theta_2, \gamma_2, \gamma_2] &= 24\beta'_7 + 168\alpha_7, \end{aligned}$$

$$\begin{aligned}[\theta_2, \theta_2, \gamma_4, \gamma_4] &= 24\beta_{11} + 168\alpha_{11}, \\ [\theta_2, \theta_2, \theta_2, \theta_2] &= -24\beta_7 - 6120\alpha_7,\end{aligned}$$

$$\begin{aligned}[\theta_2, \theta_2, \theta_2, \theta_2, \gamma_2] &= -144\beta_9 - 2160\alpha_9', \\ [\theta_2, \theta_2, \theta_2, \theta_2, \gamma_4] &= -144\beta_{11} + 2160\alpha_{11}, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= -2160\alpha_9 - 5040\alpha_9',\end{aligned}$$

$$\begin{aligned}[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= -5040\alpha_{11}, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \gamma_4] &= -2160\alpha_{13}, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \gamma_2] &= -2160\alpha_{11} \text{ and}\end{aligned}$$

$[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2] = -5040\alpha_{13}$, with the other brackets being zero. We further obtain $\delta\gamma_4 = -3\beta_3$ and $\delta\gamma_2 = \delta\theta_2 = 0$. If $L = \mathcal{D}er(\wedge V, B; \phi)$, and letting $x_i = \alpha_i^\sharp, y_j = \beta_j^\sharp, w_j = \gamma_j^\sharp$ and $z_2 = \theta_2^\sharp$, we get

$$C^\infty(s^{-1}L) = \left(\wedge (z_2, w_2, y_3, w_4, x_5, y_5, y_7, y_7', x_7, x_9, x_9', y_9, x_{11}, y_{11}, x_{13}), d \right)$$

where $d(z_2) = d(w_2) = 0$ and

$$\begin{aligned}d(y_3) &= -\frac{1}{6}w_2^2 + \frac{1}{24}w_2z_2 - \frac{1}{48}z_2^2 - \frac{1}{3}w_4 = z_4, \\ d(y_5) &= \frac{1}{24}z_2w_2^2 - \frac{1}{12}z_2^2w_2 - \frac{1}{12}z_2^3 - \frac{1}{12}z_2w_4 - \frac{1}{6}w_2^3, \\ d(x_5) &= -\frac{1}{2076}z_2w_4 + \frac{1}{168}z_2w_2^2 - \frac{1}{2076}z_2^2w_2 - \frac{1}{6030}z_2^3, \\ d(x_7) &= -\frac{1}{6120}z_2^4 - \frac{1}{2076}z_2^3w_2 - \frac{1}{2076}z_2^2w_4 + \frac{1}{168}z_2^2w_2^2, \\ d(y_7) &= \frac{1}{24}z_2^2w_4 - \frac{1}{6}w_4^2 - \frac{1}{24}z_2^4, \\ d(y_7') &= -\frac{1}{2}w_2^2w_4 + \frac{1}{24}z_2^2w_4 - \frac{1}{24}z_2^3w_2, \\ d(y_7) &= \frac{1}{24}z_2^2w_4 - \frac{1}{6}w_4^2 - \frac{1}{24}z_2^4, \\ d(y_9) &= -\frac{1}{6}w_2w_4^2 + \frac{1}{24}z_2w_4^2 - \frac{1}{144}z_2^3w_4 - \frac{1}{144}z_2^4w_2, \\ d(x_9) &= \frac{1}{168}z_2w_4^2 + \frac{1}{168}z_2^3w_4 - \frac{1}{2160}z_2^5, \\ d(x_9') &= -\frac{1}{2160}z_2^3w_4 - \frac{1}{2160}z_2^4w_2 - \frac{1}{5040}z_2^5, \\ d(y_{11}) &= \frac{1}{24}z_2^2w_4^2 - \frac{1}{6}w_4^3 - \frac{1}{144}z_2^4w_4, \\ d(x_{11}) &= \frac{1}{168}z_2^2w_4^2 + \frac{1}{2160}z_2^4w_4 - \frac{1}{5040}z_2^6 - \frac{1}{2160}z_2^5w_2, \\ d(x_{13}) &= -\frac{1}{5040}z_2^7 - \frac{1}{2160}z_2^5w_4,\end{aligned}$$

Let $z_4 = -\frac{1}{6}w_2^2 + \frac{1}{24}w_2z_2 - \frac{1}{24}z_2^2 - \frac{1}{3}w_4 \Leftrightarrow w_4 = -\frac{1}{16}z_2^2 + \frac{1}{8}w_2z_2 - \frac{1}{2}w_2^2 - 3z_4$.
Then the above Sullivan algebra is isomorphic to

$$\left(\wedge(y_3, z_4) \otimes \wedge(z_2, w_2, x_5, y_5, x_7, y_7, y_7', x_9, x_9'', y_9, x_{11}, y_{11}, x_{13}), d \right)$$

where $d(y_3) = d(z_4)$, $d(z_2) = d(w_2) = 0$

$$\begin{aligned} d(x_5) &= \frac{4531}{33382080}z_2^3 - \frac{7}{16608}w_2z_2^2 + \frac{15}{2422}z_2w_2^2 + \frac{1}{692}z_2z_4, \\ d(y_5) &= \frac{1}{576}z_2w_2^2 - \frac{1}{1152}w_2z_2^2 - \frac{23}{294}z_2^3 - \frac{1}{6}w_2^3 + \frac{1}{4}z_2z_4, \\ d(x_7) &= \frac{133019}{2117520}z_2^4 - \frac{1}{8}w_2z_2^3 + \frac{14719}{29064}z_2^2w_2^2 + \frac{1}{692}z_2^2z_4, \\ d(y_7) &= \frac{1}{768}z_2^4 - \frac{493}{79488}w_2z_2^3 + \frac{3}{16}z_2^2z_4 + 9z_4^2 - \frac{1}{24}z_2^4 + \frac{3}{4}w_2z_2z_4, \\ d(y_7') &= \frac{1}{96}w_2^2z_2^2 - \frac{1}{16}w_2^3z_2 + \frac{1}{4}w_2^4 + \frac{3}{2}w_2^2z_4 - \frac{1}{384}z_2^4 - \frac{7}{192}w_2z_2^3, \\ d(y_9) &= \frac{11861}{13824}z_2^4w_2 + \frac{1051}{3456}z_2^3w_2^2 + \frac{17}{92}z_2^2z_4w_2 + \frac{11}{18432}z_2^5, \\ d(x_9') &= \frac{421}{4440960}z_2^5 - \frac{1}{1920}w_2z_2^4 + \frac{1}{4320}z_2^3w_2^2 + \frac{1}{720}z_2^3z_4, \\ d(x_9) &= \frac{1571}{1935360}z_2^5 + \frac{373}{580608}w_2z_2^4 - \frac{9}{3584}z_2^3w_2^2 + \frac{1}{56}w_2^2z_4z_2, \\ d(x_{11}) &= \frac{79}{387072}z_2^6 - \frac{1471}{2903040}w_2z_2^5 + \frac{1}{56}w_2^2z_4z_2^2 + \frac{9}{168}z_2^2z_4^2, \\ d(y_{11}) &= -\frac{1}{4096}z_2^6 + \frac{43}{27648}w_2z_2^5 - \frac{3}{16}w_2^2z_2^2z_4 - \frac{1}{32}z_2^4z_4 - \frac{1}{64}z_2^2w_2^4, \\ d(x_{13}) &= -\frac{41}{241920}z_2^7 - \frac{1}{17280}z_2^6w_2 + \frac{1}{4320}z_2^5w_2^2 + \frac{1}{720}z_2^5z_4. \end{aligned}$$

As the ideal generated by y_3 and z_4 is acyclic, $C^\infty(s^{-1}L)$ is isomorphic to

$$\left(\wedge(z_2, w_2, y_5, x_5, y_7, y_7', x_7, x_9, x_9', y_9, x_{11}, y_{11}x_{13}), \bar{d} \right),$$

where $\bar{d}y_5 = \frac{1}{24}z_2w_2^2 - \frac{1}{12}z_2^2w_2 - \frac{1}{12}z_2^3 - \frac{1}{12}z_2w_4 - \frac{1}{6}w_2^3$.

3.3 Rational homotopy type of $Gr(2, 4) \hookrightarrow Gr(2, 8)$

A model of the inclusion is;

$$\phi : \left(\wedge V = \wedge(x_2, x_4, x_{13}, x_{15}), d \right) \longrightarrow \wedge(y_2, y_4) / (h_3, h_4) = B,$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$ and $\phi(x_{13}) = \phi(x_{15}) = 0$. Here

$$dx_{13} = 4x_2x_4^3 - 10x_2^3x_4^2 - x_2^7 + 6x_2^5x_4, \quad dx_{15} = x_4^4 - 10x_2^2x_4^3 + 15x_2^4x_4^2 - 7x_2^6x_4 + x_2^8,$$

with $h_3 = 2y_2y_4 - y_2^3$ and $h_4 = y_4^2 - 3y_2^2y_4 + y_2^4$.

$L = \text{Der}(\wedge V, B; \phi)$ is spanned by the following ϕ -derivations:

$$\alpha_{15} = (x_{15}, 1), \alpha_{13} = (x_{15}, y_2), \alpha_{11} = (x_{15}, y_4), \alpha'_{11} = (x_{15}, y_2^2), \alpha_9 = (x_{15}, y_2y_4), \alpha'_9 = (x_{15}, y_2^3), \alpha_7 = (x_{15}, y_2^4), \alpha'_7 = (x_{15}, y_2^2y_4), \alpha''_7 = (x_{15}, y_2^2y_4^2),$$

$$\beta_{13} = (x_{13}, 1), \beta_{11} = (x_{13}, y_2), \beta_9 = (x_{13}, y_4), \beta'_9 = (x_{13}, y_2^2), \beta_7 = (x_{13}, y_2y_4), \beta'_7 = (x_{13}, y_2^3), \beta_5 = (x_{13}, y_2^4), \beta'_5 = (x_{13}, y_2^2y_4), \beta''_5 = (x_{13}, y_2^2y_4^2),$$

$$\gamma_2 = (x_4, y_2), \gamma_4 = (x_4, 1) \text{ and } \theta_2 = (x_2, 1).$$

Computing the brackets, we get

$$\begin{aligned} [\theta_2, \gamma_4] &= 168\beta''_5 - 2160\beta_5, \\ [\gamma_4, \gamma_4] &= 168\beta'_7 + 24\alpha'_7 + 120\alpha''_7 + 1392\alpha_7, \\ [\gamma_2, \gamma_4] &= 168\beta_5, \\ [\theta_2, \theta_2, \theta_2] &= 168\beta'_5 - 2160\beta''_5 - 5040\beta_5, \\ [\theta_2, \gamma_4, \gamma_4] &= 168\beta_9 + 120\alpha_9 + 1392\alpha'_9, \\ [\theta_2, \gamma_2, \gamma_2] &= 168\beta_5, \\ [\theta_2, \gamma_4, \gamma_2] &= 168\beta'_7 + 1392\alpha_7 + 120\alpha''_7, \\ [\theta_2, \theta_2, \gamma_4] &= 168\beta_7 - 2160\beta'_7 + 120\alpha'_7 + 1392\alpha''_7 - 7920\alpha_7, \\ [\theta_2, \theta_2, \gamma_2] &= 168\beta''_5 - 2160\beta_5, \\ [\theta_2, \theta_2, \theta_2, \gamma_4] &= 168\beta_9 - 2160\beta'_9 + 1392\beta_9 - 7920\alpha'_9, \\ [\theta_2, \theta_2, \theta_2, \gamma_2] &= 168\beta_7 - 2160\beta'_7, \\ [\theta_2, \theta_2, \gamma_4, \gamma_4] &= 168\beta_{11} + 120\alpha_{11} + 1392\alpha'_{11}, \\ [\theta_2, \theta_2, \gamma_2, \gamma_2] &= 168\beta'_7 + 120\alpha''_7 + 1392\alpha_7, \\ [\theta_2, \theta_2, \gamma_2, \gamma_4] &= 168\beta'_9 + 120\alpha_9 + 1392\alpha'_9, \\ [\theta_2, \theta_2, \theta_2, \theta_2] &= -2160\beta_7 - 5040\beta'_7 + 1392\alpha'_7 - 7020\alpha''_7, \\ [\theta_2, \theta_2, \theta_2, \gamma_4, \gamma_4] &= 168\beta_{13} + 1392\alpha_{13}, \\ [\theta_2, \theta_2, \theta_2, \gamma_2, \gamma_2] &= 168\beta'_9 + 1392\alpha'_9, \\ [\theta_2, \theta_2, \theta_2, \gamma_2, \gamma_4] &= 168\beta_{11} + 1392\alpha''_{11}, \\ [\theta_2, \theta_2, \gamma_4, \gamma_4, \gamma_2] &= 120\alpha_{13}, \\ [\theta_2, \theta_2, \gamma_2, \gamma_2, \gamma_2] &= 120\alpha'_9, \\ [\theta_2, \theta_2, \gamma_2, \gamma_2, \gamma_4] &= 120\alpha_{11}, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= -2160\beta_9 - 5040\beta'_9 - 7920\alpha_9, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \gamma_2] &= -2160\beta'_9 + 1392\alpha'_9 - 7920\alpha'_9, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \gamma_4] &= -2160\beta_{11} + 1392\alpha'_{11} - 7920\alpha'_{11}, \end{aligned}$$

$$\begin{aligned}
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= -5040\beta_{11} - 7920\alpha_{11}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \gamma_2] &= -2160\beta_{11} - 7920\alpha'_{11}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \gamma_4] &= -2160\beta_{13} - 7920\alpha_{13}
\end{aligned}$$

and $[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2] = -5040\beta_{13}$ with $\delta\gamma_2 = \delta\gamma_4 = \delta\theta_2 = 0$ and the other brackets being zero as well. Note that $\delta\gamma_2 = \delta\gamma_4 = \delta\theta_2 = 0$ implies that $l_1 = 0$ hence for $k = 4$, the inclusion gives a minimal l_∞ algebra. Put $x_i = \alpha_i^\sharp$, $y_j = \beta_j^\sharp$, $w_j = \gamma_j^\sharp$ and $z_2 = \theta_2^\sharp$. Hence

$$C^\infty(s^{-1}L) = \left(\wedge (z_2, w_2, w_4, y_5, y_7, x_7, y_9, y'_9, x_9, y_{11}, x_{11}, x'_{11}, y_{13}, x_{13}, x_{15}), d \right),$$

where

$$\begin{aligned}
d(z_2) &= d(w_2) = 0, \\
d(y_5) &= -\frac{1}{2076}z_2w_4 - \frac{1}{5952}z_2^3 + \frac{1}{168}z_2w_2^2 - \frac{1}{2076}z_2^2w_2, \\
d(y_7) &= \frac{1}{168}w_4^2 + \frac{1}{168}z_2w_2w_4 - \frac{1}{2076}z_2^2w_4 + \frac{1}{2076}z_2^3w_2 + \frac{1}{168}z_2^2w_2^2 - \frac{1}{6120}z_2^4, \\
d(x_7) &= \frac{1}{1476}w_4^2 + \frac{1}{1452}z_2w_2w_4 - \frac{1}{7104}z_2^2w_4 - \frac{1}{7224}z_2^3w_2 + \frac{1}{1452}z_2^2w_2^2 - \frac{1}{2568}z_2^4, \\
d(y_9) &= \frac{1}{168}z_2^3w_4 + \frac{1}{168}z_2^3w_2^2 - \frac{1}{2160}z_2^5 - \frac{1}{2160}z_2^4w_2, \\
d(y'_9) &= \frac{1}{168}z_2w_4^2 - \frac{1}{2160}z_2^3w_4 + \frac{1}{168}z_2^2w_2w_4 - \frac{1}{5040}z_2^5, \\
d(x_9) &= \frac{1}{1452}z_2w_4^2 - \frac{1}{7224}z_2^3w_4 + \frac{1}{1452}z_2^2w_2w_4 + \frac{1}{1392}z_2^3w_2^2 + \frac{1}{120}z_2^2w_2^3 - \frac{1}{3960}z_2^5 - \frac{1}{7224}z_2^4w_2, \\
d(y_{11}) &= \frac{1}{168}z_2^2w_4^2 + \frac{1}{168}z_2^3w_2w_4 - \frac{1}{2160}z_2^4w_4 - \frac{1}{5040}z_2^6 - \frac{1}{2160}z_2^5w_2, \\
d(x_{11}) &= \frac{1}{120}z_2^2w_4^2 + \frac{1}{120}z_2^2w_2^2w_4 + \frac{1}{1392}z_2^4w_4, \\
d(x'_{11}) &= \frac{1}{1392}z_2^2w_4^2 + \frac{1}{1392}z_2^3w_4w_2 - \frac{1}{7920}z_2^4w_4, \\
d(y_{13}) &= -\frac{1}{2160}z_2^5w_4 - \frac{1}{5040}z_2^7 + \frac{1}{120}z_2^2w_4^2w_2, \\
d(x_{13}) &= \frac{1}{1392}z_2^3w_4^2 + \frac{1}{120}z_2^2w_4^2w_2 - \frac{1}{7920}z_2^5w_4, \\
d(x_{15}) &= \frac{1}{120}z_2^2w_4^3.
\end{aligned}$$

Observe that $[w_2]^i$ is a non zero cohomology class for $i = 0, 1, 2, \dots$. Hence, $H^*(C^\infty(s^{-1}L))$ contains a polynomial algebra. The observation of the last two examples lead us to the following results (Theorems 2 and 3).

Theorem 2. *The cohomology of map $(Gr(2, n) \hookrightarrow Gr(2, n+r), i_{n,r})$ contains a truncated algebra $\mathbb{Q}[x]/(x^{r-n+4})$, where $|x| = 2$.*

Proof. Recall that the cohomology rings of $Gr(2, n)$ and $Gr(2, n+r)$ are respectively:

$$B = H^*(Gr(2, n), \mathbb{Q}) = \wedge(x_2, x_4) / (h_{n-1}, h_n) \text{ and}$$

$$H^*(Gr(2, n+r), \mathbb{Q}) = \wedge(x_2, x_4) / (h_{n+r-1}, h_{n+r}).$$

The minimal model of $Gr(2, n+r)$ is

$(\wedge V, d) = (\wedge(x_2, x_4, x_{2n+2r-3}, x_{2n+2r-1}), d)$ with $d(x_2) = d(x_4) = 0$, $d(x_{2n+2r-3}) = h_{n+r-1}$, $d(x_{2n+2r-1}) = h_{n+r}$. A model of the inclusion is then given by

$$\phi : \wedge(x_2, x_4, x_{2n+2r-3}, x_{2n+2r-1}) \longrightarrow \wedge(y_2, y_4) / (h_{n-1}, h_n),$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$ and $\phi(x_{2n+2r-3}) = \phi(x_{2n+2r-1}) = 0$. The vector space $Der(\wedge V, B; \phi)$ is spanned by:

$$\begin{aligned} & \alpha_{2n+2r-1} = (x_{2n+2r-1}, 1), \alpha_{2n+2r-3} = (x_{2n+2r-1}, y_2), \alpha_{2n+2r-5} = (x_{2n+2r-1}, y_4), \\ & \dots, \alpha_{2n+2r-9} = (x_{2n+2r-1}, y_2^{2^{n-4}}), \beta_{2n+2r-3} = (x_{2n+2r-3}, 1), \beta_{2n+2r-5} = \\ & (x_{2n+2r-3}, y_2), \beta_{2n+2r-7} = (x_{2n+2r-3}, y_4), \dots, \beta_{2n+2r-11} = (x_{2n+2r-3}, y_2^{2^{n-4}}), \\ & \gamma_4 = (x_4, 1), \gamma_2 = (x_4, y_2) \text{ and } \theta_2 = (x_2, 1). \end{aligned}$$

As $p_1 = (\delta\gamma_4)(x_{2n+2r-3})$ and $p_2 = (\delta\gamma_4)(x_{2n+2r-1})$ are polynomials of degree at least $2n + 2r - 6$. If $r > n - 1$, then $2n + 2r - 6 > 4n - 8$. Therefore p_1 and p_2 are zero in $\wedge(x_2, x_4) / (h_{n-1}, h_n)$. Hence, $(\delta\gamma_4) = 0$. Similarly, $\delta\gamma_2 = \delta\theta_2 = 0$.

On the other hand $q_1 = l_k(\gamma_2, \dots, \gamma_2)(dx_{2n+2r-3})$ and $q_2 = l_k(\gamma_2, \dots, \gamma_2)(dx_{2n+2r-1})$ are polynomials of degree $\geq 2n + 2r - 2 - 2k$. If $k < r - n + 3$, then q_1 and q_2 are of degree $> 4n - 8$. Therefore, $l_k(\gamma_2, \dots, \gamma_2) = 0$ for $k = 2, \dots, r - n + 3$. In $C^\infty(L)$, put $\gamma_2^\# = w_2$. Then $[w_2^i] \neq 0$ for $i = 1, \dots, r - n + 3$. $\square$

Theorem 3. Consider the inclusion $i_{n,r} : Gr(2, n) \hookrightarrow Gr(2, n+r)$. If $r > 3n - 6$, then the cohomology of map $(Gr(2, n), Gr(2, n+r))$ contains a polynomial algebra over a generator of degree 2.

Proof. Let $\phi : (\wedge(x_2, x_4, x_{2n+2r-3}, x_{2n+2r-1}), d) \longrightarrow \wedge(y_2, y_4) / (h_{n-1}, h_n)$ be a model of $i_{n,r}$. Assume $n \geq 4$, in particular $r \geq 3n - 8$, then any odd derivation is of degree at least $(2n + 2r - 3) - (4n - 8) = 2r - 2n + 5 \geq 5$. Hence $L_3 = L_1 = 0$. Moreover, $\gamma_2 = (x_4, y_2)$ is automatically a cycle (as $L_1 = 0$) which is not a boundary (as $L_3 = 0$). It is sufficient to show that $l_m(\gamma_2, \dots, \gamma_2) = 0$ for $m \geq 2$, where l_m denotes the bracket length m . We consider the case when $n+r$ is even, the other case is dealt with in a

similar way.

For $m > \frac{n+r}{2}$, $l_m = 0$. Assume $2 \leq m \leq \frac{n+r}{2}$,

$$l_m(\gamma_2, \dots, \gamma_2)(x_{2n+2r-1}) = \sum_{i \geq m} \alpha_i y_4^{i-m} \cdot y_2^m y_2^{n+r-2i} \quad (3.1)$$

$$l_m(\gamma_2, \dots, \gamma_2)(x_{2n+2r-3}) = \sum_{i \geq m} \beta_i y_4^{i-m} \cdot y_2^m \dots y_2^{n+r-1-2i} \quad (3.2)$$

The right hand side of (3.1) and (3.2) are polynomials of total degree $\geq 2n + 2r - 2m - 2$. Assume now that $r > 3n - 6$:

$$\begin{aligned} \Rightarrow n + r - 2 &> 4n - 8 \\ \Rightarrow 2n + 2r - 2 - (n + r) &> 4n - 8 \\ \Rightarrow 2n + 2r - 2m - 2 &> 4n - 8 \end{aligned}$$

as $m \leq \frac{n+r}{2}$. Therefore, $l_m(\gamma_2, \dots, \gamma_2) = 0$ for all $m \geq 0$.

□

3.4 Inclusion of $Gr(3, 6)$ in $Gr(3, 7)$

Consider the inclusion $Gr(3, 6) \hookrightarrow Gr(3, 7)$ i.e, $k = 3, n = 6$ and $r = 1$. A Sullivan model of $Gr(3, 7)$ is $(\wedge(x_2, x_4, x_6, x_9, x_{11}, x_{13}), d)$ with $d(x_2) = d(x_4) = dx_6 = 0$ and

$$\begin{aligned} d(x_9) &= 4x_2^3 x_4 - 3x_2 x_4^2 - x_2^5 - 3x_2^2 x_6 + x_4 x_6, \\ d(x_{11}) &= -x_4^3 + 6x_2^2 x_4^2 - 5x_2^4 x_4 + x_2^6 + x_6^2 + 4x_2^3 x_6, \end{aligned}$$

and

$$d(x_{13}) = 4x_2 x_4^3 - 10x_2^3 x_4^2 - x_2^7 + 6x_2^5 x_4 - 3x_2 x_6^2 - 5x_2^4 x_6.$$

Furthermore, the cohomology ring of $Gr(3, 6)$ is given by

$$H^*(Gr(3, 6), \mathbb{Q}) = \wedge(y_2, y_4, y_6) / (h_4, h_5, h_6)$$

where

$$\begin{aligned} h_4 &= y_4^2 - 3y_2^2 y_4 + y_2^4 + 2y_2 y_6, \\ h_5 &= 4y_2^3 y_4 - 3y_2 y_4^2 - y_2^5 - 3y_2^2 y_6 + 2y_4 y_6 \end{aligned}$$

and

$$h_6 = -y_4^3 + 6y_2^2 y_4^2 - 5y_2^4 y_4 + y_2^6 + y_2^3 y_6 + y_2 y_4 y_6.$$

A model of the inclusion is hence given by

$$\phi : (\wedge V, d) = \left(\wedge(x_2, x_4, x_6, x_9, x_{11}, x_{13}), d \right) \longrightarrow B = \wedge(y_2, y_4, y_6) / (h_4, h_5, h_6)$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_6) = y_6$ and $\phi(x_9) = \phi(x_{11}) = \phi(x_{13}) = 0$. Define the following derivations

$$\alpha_5 = (x_{13}, y_4^2), \alpha_5' = (x_{13}, y_2^4), \alpha_5'' = (x_{13}, y_2^2 y_4), \alpha_3 = (x_{13}, y_2^5), \alpha_3' = (x_{13}, y_2^3 y_4), \alpha_3'' = (x_{13}, y_2^2 y_4^2), \alpha_1 = (x_{13}, y_2^6), \alpha_1' = (x_{13}, y_6^2), \alpha_1'' = (x_{13}, y_2^2 y_4^2),$$

$$\beta_5 = (x_{11}, y_6), \beta_5' = (x_{11}, y_2 y_4), \beta_5'' = (x_{11}, y_2^3), \beta_3 = (x_{11}, y_2^4), \beta_3' = (x_{11}, y_4^2), \beta_3'' = (x_{11}, y_2^2 y_4), \beta_1 = (x_{11}, y_2^5), \beta_1' = (x_{11}, y_2^3 y_4), \beta_1'' = (x_{11}, y_2 y_4^2),$$

$$\xi_5 = (x_9, y_4), \xi_5' = (x_9, y_2^2), \xi_3 = (x_9, y_6), \xi_3' = (x_9, y_2^3), \xi_3'' = (x_9, y_2 y_4), \xi_1 = (x_9, y_2^4), \xi_1' = (x_9, y_4^2), \xi_1'' = (x_9, y_2^2 y_4),$$

$$\theta_2 = (x_2, 1), \theta_4' = (x_4, 1), \gamma_2' = (x_6, y_2^2), \theta_2' = (x_4, y_2), \gamma_2 = (x_6, y_4), \omega_6 = (x_6, 1) \text{ and } \psi_4 = (x_6, y_2).$$

Moreover,

$$\begin{aligned} \delta\theta_2 &= 11\xi_1 + 9\xi_1' + 6\xi_1'' - 18\beta_1 + 8\beta_1' - 24\beta_1'' - 20\alpha_1 - 21\alpha_1' + 9\alpha_1'', \\ \delta\theta_2' &= -6\xi_1 - 2\xi_1' + 4\xi_1'' + 5\beta_1 - 6\beta_1' + 3\beta_1'' + 14\alpha_1 + 20\alpha_1' + 8\alpha_1'', \\ \delta\gamma_2 &= 0\xi_1 - 2\xi_1' + 3\xi_1'' - 2\beta_1 - 6\beta_1' - 2\beta_1'' + 11\alpha_1 + 11\alpha_1' + 11\alpha_1'', \\ \delta\gamma_2' &= 3\xi_1 + 0\xi_1' - 2\xi_1'' - 3\beta_1 - 3\beta_1' - 3\beta_1'' + 11\alpha_1 + 11\alpha_1' + 11\alpha_1'', \\ \delta\psi_4 &= 3\xi_3 + 0\xi_3' - 2\xi_3'' - 6\beta_3 - 2\beta_3' - 2\beta_3'' + 8\alpha_3 + 5\alpha_3' + 5\alpha_3'', \\ \delta\theta_4' &= -2\xi_3 - 4\xi_3' + 4\xi_3'' + 5\beta_3 - 3\beta_3' - 12\beta_3'' - 6\alpha_3 + 20\alpha_3' - 20\alpha_3'' \text{ and} \\ \delta\omega_6 &= -2\xi_5 + 3\xi_5' + 0\xi_5'' - 2\beta_5 + 0\beta_5' - 4\beta_5'' + 6\alpha_5 + 11\alpha_5' + 6\alpha_5''. \end{aligned}$$

We observe that $\delta : L_{2i} \longrightarrow L_{2i-1}$ is injective for $i = 1, 2$ and 3 as $\delta\theta_2, \delta\theta_2', \delta\gamma_2, \delta\gamma_2'$ are linearly independent, as are $\delta\psi_4$ and $\delta\theta_4'$. We therefore conclude that $H_*(L)$ is concentrated in odd degrees.

Hence, $C^\infty(L)$ is quasi-isomorphic to $(\wedge(X^{odd}), 0)$ and the inclusion $Gr(3, 6) \hookrightarrow Gr(3, 7)$ has the rational homotopy type of a product of odd spheres. We conjecture that $map(Gr(k, n), Gr(k, n+1); i_n)$ has the rational homotopy type of a product of odd spheres. We consider now the inclusion $Gr(3, n) \hookrightarrow Gr(3, n+r)$.

Theorem 4. *If $r > 2n - 4$, then the cohomology of $\text{map}\left(Gr(3, n), Gr(3, n + r); i_{n,r}\right)$ contains a truncated algebra $\mathbb{Q}[x] / (x^{r-2n+9})$, where $|x| = 2$.*

Proof. The cohomology classes of $Gr(3, n)$ and $Gr(3, n + r)$ are

$$H^*\left(Gr(3, n), \mathbb{Q}\right) = \wedge(x_2, x_4, x_6) / (h_{n-2}, h_{n-1}, h_n) \text{ and}$$

$$H^*\left(Gr(3, n + r), \mathbb{Q}\right) = \wedge(x_2, x_4, x_6) (h_{n+r-2}, h_{n+r-1}, h_{n+r}).$$

The minimal model of $Gr(3, n + r)$ is

$(\wedge V, d) = \left(\wedge(x_2, x_4, x_6, x_{2n+2r-5}, x_{2n+2r-3}, x_{2n+2r-1}, x_{2n+2r-1}), d\right)$ with $d(x_2) = d(x_4) = d(x_6) = 0$, $d(x_{2n+2r-5}) = h_{n+r-2}$, $d(x_{2n+2r-3}) = h_{n+r-1}$ and $d(x_{2n+2r-1}) = h_{n+r}$. A model of the inclusion is given by

$$\begin{aligned} \phi : \wedge V &= \left(\wedge(x_2, x_4, x_6, x_{2n+2r-5}, x_{2n+2r-3}, x_{2n+2r-1}), d\right) \\ &\longrightarrow B = \wedge(y_2, y_4, y_6) / (h_{n-2}, h_{n-1}, h_n). \end{aligned}$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_6) = y_6$ and $\phi(x_{2n+2r-5}) = \phi(x_{2n+2r-3}) = \phi(2n + 2r - 1) = 0$. Let $L = \text{Der}(\wedge V, B; \phi)$. The only even degree ϕ -derivations are $\theta_2 = (x_2, 1)$, $\theta'_2 = (x_4, y_2)$, $\gamma_2 = (x_6, y_4)$, $\gamma'_2 = (x_6, y_2^2)$, $\psi_4 = (x_6, y_2)$, $\theta'_4 = (x_4, 1)$ and $\omega_6 = (x_6, 1)$.

As $p_1 = (\delta\omega_6)(x_{2n+2r-5})$, $p_2 = (\delta\omega_6)(x_{2n+2r-3})$ and $p_3 = (\delta\omega_6)(x_{2n+2r-1})$ are polynomials of degree at least $2n + 2r - 10$. If $r > 2n - 4$, then $2n + 2r - 10 > 6n - 18$. Therefore, p_1 , p_2 and p_3 are zero in B . Hence all even degree derivations are zero.

On the other hand,

$q_1 = l_k(\gamma_2, \dots, \gamma_2)(d_{2n+2r-5})$, $q_2 = l_k(\gamma_2, \dots, \gamma_2)(dx_{2n+2r-3})$ and $q_3 = l_k(\gamma_2, \dots, \gamma_2)(dx_{2n+2r-1})$ are polynomials of degree at least $2n + 2r - 2k - 4$. If $k < r - 2n + 7$, then q_1 , q_2 and q_3 will be of degree $> 6n - 18$. Therefore, $l_k(\gamma_2, \dots, \gamma_2) = 0$ for $k = 2, \dots, r - 2n + 7$. Hence in $C^\infty(L)$, $\gamma_2^\# = \rho_2$. Then $[\rho_2^i] \neq 0$ for $i = 1, \dots, r - 2n + 7$. $\square$

Theorem 5. *Consider the inclusion $i_{n,r} : Gr(3, n) \hookrightarrow Gr(3, n + r)$. If $r > 5n - 14$, then the cohomology of $\text{map}\left(Gr(3, n), Gr(3, n + r); i_{n,r}\right)$ contains a polynomial algebra over a generator of degree 2.*

Proof. Let

$$\begin{aligned} \phi : \left(\wedge(x_2, x_4, x_6, x_{2n+2r-5}, x_{2n+2r-3}, x_{2n+2r-1}), d\right) \\ \longrightarrow \wedge(y_2, y_4, y_6) / (h_{n-2}, h_{n-1}, h_n), \end{aligned}$$

be a model of the inclusion. Assume $n \geq 6$, in particular $r \geq 5n-18$, then any odd derivation is of degree at least $2n+2r-5-(6n-18) = 2r-4n+13 \geq 13$. Hence, $L_{11} = L_9 = L_7 = \cdots = L_1 = 0$.

Let $\gamma_2 = (x_4, y_2)$. It is sufficient to show that $l_m(\gamma_2, \gamma_2, \cdots, \gamma_2) = 0$ for $m \geq 2$, where l_m denotes bracket length m .

Consider the case when $n+r$ is even, the other case is dealt with in a similar way. For $m > \frac{n+r}{2}$, $l_m = 0$. Assume $2 \leq m \leq \frac{n+r}{2}$,

$$l_m(\gamma_2, \gamma_2, \cdots, \gamma_2)(x_{2n+2r-5}) = r$$

$$l_m(\gamma_2, \gamma_2, \cdots, \gamma_2)(x_{2n+2r-3}) = s \text{ and}$$

$$l_m(\gamma_2, \gamma_2, \cdots, \gamma_2)(x_{2n+2r-1}) = t.$$

The polynomials r, s and t are of total degree at least $2n + 2r - 2m - 4$. Assume now that $r > 5n - 14$, then $2n + 2r - 2m - 4 > 6n - 14$. Therefore, $l_m(\gamma_2, \gamma_2, \cdots, \gamma_2) = 0$ for all $m \geq 0$. Hence $H^*(C^\infty(sL))$ contains a polynomial algebra over a generator of degree 2. $\square$

Chapter 4

Gottlieb Groups of Function Spaces

4.1 Introduction

Given a pointed topological space (X, x_0) , the n^{th} Gottlieb group of X , also called the evaluation subgroup of $\pi_n(X)$ and denoted by $G_n(X)$, consists of those $\alpha \in \pi_n(X)$ for which there is a map $F : X \times S^n \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times S^n & \xrightarrow{F} & X \\ J \uparrow & & \nabla \uparrow \\ X \vee S^n & \xrightarrow{1_X \vee f} & X \vee X, \end{array}$$

where $f : S^n \rightarrow X$ is a representative of α and ∇ is the folding map. Thus for every $\alpha \in G_n(X, x_0)$, there exists at least one map $F : X \times S^n \rightarrow X$ such that $F(x_0, s) = f(s)$. We say that F is an affiliated map to α [9]. If X has a base point x_0 and $aut X$ denotes the monoid of self homotopy equivalences of X with $ev : aut X \rightarrow X$ the evaluation map at x_0 , then it follows from the definition that

$$G_n(X) = im\left(ev_{\sharp} : \pi_n(aut X, 1_X) \rightarrow \pi_n(X, x_0)\right).$$

Similarly, if $f : X \rightarrow Y$ is a based map between simply connected CW complexes and $map(X, Y; f)$, the space of maps from X to Y which are homotopic to f , then $G_n(Y, X; f) = im\left(ev_{\sharp} : map(X, Y; f) \rightarrow \pi_n(Y)\right)$ is the n^{th} evaluation subgroup of f [26]. In [27] Woo and Lee defined relative evaluation subgroups $G_n^{rel}(X, Y; f)$ and showed that they fit in a sequence

$$\cdots \rightarrow G_{n+1}^{rel}(Y, X; f) \rightarrow G_n(X) \rightarrow G_n(Y, X; f) \rightarrow \cdots$$

called the G -sequence of f .

Let $\phi : (A, d) \rightarrow (B, d)$ be a morphism between $cdga$'s and consider the vector space $Der_*(A, B; \phi)$ of ϕ -derivations.

Pre-composition with ϕ , respectively post-composition with the augmentation $\varepsilon : B \rightarrow \mathbb{Q}$, gives a map of chain complexes $\phi^* : Der_*(B, B; 1) \rightarrow Der_*(A, B; \phi)$, respectively $\varepsilon_* : Der_*(A, B; \phi) \rightarrow Der_*(A, \mathbb{Q}; \varepsilon)$.

Let $f : C \rightarrow D$ be a map of differential graded vector spaces. Define a commutative differential graded vector space, called the mapping cone and denoted by $Rel_*(\phi)$, as follows $Rel_n(\phi) = C_{n-1} \oplus D_n$ with differential $\delta (= \delta_\phi)$ of degree -1 given by $\delta(a, b) = (-d_A(a), \phi(a) + d_D(b))$. Further, define chain maps $J : D_n \rightarrow Rel_n(\phi)$ and $P : Rel_n(\phi) \rightarrow A_{n-1}$ by $J(b) = (0, b)$ and $P(a, b) = a$. These give short exact sequence of chain complexes

$$0 \longrightarrow D_* \xrightarrow{J} Rel_*(f) \xrightarrow{P} C_{*-1} \longrightarrow 0.$$

Which leads to a long exact sequence in homology

$$\cdots \rightarrow H_{n+1}(Rel(\phi)) \xrightarrow{H(P)} H_n(C) \xrightarrow{H(\phi)} H_n(D) \xrightarrow{H(J)} H_n(Rel(\phi)) \rightarrow \cdots$$

Whose connecting homomorphism is $H(\phi)$. We refer to this sequence as the long exact homology sequence of ϕ .

Let $f : X \rightarrow Y$ be a map between simply connected CW -complexes with X finite and $\phi : (\wedge V, d) \rightarrow (\wedge W, d)$ a Sullivan model of f . The long exact sequence induced by $f_* : map(X, X; 1) \rightarrow map(X, Y; 1)$ on rational homotopy groups is equivalent to the long exact homology sequence of the map

$$\phi^* : Der(\wedge W, \wedge W; 1) \rightarrow Der(\wedge V, \wedge W; \phi)[16].$$

Suppose given a commutative differential graded algebra map $\phi : A \rightarrow B$, from this map, the following commutative diagram can be formed;

$$\begin{array}{ccc} Der_*(B, B; 1) & \xrightarrow{\phi^*} & Der_*(A, B; \phi) \\ \downarrow \varepsilon_* & & \downarrow \varepsilon_* \\ Der_*(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi^*}} & Der_*(A, \mathbb{Q}; \varepsilon). \end{array}$$

Here, ε denotes the augmentation of either A or B . On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder ($n \geq 2$):

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\phi^*)) & \xrightarrow{H(P)} & H_n(\text{Der}(B, B; 1)) & \xrightarrow{H(\phi^*)} & H_n(\text{Der}(A, B; \phi)) \cdots \\
& & \downarrow H(\varepsilon_* \varepsilon_*) & & \downarrow H(\varepsilon_*) & & \downarrow H(\varepsilon_*) \\
\cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\widehat{\phi^*})) & \xrightarrow{H(\widehat{P})} & H_n(\text{Der}(B, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\widehat{\phi^*})} & H_n(\text{Der}(A, \mathbb{Q}; \varepsilon)) \cdots
\end{array}$$

Suppose $\phi : A \rightarrow B$ is a map of commutative differential graded algebras, we define the evaluation subgroups of ϕ by;

$$G_n(A, B; \phi) = \text{im}\{H(\varepsilon_*) : H_n(\text{Der}(A, B; \phi)) \rightarrow H_n(\text{Der}(A, \mathbb{Q}; \varepsilon))\}.$$

In the special case $A = B$ and $\phi = 1_B$, we refer to the Gottlieb group of B , and use the notation $G_n(B)$. The n^{th} relative evaluation subgroup of ϕ is defined by;

$$G_n^{\text{rel}}(A, B; \phi) = \text{im}\{H(\varepsilon_*, \varepsilon_*) : H_n(\text{Rel}(\phi^*)) \rightarrow H_n(\text{Rel}(\widehat{\phi^*}))\}$$

Moreover, $G_n(B) \cong G_n(Y) \otimes \mathbb{Q}$ and $G_n(A, B; \phi) \cong G_n(Y, X; f) \otimes \mathbb{Q}$ [16], and $G_*^{\text{rel}}(Y, X; \phi)$ is the relative Gottlieb group of the mapping cylinder of $f : X \rightarrow Y$ [27]. Then the image of the upper long exact sequence in the lower, of the ladder above, gives a (not necessarily exact) sequence

$$\cdots \xrightarrow{H(\widehat{J})} G_{n+1}^{\text{rel}}(A, B; \phi) \xrightarrow{H(\widehat{P})} G_n(B) \xrightarrow{H(\widehat{\phi^*})} G_n(A, B; \phi) \xrightarrow{H(\widehat{J})} \cdots$$

that we terminate in $G_2(A, B; \phi)$. We refer to this sequence as the G -sequence of the map $\phi : A \rightarrow B$.

4.2 The inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$

Recal that the cohomology rings of $Gr(2, n)$ and $Gr(2, n+1)$ are:

$$H^*(Gr(2, n), \mathbb{Q}) = \wedge(y_2, y_4) / (h_{n-1}, h_n) \text{ and}$$

$$H^*(Gr(2, n+1), \mathbb{Q}) = \wedge(x_2, x_4) / (h_n, h_{n+1}) \text{ respectively.}$$

The minimal Sullivan model of $Gr(2, n)$ is

$\left(\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d \right)$ with $d(y_2) = d(y_4) = 0$, $d(y_{2n-3}) = h_{n-1}$, $d(y_{2n-1}) = h_n$. In the same way, a minimal Sullivan model of $Gr(2, n+1)$ is given by $\left(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d \right)$ with $dx_2 = dx_4 = 0$, $dx_{2n-1} = h_n$ and $dx_{2n+1} = h_{n+1}$.

Lemma 1. $h_{n+1} = -x_2h_n - x_4h_{n-1}$.

Proof. Write the Taylor series $(1 + x_2 + x_4)^{-1} = 1 + h_1 + h_2 + \dots$ where, $|h_i| = 2i$. From $(1 + x_2 + x_4)(1 + x_2 + x_4)^{-1} = 1$, one gets the relation $h_{n+1} = -x_2h_n - x_4h_{n-1}$ $\square$

In particular, h_{n+1} is co-boundary in $\left(\wedge(x_2, x_4, x_{2n-3}, x_{2n-1}), d \right)$, that is, there exists α of degree $2n+1$ such that $d\alpha = h_{n+1}$.

Theorem 6. Let $B = \left(\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d \right)$. Then $G_n(B) = \langle [y_{2n-3}^*], [y_{2n-1}^*] \rangle$.

Proof. Let $\alpha_{2n-1} = (y_{2n-1}, 1)$ and $\alpha_{2n-3} = (y_{2n-3}, 1)$. Then $\delta\alpha_{n-1} = \delta\alpha_{2n-3} = 0$. Moreover, α_{2n-3} and α_{2n-1} can not be boundaries for degree reason. Therefore, $[\alpha_{2n-3}]$ and $[\alpha_{2n-1}]$ are non zero homology classes in $H_*(Der(B, B; 1))$. Further, $\varepsilon_*(\alpha_{2n-3}) = y_{2n-3}^*$ and $\varepsilon_*(\alpha_{2n-1}) = y_{2n-1}^*$.

As $Gr(2, n)$ is a finite CW-complex then $G_{even}(B) = 0$ [5, Pg.379]. Hence, $G_n(B) = \langle [y_{2n-3}^*], [y_{2n-1}^*] \rangle$. $\square$

The inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ has a model of the form

$$\phi : \wedge V = \left(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d \right) \longrightarrow \left(\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d \right) = B,$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_{2n-1}) = y_{2n-1}$ and $\phi(x_{2n+1}) = \alpha$, where $d\alpha = h_{n+1}$ by Lemma 1.

Theorem 7. Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ and $\phi : (\wedge V, d) \longrightarrow (B, d)$ its Sullivan model, then $G_*(\wedge V, B; \phi) \cong \langle [x_{2n-1}^*], [x_{2n+1}^*] \rangle$.

Proof. As $Gr(2, n)$ is formal, $Der(\wedge V, B; \phi) \xrightarrow{\cong} Der(\wedge V, H(B); f \circ \phi)$, where $f : B \xrightarrow{\cong} H(B)$ is a quasi isomorphism. Similarly, since B is formal $Der(B, B; 1) \simeq Der(B, H(B); f)$. Define $\theta_{2n-1} = (x_{2n-1}, 1)$, $\theta_{2n+1} = (x_{2n+1}, 1)$ in $Der(\wedge V, H(B); f \circ \phi)$. Then $\delta\theta_{2n-1} = \delta\theta_{2n+1} = 0$. Moreover, $[\theta_{2n-1}]$ and $[\theta_{2n+1}]$ are nonzero co-homology classes in $H_*(Der(\wedge V, H^*(B); f \circ \phi))$.

We note that, $\theta_2 = (x_2, 1)$ and $\theta_4 = (x_4, 1)$ are not cycles in

$$Der(\wedge V, H(B); f \circ \phi) \quad [20].$$

Further, $H(\varepsilon_*)([\theta_{2n-1}]) = [x_{2n-1}^*] \in G_{2n-1}(\wedge V, H(B); f \circ \phi)$. In a similar way, $H(\varepsilon_*)([\theta_{2n+1}]) = [x_{2n+1}^*] \in G_{2n+1}(\wedge V, H(B); f \circ \phi)$. It then follows that $G_*(\wedge V, H(B); f \circ \phi) = \langle [x_{2n-1}^*], [x_{2n+1}^*] \rangle$. $\square$

We can now compute the relative Gottlieb group of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$.

Theorem 8. *Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ and*

$$\begin{aligned} \phi : (\wedge V, d) = \left(\wedge (x_2, x_4, x_{2n-1}, x_{2n+1}), d \right) \longrightarrow \\ \left(\wedge (y_2, y_4, y_{2n-3}, y_{2n-1}), d \right) = B \end{aligned}$$

its Sullivan model, then $G_^{rel}(\wedge V, B; \phi) = \langle [(x_{2n-3}^*, 0)], [(0, y_{2n+1}^*)] \rangle$.*

Proof. Consider the diagram below [16].

$$\begin{array}{ccccc} Der(B, H(B); f) & \xrightarrow{\phi^*} & Der(\wedge V, H(B); f \circ \phi) & \xrightarrow{J} & Rel(\phi^*) \\ \downarrow \varepsilon_* & & \downarrow \varepsilon_* & & \downarrow (\varepsilon_*, \varepsilon_*) \\ Der(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi^*}} & Der(\wedge V, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{J}} & Rel(\widehat{\phi^*}). \end{array}$$

Let $\alpha_{2n-1} = (y_{2n-1}, 1)$, $\alpha_{2n-3} = (y_{2n-3}, 1) \in Der(B, H(B); f)$ and $\theta_{2n-1}, \theta_{2n+1} \in Der(\wedge V, H^*(B); \phi)$ as defined above. Then $\phi^*(\alpha_{2n-1}) = \theta_{2n-1}$ and $\phi^*(\alpha_{2n-3}) = 0$.

Further, $D(s\alpha_{2n-1}, 0) = (0, \theta_{2n-1})$, $D(s\alpha_{2n-3}, 0) = (0, 0)$ and $D(0, \theta_{2n-1}, 0) = 0 = D(0, \theta_{2n+1})$. Therefore, $[(\alpha_{2n-3}, 0)]$ and $[(0, \theta_{2n+1})]$ are non zero homology classes in $H_*(Rel(\phi^*))$. Moreover, $H(\varepsilon_*, \varepsilon_*)[(\alpha_{2n-3}, 0)] = [(x_{2n-3}^*, 0)]$ and $H(\varepsilon_*, \varepsilon_*)[(0, \theta_{2n+1})] = [(0, y_{2n+1}^*)]$. A straightforward computation shows that $[(x_{2n-3}^*, 0)]$ and $[(0, y_{2n+1}^*)]$ span $H(\varepsilon_*, \varepsilon_*)$. $\square$

The G -sequence reduces to

$$\begin{aligned} 0 &\longrightarrow G_{2n+1}(\wedge V, B; \phi) \xrightarrow{H(J)} G_{2n+1}^{rel}(\wedge V, B; \phi) \longrightarrow 0 \cdots \\ \cdots 0 &\longrightarrow G_{2n-1}(B) \xrightarrow{H(\phi^*)} G_{2n-1}(\wedge V, B; \phi) \longrightarrow 0 \cdots \\ \cdots 0 &\longrightarrow G_{2n-2}^{rel}(\wedge V, B; \phi) \xrightarrow{H(P)} G_{2n-3}(B) \longrightarrow 0. \end{aligned}$$

and is exact.

Example 1. Consider $Gr(2, 4) \hookrightarrow Gr(2, 5)$. A model of the inclusion is given by

$$\phi : \wedge V = \left(\wedge (x_2, x_4, x_7, x_9), d \right) \longrightarrow \left(\wedge (y_2, y_4, y_5, y_7), d \right) = B,$$

where $dx_2 = dx_4 = 0, dx_7 = x_4^2 - 3x_2^2x_4 + x_2^4, dx_9 = 4x_2^3x_4 - 3x_2x_4^2 - x_2^5$
 $dy_2 = dy_4 = 0, dy_5 = 2y_2y_4 - y_2^3$ and $dy_7 = y_4^2 - 3y_2^2y_4 + y_2^4$.

Moreover, $\phi(x_2) = y_2, \phi(x_4) = y_4, \phi(x_7) = y_7$ and $\phi(x_9) = -y_2y_7 - y_4y_4$.

We compute $G_*^{rel}(\wedge V, B; \phi)$. Let $\alpha_7 = (y_7, 1), \alpha_5 = (y_5, 1) \in Der_7(B, H(B); f)$ where $f : B \xrightarrow{\cong} H(B)$ and $\theta_7 = (x_7, 1), \theta_9 = (x_9, 1) \in Der(\wedge V, H^*(B); f \circ \phi)$ then $\phi^*(\alpha_7) = \theta_7$ and $\phi^*(\alpha_5) = 0$. Moreover, $D(s\alpha_7, 0) = (0, \theta_7), D(s\alpha_5, 0) = (0, 0)$ and $D(0, \theta_7) = D(0, \theta_9) = (0, 0)$. Hence $[(s\alpha_5, 0)]$ and $[(0, \theta_9)]$ are non zero homology classes. Moreover, $(\epsilon_*, \epsilon_*)(s\alpha_5, 0) = (sy_5^*, 0), (\epsilon_*, \epsilon_*)(0, \theta_9) = (0, y_9^*)$. Therefore

$$G_*^{rel}(\wedge V, B; \phi) = \langle [(0, x_9^*)], [(sy_5^*, 0)] \rangle.$$

4.3 The inclusion $Gr(2, n) \hookrightarrow Gr(2, n + r)$ for $r > 1$

The cohomology rings of $Gr(2, n + r)$ is:

$$H^*(Gr(2, n + r), \mathbb{Q}) = \wedge(y_2, y_4) / (h_{n+r-1}, h_{n+r}).$$

The minimal model of $Gr(2, n + r)$ is

$\left(\wedge (y_2, y_4, y_{2n+2r-3}, y_{2n+2r-1}), d \right)$ with $d(y_2) = d(y_4) = 0, d(y_{2n+2r-3}) = h_{n+r-1}, d(y_{2n+2r-1}) = h_{n+r}$. A model of the inclusion is then given by

$$\phi : \wedge V = \wedge(y_2, y_4, y_{2n+2r-3}, y_{2n+2r-1}) \longrightarrow \wedge(x_2, x_4, x_{2n-3}, x_{2n-1}) = B,$$

where, $\phi(y_2) = x_2, \phi(y_4) = x_4, \phi(y_{2n+2r-3}) = \beta$ and $\phi(y_{2n+2r-1}) = \varrho$ with $d\beta = h_{n+r-1}$ and $d\varrho = h_{n+r}$ respectively.

Theorem 9. Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n + r), r > 1$, then $G_*(\wedge V, B; \phi) = \langle y_{2n+2r-3}^*, y_{2n+2r-1}^* \rangle = G_n(\wedge V)$ and $G_*^{rel}(\wedge V, B; \phi) = \langle (sx_{2n-1}^*, 0), (sx_{2n-3}^*, 0), (0, y_{2n+2r-1}^*), (0, y_{2n+2r-3}^*) \rangle$.

Proof. In $sDer(B, \mathbb{Q}; \varepsilon) \oplus Der(\wedge V, \mathbb{Q}; \varepsilon)$ we have $(sx_{2n-1}^*, 0)$, $(sx_{2n-3}^*, 0)$, $(0, y_{2n+2r-1}^*)$ and $(0, y_{2n+2r-3}^*)$ which are all cycles.

Hence, $G_*(\wedge V, B; \phi) = \langle y_{2n+2r-3}^*, y_{2n+2r-1}^* \rangle = G_n(\wedge V)$ and $G_*^{rel}(\wedge V, B; \phi) = \langle (x_{2n-1}^*, 0), (x_{2n-3}^*, 0), (0, y_{2n+2r-1}^*), (0, y_{2n+2r-3}^*) \rangle$. $\square$

Example 2. Consider $Gr(2, 4) \hookrightarrow Gr(2, 6)$. A model of the inclusion is given by

$$\phi : \wedge V = \wedge(y_2, y_4, y_9, y_{11}) \longrightarrow \wedge(x_2, x_4, x_5, x_7) = B,$$

where, $\phi(y_2) = x_2$, $\phi(y_4) = x_4$, $\phi(y_9) = h_5 = 4x_2^3x_4 - 3x_2x_4^2 - x_2^5$ and $\phi(y_{11}) = h_6 = -x_4^3 + 6x_2^2x_4^2 - 5x_2^4x_4 + x_2^6$.

In $sDer(B, \mathbb{Q}; \varepsilon) \oplus Der(\wedge V, \mathbb{Q}; \varepsilon)$ we have $(sx_5^*, 0)$, $(sx_7^*, 0)$, $(0, y_9^*)$ and $(0, y_{11}^*)$ which are all cycles.

Hence, $G_*^{rel}(\wedge V, H^*(B)) \cong \langle (x_5^*, 0), (x_7^*, 0), (0, y_9^*), (0, y_{11}^*) \rangle$ and $G_*(\wedge V, H^*(B)) = \langle y_9^*, y_{11}^* \rangle$.

4.4 The Inclusion $Gr(2, n) \hookrightarrow \mathbb{C}P^N$

The Plücker embedding, $i : Gr(2, n) \hookrightarrow \mathbb{C}P^N$ is the inclusion of the Grassmannian of 2-subspaces of $\mathbb{C}^n$ into the complex projective space of dimension $N - 1$ where $N = \frac{n(n-1)}{2}$. The inclusion $i : Gr(2, n) \hookrightarrow \mathbb{C}P^N$ has a Sullivan model given by

$$\phi : (\wedge(x_2, x_{2N-1}), d) = (\wedge V, d) \longrightarrow (\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d) = (\wedge W, d).$$

A direct computation shows $G_*(\wedge W) = \langle [y_{2n-3}^*], [y_{2n-1}^*] \rangle$.

Lemma 2. Let $N = \frac{n(n-1)}{2}$. Then $2(N - 1) > 4n - 8$ for $n \geq 4$.

Proof. $2(N - 1) = n(n - 1) - 2$. The inequality $n(n - 1) > 4n - 8$ holds for $n = 4$. For $n \geq 5$, then $n(n - 1) \geq 4n$, hence $n(n - 1) - 2 > 4n - 8$. $\square$

Lemma 3. Let $\phi : (\wedge(x_2, x_{2N-1}), d) \longrightarrow (\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d)$, where $\phi(x_2) = y_2$, $\phi(x_{2N-1}) = y$ such that $dy = y_2^N$. There is a ϕ -derivation α_2 such that $\alpha_2 = (x_2, 1)$ and which is a cycle.

Proof. Define α_2 on x_{2N-1} so that $d\alpha_2(x_{2N-1}) - \alpha_2(dx_{2N-1}) = 0$. Hence

$$\begin{aligned} d\alpha_2(x_{2N-1}) &= d\alpha_2(x_{2N-1}) - \alpha_2(x_2^N) \\ &= d\alpha_2(x_{2N-1}) - Ny_2^{N-1}. \end{aligned}$$

As the dimension of $Gr(2, n)$ is less than $2(N-1)$ by Lemma 2, then Ny_2^{N-1} is a coboundary, that is $Ny_2^{N-1} = z$ with $z = \alpha_2(x_{2N-1})$. Moreover, $\delta\alpha_2 = 0$. As $\alpha_2 = (x_2, 1)$ can not be a boundary, it hence represents a non zero homology class in $H_*(Der(\wedge V, \wedge W; \phi), \delta)$. $\square$

Further, the ϕ -derivation $\alpha_{2N-1} = (x_{2N-1}, 1)$ represents a non zero homology class as well. As $H(\varepsilon_*)([\alpha_2]) = [x_2^*]$ and $H(\varepsilon_*)([\alpha_{2N-1}]) = [x_{2N-1}^*]$, we get the result below.

Proposition 1. $G_*(\wedge V, \wedge W; \phi) = \mathbb{Q} \langle [x_2^*], [x_{2N-1}^*] \rangle$.

We decompose $Rel_*(\phi^*) = Ker(\varepsilon_*, \varepsilon_*) \oplus K$. As K is isomorphic to $im(\varepsilon_*, \varepsilon_*)$, a non zero element in $G_*^{rel}(\wedge V, \wedge W; \phi)$ is represented by an element in K modulo the kernel of $(\varepsilon_*, \varepsilon_*)$. Moreover, the vector space K is spanned by

$$\{(0, \alpha_2), (0, \alpha_{2N-1}), (s\beta_2, 0), (s\beta_4, 0), (s\beta_{2N-3}, 0), (s\beta_{2N-1}, 0)\},$$

where $\beta_j = (y_j, 1)$.

Lemma 4. $(0, \alpha_2) = J(\alpha_2)$ represents a nonzero homology class in $H_2(Rel_*(\phi^*))$.

Proof. As $(0, \alpha_2)$ is a cycle, it remains to show that it is not a boundary. Otherwise, there exists $\beta'_2 \in Der(\wedge W)$ and $\alpha_3 \in Der(\wedge V, \wedge W; \phi)$ such that $d(s\beta'_2, \alpha_3) = (-s\beta'_2, \delta\alpha_3 + \phi^*(\beta'_2)) = (0, \alpha_2)$. Hence $\delta\beta'_2 = 0$ and $\phi^*(\beta'_2) = \alpha_2 - \delta\alpha_3$. For degree reasons $\alpha_3(x_2) = 0$, hence $(\delta\alpha_3)(x_2) = 0$. Therefore, $\phi'(\beta'_2)(x_2) = \alpha_2(x_2) = 1$. We deduce that $\beta'_2(y_2) = 1$. Moreover, β'_2 cannot be a cycle, otherwise $H(\varepsilon_*)([\beta'_2]) = [y_2^*] \neq 0$ would be a non zero Gottlieb element in $\pi_2(X_0)$, which is impossible, [5, Proposition 28.8]. We conclude that $[(0, \alpha_2)]$ is a nonzero homology class in $H_2(Rel(\phi^*))$. $\square$

We can now determine $G_*^{rel}(\wedge V, \wedge W; \phi)$.

Theorem 10. The relative Gottlieb group $G_*^{rel}(\wedge V, \wedge W; \phi)$ is isomorphic to the direct sum $sG_*(\wedge W) \oplus G_{2N-1}(\wedge V, \wedge W; \phi)$. Moreover, the sequence

$$0 = G_2(\wedge W) \longrightarrow G_2(\wedge V, \wedge W; \phi) \xrightarrow{H(\widehat{J})} G_2^{rel}(\wedge V, \wedge W; \phi) \longrightarrow 0$$

is not exact.

Proof. For degree reasons, $d\beta_2 = 0 = d\beta_4$. Moreover, $(s\beta_{2n-3}, 0)$ and $(s\beta_{2n-1}, 0)$ represent nonzero homology classes in $Rel_*(\phi^*)$. However, in $Rel_*(\phi^*)$, one

gets $d(sy_2^*, 0) = (0, \widehat{\phi^*}(y_2^*)) = (0, x_2^*)$. Hence $H(\varepsilon_*, \varepsilon_*)([0, \alpha_2]) = [(0, x_2^*)]$ is zero in $H(Rel_*(\widehat{\phi^*}))$. We conclude that $G_2^{rel}(\wedge V, \wedge W; \phi) = 0$. Therefore

$$\begin{aligned} G_*^{rel}(\wedge V, \wedge W; \phi) &= im H(\varepsilon_*, \varepsilon_*) \\ &= \langle [(sy_{2n-3}^*, 0)], [(sy_{2n-1}^*, 0)], [(0, x_{2N-1}^*)] \rangle \\ &= sG_*(\wedge W) \oplus G_{2N-1}(\wedge V, \wedge W; \phi). \end{aligned}$$

□

The sequence is not exact as $H(\widehat{J})([x_2^*]) = 0$. Hence $H(\widehat{J})$ is not injective.

Remark 1. *It was incorrectly stated in [14] that $G_*^{rel}(\wedge V, \wedge W; \phi) = 0$.*

Chapter 5

Rational Homotopy Type of $Gr(k, n) \hookrightarrow Gr(k, n + r)$

We can now generalise Theorem 2 as follows.

Theorem 11. *If $r > nk - k^2 - n + 2k - 1$, then the rational cohomology of $map(Gr(k, n), Gr(k, n+r); i_{n,r})$ contains a truncated algebra $\mathbb{Q}[x] / x^{r+n+k^2-nk}$, where $|x| = 2$, for $k \geq 2$ and $n \geq 4$.*

Proof. Consider $Gr(k, n) \hookrightarrow Gr(k, n + r)$. A Sullivan model of the inclusion is given by

$$\begin{aligned} \phi : \wedge V &= \left(\wedge (x_2, x_4, x_6, \dots, x_{2k}, x_{2(n+r-k)+1}, \dots, x_{2(n+r)-1}), d \right) \\ &\longrightarrow B = \wedge(y_2, y_4, y_6, \dots, y_{2k}) / (h_{n-k+1}, \dots, h_k), \end{aligned}$$

where $\left(\wedge (x_2, x_4, x_6, \dots, x_{2k}, x_{2(n+r-k)+1}, \dots, x_{2(n+r)-1}), d \right)$ is the minimal Sullivan model of $Gr(k, n + r)$, with $d(x_2) = d(x_4) = d(x_6) = \dots = d(x_{2k}) = 0$.

$$H^*(Gr(k, n), \mathbb{Q}) = B = \wedge(y_2, y_4, y_6, \dots, y_{2k}) / (h_{n-k+1}, \dots, h_k).$$

Note that $\phi(x_2) = y_2, \phi(x_4) = y_4, \phi(x_6) = y_6, \dots, \phi(x_{2k}) = y_{2k}$ and $\phi(x_{2n+2r-2k+1}) = \dots = \phi(x_{2n+2r-1}) = 0$. In $L = Der(\wedge V, B; \phi)$ we consider the derivation $\alpha_{2k} = (x_{2k}, 1)$, the highest even degree derivation. Then

$$\begin{aligned} a_1 &= (\delta \alpha_{2k})(x_{2n+2r-2k+1}), \\ &\vdots \\ a_k &= (\delta \alpha_{2k})(x_{2n+2r-1}) \end{aligned}$$

are polynomials of degree at least $2n + 2r - 4k + 2$.

If $r > nk - k^2 - n + 2k - 1$, then $2n + 2r - 4 + 2 > 2nk - 2k^2$. Hence $a_1, a_2, \dots, a_k$ are zero in B . Therefore, α_{2k} is a cycle.

On the other hand, let $\gamma_2 = (x_4, y_2)$. Then

$$\begin{aligned} b_1 &= l_m(\gamma_2, \gamma_2, \dots, \gamma_2), \\ &\vdots \\ b_k &= l_m(\gamma_2, \gamma_2, \dots, \gamma_2) \end{aligned}$$

are polynomials of degree $\geq 2n + 2r - 2k + 2 - 2m$. If $m < r + n + k^2 + 1 - nk - k$, then $b_1, b_2, \dots, b_k$ are of degree $> 2k(n - k)$.

Therefore, $l_m(\gamma_2, \gamma_2, \dots, \gamma_2) = 0$ for $m = 2, \dots, (r + n + k^2 + 1 - nk - k)$. In $C^\infty(s^{-1}L)$, let $\gamma_2^* = \varphi_2$, then $[\varphi_2^i] \neq 0$ for $i = 1, \dots, (r + n + k^2 + 1 - nk - k)$. $\square$

Theorem 12. *If $r > 2nk - n - 2k^2 + 2k - 2$, then the cohomology of $\text{map}(Gr(k, n), Gr(k, n + r); i_{n,r})$ contains a polynomial algebra over a generator of degree 2.*

Proof. Let

$$\begin{aligned} \phi : \wedge V &= \left(\wedge (x_2, x_4, x_6, \dots, x_{2k}, x_{2(n+r-k)+1}, \dots, x_{2(n+r)-1}), d \right) \\ &\longrightarrow B = \wedge(y_2, y_4, y_6, \dots) / (h_{n-k+1}, \dots, h_k), \end{aligned}$$

be a model of the inclusion and $L = \text{Der}(\wedge V, B; \phi)$. Assume $n \geq 4$ in particular

$r < 2nk - n - 2k^2$ then any odd derivation is of degree at least $2n + 2r - 2k - 2nk + 2k^2 + 1$. Hence

$L_1 = L_3 = \dots = L_f = 0$ where $f = 2n + 2r - 2k - 2nk + 2k^2 - 1$. It suffices to show that $l_m(\gamma_2, \gamma_2, \dots, \gamma_2) = 0$ for $m \geq 2$. Consider $n + r$ even, the other case is dealt with in a similar way.

For $m > \frac{n+r}{2}$, $l_m = 0$. Assume $2 \leq m \leq \frac{n+r}{2}$,

$$\begin{aligned} l_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2n+2r-2k+1}) &= c_1, \\ &\vdots \\ l_m(\gamma_2, \gamma_2, \dots, \gamma_2)(x_{2n+2r-1}) &= c_k \end{aligned}$$

are polynomials of total degree $\geq 2n + 2r - 2k + 2 - 2m$. If $r > 2k + nk - k^2 - n - 1$, then $2n + 2r - 2k + 2 - 2m > k(n - k)$. $\square$

Chapter 6

Conclusions and Recommendations

6.1 Summary

In Chapter one, we laid the ground work by defining concepts that were to be used in the course of the research. Precisely, the Sullivan model of the Grassmann manifold from which we defined the model of the inclusion. We recalled the concepts of *cdga's* and L_∞ -algebras which formed an integral part of the research.

In Chapter two, we gave a historical outline of the research in function spaces. Specific to this research was the paper by [2] due to the following result,
Let $\phi : (\wedge V, d) \longrightarrow B$ be a cdga model of $(X, Y; f)$ in which $(\wedge V, d)$ is the minimal Sullivan model of Y and B is a connected finite dimensional cdga model of the finite CW-complex X . If $L = s^{-1}V^\sharp$, then $\phi^\sharp : B^\sharp \longrightarrow C_\infty(L)$ is a commutative differential graded co-algebra morphism modeling f which induces an L_∞ structure on $\text{Hom}(B^, L)$ with brackets $[\cdots]_{\phi^\sharp}$. It is from the structure of $C_\infty(L)$ that we based our research to obtain our results.*

In Chapter three, we studied using numerical examples the inclusion of $Gr(2, 4)$ in $Gr(2, 5)$, $Gr(2, 4)$ in $Gr(2, 7)$, $Gr(2, 4)$ in $Gr(2, 8)$ and $Gr(3, 6)$ in $Gr(3, 7)$ which led to theorems 1, 2, 3, 4 and 5. For the generalisation $Gr(2, n) \hookrightarrow Gr(2, n+1)$ it was possible to deduce that $\delta : L_{2i} \longrightarrow L_{2i-1}$ is injective for $i = 1, 2$ as there were only two cohomology classes hence; it was relatively easy to obtain their equivalence classes. Hence theorem 1 could be proved. For the case of $Gr(3, n) \hookrightarrow Gr(3, n+1)$, it was not possible to determine all the equivalent cohomology classes, hence rendering many redundant derivations that would otherwise have been reduced. It was hence possible

to deduce that $i : Gr(2, n) \hookrightarrow Gr(2, n+1)$ has the rational homotopy type of a product of odd spheres as was also the case with $Gr(3, n) \hookrightarrow Gr(3, n+1)$.

In chapter four, we studied rationalized evaluation subgroups of mapping spaces between complex Grassmannian and G -Sequences and expressed $G_*^{rel}(Gr(2, n+r), Gr(2, n); i_{n,r}) \otimes \mathbb{Q}$ in terms of $G_*(Gr(2, n+r), Gr(2, n); i_{n,r}) \otimes \mathbb{Q}$ and $G_*(Gr(2, n+r)) \oplus \mathbb{Q}$.

In Chapter five, we related the results of Chapter 3 and 4 to the general case $Gr(k, n) \hookrightarrow Gr(k, n+r)$ and in particular it was possible to relate Theorems 2 and 3 to Theorems 6 and 7. Further, we showed that the cohomology of map $(Gr(2, n), Gr(2, n+r); i_{n,r})$ contains a truncated algebra $\mathbb{Q}[x]/(x^{r-n+4})$, where $|x| = 2$.

6.2 Future work

For the case of $Gr(3, n) \hookrightarrow Gr(3, n+1)$ and in general $Gr(k, n) \hookrightarrow Gr(k, n+r)$, it was not possible to determine cohomology algebra given mapping spaces.

Rationalized evaluation subgroups of mapping spaces between complex Grassmannians for the case when $r \geq 1$. We only determined the evaluation subgroups for the case $r = 1$, that is $Gr(2, n) \hookrightarrow Gr(2, n+1)$. An even interesting case would be the case when $k > 2$.

Another area of research that immediately crops up would be the study of rational cohomology algebra of the Plücker embedding of some complex Grassmannians. Here, we only managed to determine the evaluation subgroups of the said embedding.

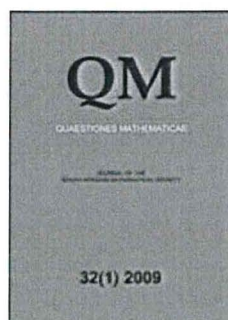
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Appendix Publications



Rational homotopy of mapping spaces between complex Grassmannian.

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RATIONAL HOMOTOPY OF MAPPING SPACES BETWEEN COMPLEX GRASSMANNIAN

PAUL ANTONY OTIENO, JEAN BAPTISTE GATSINZI,
AND VITALIS ONYANGO-OTIENO

ABSTRACT. The complex Grassmann $Gr(k, n)$ is the space of k dimensional subspaces of $\mathbb{C}^n$. It is a complex manifold of complex dimension $k(n - k)$. There is a natural inclusion $i_{k,n} : Gr(k, n) \hookrightarrow Gr(k, n + r)$. In this paper, we use Sullivan models to compute the rational homotopy type of the component of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n + r)$ in the space of mappings from $Gr(2, n)$ to $Gr(2, n + r)$, $r \geq 1$. We show in particular that $map(Gr(2, n) \rightarrow Gr(2, n + 1), i_n)$ has the rational homotopy type of odd spheres.

1. INTRODUCTION

The study of the rational homotopy type of spaces started with Thom in the case where the codomain is an Eilenberg-MacLane space [10]. The first description of a Sullivan model of function spaces is due to Haefliger [4]. Moreover, a model of function spaces between complex projective spaces was given by Møller and Raussen using the Postnikov tower [9]. However, there is no explicit and complete description of the homotopy type of the component of the inclusion $Gr(k, n) \hookrightarrow Gr(k, n + r)$ in the space of mappings from $Gr(2, n)$ to $Gr(2, n + r)$, $r \geq 1$. We begin our work with a model of the inclusion.

The Grassmannian $Gr(k, n)$ is the set of k dimensional subspaces of $\mathbb{C}^n$. Particularly, $Gr(1, n)$ is the collection of lines through the origin. It is called the complex projective space and is denoted by $\mathbb{C}P^{n-1}$.

The Grassmann manifold is endowed with the quotient topology as defined by the function $(v_1, v_2, \dots, v_k) \rightarrow \langle v_1, v_2, \dots, v_k \rangle$ of the Stiefel manifold $V_k(\mathbb{C}^n)$ onto $Gr(k, n)$, where the Stiefel manifold of orthonormal k frames in $\mathbb{C}^n$ is the subspace of $(v_1, v_2, \dots, v_k) \in (S^{2n-1})^k$ such

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that $\langle v_i, v_j \rangle = \delta_{ij}$ [6, Pg.13]. Moreover,

$$Gr(k, n) \cong U(n) / U(k) \times U(n - k),$$

where $U(n)$ is the unitary group [8, Chap.18]. There is a canonical inclusion $i_n : Gr(k, n) \hookrightarrow Gr(k, n + 1)$ which is induced by $i_n : \mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ defined by $i_n(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$.

2. SULLIVAN MODELS

Definition 1. A differential graded algebra (dga) is a graded algebra $A = \bigoplus_{n \geq 0} A^n$ together with a derivation d of degree $+1$. This means that for each n , there is a vector space homomorphism $d = d_n : A^n \rightarrow A^{n+1}$ such that $d \circ d = 0$ and $d(a \cdot b) = d(a) \cdot b + (-1)^n a \cdot d(b)$. Then (A, d) is called a cochain algebra. A graded algebra A is commutative if $a \cdot b = (-1)^{\deg a \cdot \deg b} b \cdot a$ for $a, b \in A$ [3, Chap. 3].

Definition 2. A Sullivan algebra is a commutative cochain algebra of the form $(\wedge V, d)$ where $V = \{V^p\}_{p \geq 2}$ and $\wedge V$ denotes the graded free commutative algebra on V . A Sullivan model for a commutative cochain algebra (A, d) is a quasi isomorphism $m : (\wedge V, d) \rightarrow (A, d)$ from a Sullivan algebra $(\wedge V, d)$. A Sullivan algebra is said to be minimal if the differential is decomposable, that is $Im d \subset \wedge^+ V \cdot \wedge^+ V$. Moreover, if $H^0(A) = \mathbb{F}$, then (A, d) has a minimal model which is determined up to isomorphism. If X is a nilpotent space and $A_{PL}(X)$ the commutative differential graded algebra $(cdga)$ of piecewise linear forms on X , then a Sullivan model of X is a Sullivan model of $A_{PL}(X)$ [3, Chap.12]. A space X is formal if there is a quasi isomorphism $(\wedge V, d) \rightarrow H^*(X, \mathbb{Q})$. Moreover, complex Grassmann manifolds are formal [2].

The cohomology ring $H^*(Gr(k, n), \mathbb{Q})$ of $Gr(k, n)$ is generated by the Chern classes $c_i \in H^{2i}(Gr(k, n), \mathbb{Q})$, for $1 \leq i \leq k$. Further, the cohomology ring has a presentation

$$H^*(Gr(k, n), \mathbb{Q}) = \wedge(c_1, c_2, \dots, c_k) / (h_{n-k+1}, \dots, h_n)$$

as the quotient of the polynomial ring generated by $c_1, c_2, \dots, c_k$, $|c_i| = 2i$, modulo the ideal generated by the elements $h_j, n - k + 1 \leq j \leq n$. Here, h_j is defined as the $2j^{th}$ degree term in the Taylor's expansion of $(1 + c_1 + c_2 + c_3 + c_k)^{-1}$ where, $(1 + c_1 + c_2 + c_3 + c_k)$ is the total Chern class [5].

For instance, $H^*(Gr(2, 4), \mathbb{Q}) = \wedge(x_2, x_4) / (h_3, h_4)$ where $h_3 = 2x_2x_4 -$

x_2^3 and $h_4 = x_4^2 - 3x_2^2x_4 + x_2^4$. As $(2x_2x_4 - x_2^3, x_4^2 - 3x_2^2x_4 + x_2^4)$ forms a regular sequence, the Sullivan model of $Gr(2, 4)$ is hence given by $(\wedge(x_2, x_4, x_5, x_7), d)$ with $dx_2 = dx_4 = 0$, $dx_5 = 2y_2y_4 - y_2^3$ and $dx_7 = y_4^2 - 3y_2^2y_4 + y_2^4$.

Definition 3. On a graded vector space L , an L_∞ structure, usually denoted by $(L, \{l_k\}_{k \geq 1})$, is a collection of linear maps, $\{l_k\}_{k \geq 1}$ called brackets, where $l_k = [\cdot, \dots, \cdot] : \otimes^k L \rightarrow L$ such that the following conditions are satisfied:

- (1) l_k are graded skew symmetric, that is, for any k permutation σ

$$[x_{\sigma(1)}, \dots, x_{\sigma(k)}] = \text{sgn}(\sigma) \epsilon_\sigma [x_1, \dots, x_k],$$

where ϵ_σ is the sign given by the Koszul convention.

- (2) The generalised Jacobi identity holds, that is,

$$\sum_{i+j=n+1} \sum_{\sigma \in S(i, n-i)} \text{sgn}(\sigma) \epsilon_\sigma (-1)^{i(j-1)} [x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_{\sigma(1)}, \dots, x_{\sigma(n)}] = 0,$$

where $S(i, n-i)$ denotes the $(i, n-i)$ shuffles which are permutations σ such that $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(n) \in S_n$.

An L_∞ algebra $(L, \{l_k\}_{k \geq 1})$ is minimal if $l_1 = 0$. Moreover L_∞ structures are in one to one correspondence with codifferentials on the non unital, free commutative coalgebra $\Lambda^+ sL$ in which s denotes *suspension*, that is $(sL)_k = L_{k-1}$ [7].

Note that if L is a graded vector space of finite type, an L_∞ structure on L induces a commutative differential graded algebra structure $C^\infty(L) = (\wedge (sL)^\#, d)$, $d = \sum_{j \geq 1} d_j$, where $d_j : V \rightarrow \wedge^{j+1} V$ is defined by $\langle d_j v; sx_1 \wedge \dots \wedge sx_j \rangle = (-1)^\epsilon \langle v; s[x_1, \dots, x_j] \rangle$ and $\epsilon = (-1)^{|v| + \sum_{k=1}^{j-1} (j-k)|x_k|}$ [7]. This is a generalisation of the Quillen cochain functor to L_∞ algebras [3, Chap. 23].

Further, an L_∞ algebra is an L_∞ model of a simply connected space X , if it is a model of the differential graded Lie algebra $\lambda(X)$. Here λ denotes the Quillen functor that associates to any simply connected space X , a differential graded Lie algebra $\lambda(X)$, which determines an equivalence between the homotopy categories of rational simply connected spaces and that of reduced differential graded Lie algebras over the rationals. If X is simply connected and of finite type, and $\mathbb{L}$ denotes the Quillen model of $\lambda(X)$, then $C^\infty(\mathbb{L})$ is a Sullivan model of X [3, Chap. 22].

Definition 4. Let $\phi : (A, d_A) \rightarrow (B, d_B)$ be a morphism of cochain algebras. A ϕ -derivation θ of degree n is a linear map $A^* \rightarrow A^{*-n}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{n|a|}\phi(a)\theta(b)$. We denote by $Der_n(A, B, \phi)$ the vector space of all ϕ -derivations of degree n . Define a differential of chain complexes $\delta : Der_n(A, B, \phi) \rightarrow Der_{n-1}(A, B, \phi)$, by $\delta\theta = d_B\theta - (-1)^{|\theta|}\theta d_A$.

If $\phi : (\wedge V, d) \rightarrow (B, d)$ is a morphism of commutative differential graded algebras, then there is an isomorphism of vector spaces

$$Der(\wedge V, B, \phi) \xrightarrow{\sim} Hom(V, B).$$

Moreover, if $\{v_1, \dots, v_n\}$ is a basis for V , we will denote by (v_i, b_i) the ϕ -derivation θ such that:

$$\begin{cases} \theta(v_i) = b_i \\ \theta(v_j) = 0 \quad i \neq j, b_i \in B. \end{cases}$$

We now follow [1] for this definition.

Consider positive derivations defined by:

$$\widetilde{Der}_i(\wedge V, B, \phi) = \begin{cases} Der_i(\wedge V, B, \phi) & \text{if } i \geq 2 \\ Z(Der_i(\wedge V, B, \phi)) & \text{if } i = 1. \end{cases}$$

If $(A, d) = (\wedge V, d)$ is a Sullivan algebra, then given $j \geq 2$ and $\varphi_1, \dots, \varphi_j \in \widetilde{Der}(\wedge V, B, \phi)$, of degrees $p_1, \dots, p_j$, we define their bracket of length j , $[\varphi_1, \dots, \varphi_j] \in \widetilde{Der}(\wedge V, B, \phi)$ by

$$[\varphi_1, \dots, \varphi_j](v) = (-1)^{p_1 + \dots + p_j - 1} \sum_{i_1, \dots, i_j} \epsilon \phi(v_1 \dots \widehat{v}_{i_1} \dots \widehat{v}_{i_j} \dots v_k) \varphi(v_{i_1}) \dots \varphi(v_{i_j}),$$

where $dv = \sum v_1 \dots v_k$ and ϵ is the sign given by the Koszul convention. These operations may be *desuspended* to define a set of linear maps $\{l_j\}_{j \geq 1}$ each of degree $j - 2$ on $s^{-1}\widetilde{Der}(\wedge V, B, \phi)$ as follows :

For $j = 1$

$$l_1 : s^{-1}\widetilde{Der}(\wedge V, B, \phi) \rightarrow s^{-1}\widetilde{Der}(\wedge V, B, \phi), \quad l_1(s^{-1}\theta) = -s^{-1}\delta\theta.$$

For $j = 2$ define

$$l_j : s^{-1}\widetilde{Der}(\wedge V, B, \phi) \otimes \dots \otimes s^{-1}\widetilde{Der}(\wedge V, B, \phi) \rightarrow s^{-1}\widetilde{Der}(\wedge V, B, \phi)$$

by

$$l_j(s^{-1}\varphi_1, \dots, s^{-1}\varphi_j) = (-1)^\alpha s^{-1}[\varphi_1, \dots, \varphi_j]$$

and $\alpha = \sum_{n=1}^{j-1} (j - n) + |\varphi_n|$.

Let $f : X \rightarrow Y$ be a continuous function between 1-connected spaces

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and $map(X, Y; f)$ the component of f in the space of continuous maps from X to Y . Let $\phi : (\wedge V, d) \longrightarrow B$ be a *cdga* model of f . Lemma 3.3 of [1] proves that $(s^{-1}\widetilde{Der}(\wedge V, B, \phi), \{l_j\}_{j \geq 1})$ is an L_∞ model of $map(X, Y; f)$.

3. INCLUSION OF $Gr(2, 4)$ IN $Gr(2, 5)$

Recall that

$$H^*(Gr(2, 4), \mathbb{Q}) = \wedge(y_2, y_4) / (2y_2y_4 - y_2^3, y_2^4 - 3y_2^2y_4 + y_4^2).$$

Moreover,

$$H^*(Gr(2, 5), \mathbb{Q}) = \wedge(x_2, x_4) / (x_2^4 - 3x_2^2x_4 + x_4^2, 4x_2^3x_4 - 3x_2x_4^2 - x_2^5).$$

Hence the minimal Sullivan model of $Gr(2, 5)$ is given by $(\wedge(x_2, x_4, x_7, x_9), d)$ where $dx_2 = dx_4 = 0$, $dx_7 = x_2^4 - 3x_2^2x_4 + x_4^2$ and $dx_9 = 4x_2^3x_4 - 3x_2x_4^2 - x_2^5$.

The inclusion $Gr(2, 4) \hookrightarrow Gr(2, 5)$ has a model given by

$$\phi : \wedge((x_2, x_4, x_7, x_9), d) \longrightarrow \wedge(y_2, y_4) / (2y_2y_4 - y_2^3, y_2^4 - 3y_2^2y_4 + y_4^2),$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_7) = 0$ and $\phi(x_9) = 0$. Let $(\wedge V, d) = \wedge(x_2, x_4, x_7, x_9)$ and $B = \wedge(y_2, y_4) / (2y_2y_4 - y_2^3, y_2^4 - 3y_2^2y_4 + y_4^2)$. The vector space $Der(\wedge V, B, \phi)$ is generated by:

$$\begin{aligned} \alpha_9 &= (x_9, 1), \alpha_7 = (x_9, y_2), \alpha_5 = (x_9, y_4), \alpha'_5 = (x_9, y_2^2), \alpha_3 = \\ &= (x_9, y_2y_4), \alpha_1 = (x_9, y_2^4 - 2y_2^2y_4), \\ \beta_7 &= (x_7, 1), \beta_5 = (x_7, y_2), \beta_3 = (x_7, y_4), \beta'_3 = (x_7, y_2^2), \beta_1 = \\ &= (x_7, y_2y_4), \\ \gamma_4 &= (x_4, 1), \gamma_2 = (x_4, y_2), \\ \theta_2 &= (x_2, 1). \end{aligned}$$

Moreover, $\delta\theta_2 = -4\alpha_1 + 2\beta_1$, $\delta\gamma_2 = -\alpha_1 - 4\beta_1$ and $\delta\gamma_4 = -\alpha_3 + 4\beta'_3$ and $\delta\alpha_i = \delta\beta_j = 0$.

Computing the Lie brackets we get

$$\begin{aligned} [\theta_2, \theta_2] &= -96\alpha_3 + 24\beta'_3 - 6\beta_3, [\theta_2, \gamma_2] = 18\alpha_3 - 6\beta'_3, \\ [\theta_2, \gamma_4] &= 24\alpha'_5 - 6\alpha_5 - 6\beta_5, [\gamma_2, \gamma_2] = -6\alpha_3 + 2\beta'_3, \\ [\gamma_2, \gamma_4] &= -6\alpha'_5 + 2\beta_5, [\gamma_4, \gamma_4] = 2\beta_7, \\ [\theta_2, \theta_2, \theta_2] &= 24\alpha_5 - 120\alpha'_5 + 24\beta_5, [\theta_2, \theta_2, \gamma_2] = 24\alpha'_5 - 6\beta_5, \\ [\theta_2, \theta_2, \gamma_4] &= 24\alpha_7 - 6\beta_7, [\theta_2, \gamma_2, \gamma_2] = -6\alpha'_5, \\ [\theta_2, \gamma_4, \gamma_4] &= -6\alpha_9, [\theta_2, \gamma_2, \gamma_4] = -6\alpha_7, \\ [\theta_2, \theta_2, \theta_2, \theta_2] &= -120\alpha_7 + 24\beta_7, \\ [\theta_2, \theta_2, \theta_2, \gamma_2] &= 24\alpha_7, [\theta_2, \theta_2, \theta_2, \gamma_4] = 24\alpha_9, \end{aligned}$$

$$[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2] = -120\alpha_9,$$

the other brackets being zero.

If $L = s^{-1}\widetilde{Der}(\wedge V, B, \phi)$ and letting $x_i = \alpha_i^\sharp, y_j = \beta_j^\sharp, w_j = \gamma_j^\sharp$ and $z_2 = \theta_2^\sharp$,

$$C^\infty(L) = \left(\wedge (x_1, y_1, w_2, z_2, x_3, y_3, y_3', w_4, x_5, x_5', y_5, x_7, y_7, x_9), d \right),$$

where; $d(z_2) = d(w_2) = 0, d(x_1) = -\frac{1}{4}z_2 - w_2, d(y_1) = \frac{1}{2}z_2 - \frac{1}{4}w_2,$
 $d(x_3) = w_4 - \frac{1}{96}z_2^2 + \frac{1}{18}w_2z_2 - \frac{1}{6}w_2^2, d(y_3) = \frac{1}{4}w_4 - \frac{1}{6}z_2^2,$
 $d(x_5) = \frac{1}{24}z_2^3 - \frac{1}{6}z_2w_4, d(x_5') = -\frac{1}{120}z_2^3 + \frac{1}{24}z_2^2w_2 + \frac{1}{24}z_2w_4 - \frac{1}{6}z_2w_2^2 - \frac{1}{6}w_2w_4,$
 $d(y_5) = \frac{1}{24}z_2^3 - \frac{1}{6}z_2^2w_2 - \frac{1}{6}z_2w_4 + \frac{1}{2}w_2w_4,$
 $d(x_7) = \frac{1}{24}z_2^2w_4 + \frac{1}{24}z_2^3w_2 + \frac{1}{120}z_2^4 - \frac{1}{6}z_2w_2w_4,$
 $d(y_7) = \frac{1}{24}z_2^4 - \frac{1}{6}z_2^2w_4 + \frac{1}{24}w_4^2, d(x_9) = -\frac{1}{120}z_2^5 - \frac{1}{24}z_2^3w_4 - z_2w_4^2.$

Note that the ideal generated by $\langle x_1, y_1, w_2, z_2 \rangle$ is acyclic, hence $C^\infty(L)$ is quasi isomorphic to

$$\wedge (x_3, y_3', x_5, x_5', y_5, x_7, y_7, x_9), 0).$$

Hence, the inclusion $Gr(2, 4) \hookrightarrow Gr(2, 5)$ has the rational homotopy type of $S^3 \times S^3 \times S^5 \times S^5 \times S^5 \times S^7 \times S^7 \times S^9$.

We can hence generalise the result as follows:

Theorem 1. *Consider the inclusion $i_n : Gr(2, n) \hookrightarrow Gr(2, n+1)$. The mapping space $(Gr(2, n), Gr(2, n+1), i_n)$ has the rational homotopy type of a product of odd spheres.*

Proof. Since $H^*(Gr(k, n), \mathbb{Q}) = \wedge(c_1, c_2, \dots, c_k) / (h_{n-k+1}, \dots, h_n), |c_i| = 2i$, then it follows that

$$H^*(Gr(2, n), \mathbb{Q}) = \wedge(x_2, x_4) / (h_{n-1}, h_n) \text{ and}$$

$$H^*(Gr(2, n+1), \mathbb{Q}) = \wedge(x_2, x_4) / (h_n, h_{n+1}).$$

The minimal model of $Gr(2, n+1)$ is $(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d)$ where $dx_2 = dx_4 = 0, dx_{2n-1} = h_n$ and $dx_{2n+1} = h_{n+1}$. Assume $n \geq 4$. Hence a model of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ is given by;

$$\phi : (\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d) \longrightarrow \wedge(y_2, y_4) / (h_{n-1}, h_n),$$

where $\phi(x_2) = y_2, \phi(x_4) = y_4$ and $\phi(x_{2n-1}) = \phi(x_{2n+1}) = 0$. Moreover, $Gr(2, n)$ is a complex manifold of dimension $2(n-2)$, and of real dimension $2(2(n-2)) = 4n-8$.

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The Grassmann manifold $Gr(2, n)$ being a symplectic manifold has a symplectic class represented by $y_2 \in H^2(Gr(2, n), \mathbb{Q})$. Further, for $1 \leq i \leq (2n - 4)$, y_2^i is a non zero cohomology class.

Let

$$L = Der\left(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d, \wedge(y_2, y_4) / (h_{n-1}, h_n), \phi\right).$$

The only even degree ϕ -derivations are $\theta_2 = (x_2, 1)$, $\gamma_2 = (x_4, y_2)$ and $\gamma_4 = (x_4, 1)$. Let $\alpha_1 = (x_{2n-1}, y_2^{n-1})$ and $\beta_1 = (x_{2n+1}, y_2^n)$. Moreover,

$$\begin{aligned} dx_{2n-1} &= a_n x_2^n + a_{n-2} x_2^{n-2} x_4 + \cdots, a_n \neq 0 \quad \text{and} \\ dx_{2n+1} &= b_n x_2^{n+1} + b_{n-2} x_2^{n-1} x_4 + \cdots, b_n \neq 0. \end{aligned}$$

Then,

$$\begin{aligned} (\delta\theta_2)(x_{2n-1}) &= a_n \cdot n \cdot y_2^{n-1} + b_{n-2} \cdot (n-2) \cdot y_4 y_2^{n-3} + \cdots \\ (\delta\theta_2)(x_{2n+1}) &= b_n \cdot n \cdot y_2^n + b_{n-2} \cdot (n-1) \cdot y_4 y_2^{n-2} + \cdots \\ (\delta\gamma_2)(x_{2n-1}) &= a_{n-2} \cdot (n-2) \cdot y_2 y_4^{n-3} + \cdots \\ (\delta\gamma_2)(x_{2n+1}) &= b_{n-2} \cdot (n-1) \cdot y_2^{n-1} + \cdots \\ (\delta\gamma_4)(x_{2n-1}) &= a_{n-2} \cdot (n-2) \cdot y_2^{n-3} + \cdots \\ (\delta\gamma_4)(x_{2n+1}) &= b_{n-2} \cdot (n-1) \cdot y_2^{n-2} + \cdots \end{aligned}$$

Hence,

$$\begin{aligned} \delta\theta_2 &= n \cdot a_n \cdot \alpha_1 + n \cdot b_n \cdot \beta_1 + \cdots \\ \delta\gamma_2 &= b_{n-2}(n-1)\alpha_1 + 0\beta_1 + \cdots \end{aligned}$$

We deduce that $\delta : L_2 \rightarrow L_1$ is injective as $\delta\theta_2$ and $\delta\gamma_2$ are linearly independent. Therefore, $H_2(L, \delta) = 0$. In the same way $H_4(L, \delta) = 0$. Moreover, all brackets with odd dimension derivations are zero. We conclude that $H_*(L)$ is concentrated in odd degrees with all brackets being zero.

Hence, $C^*(L)$ has a minimal model of the form $(\wedge X, 0)$, where $X = X^{odd}$. Therefore map $(Gr(2, n), Gr(2, n+1); i_n)$ has the rational homotopy type of a product of odd spheres.

□

4. THE INCLUSION $Gr(2, n) \hookrightarrow Gr(2, n+r)$, $r \geq 2$

Having established that map $(Gr(2, n), Gr(2, n+1); i_n)$ has the rational homotopy type of a product of odd spheres, we now set to consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+r)$, $r \geq 2$.

We start with a computational example. Consider the inclusion $Gr(2, 4) \hookrightarrow Gr(2, 8)$. Recall that $H^*(Gr(2, 4), \mathbb{Q}) = \wedge(y_2, y_4) / (h_3, h_4)$. A Sullivan

model of $Gr(2, 8)$ is $\wedge((x_2, x_4, x_{13}, x_{15}), d)$ with $d(x_2) = d(x_4) = 0$, $dx_{13} = -x_2^7 + 6x_4x_2^5 - 10x_2^3x_4^2 + 4x_2x_4^3$ and $dx_{15} = x_2^8 - 7x_2^6x_4 + 15x_2^4x_4^2 - 10x_2^2x_4^3 + x_4^4$. A model of the inclusion is thus given by;

$$\phi : (\wedge V, d) = \left(\wedge (x_2, x_4, x_{13}, x_{15}), d \right) \longrightarrow \wedge(y_2, y_4) / (h_3, h_4) = B$$

where; $\phi(x_2) = y_2$, $\phi(x_4) = y_4$ and $\phi(x_{13}) = \phi(x_{15}) = 0$. The vector space $Der(\wedge V, B, \phi)$ is spanned by:

$$\begin{aligned} \alpha_{15} &= (x_{15}, 1), \alpha_{13} = (x_{15}, y_2), \alpha_{11} = (x_{15}, y_4), \alpha'_{11} = (x_{15}, y_2^2), \\ \alpha_9 &= (x_{15}, y_2^3 = 2y_2y_4), \alpha_7 = (x_{15}, y_4^2 = 2y_2^2y_4 = y_2^4), \\ \beta_{13} &= (x_{13}, 1), \beta_{11} = (x_{13}, y_2), \beta_9 = (x_{13}, y_4), \beta'_9 = (x_{13}, y_2^2), \\ \beta_7 &= (x_{13}, y_2^3 = 2y_2y_4), \beta_5 = (x_{13}, y_4^2 = 2y_2^2y_4 = y_2^4), \\ \gamma_2 &= (x_4, y_2), \gamma_4 = (x_4, 1) \text{ and} \\ \theta_2 &= (x_2, 1). \end{aligned}$$

Computing the Lie brackets, we get

$$\begin{aligned} [\theta_2, \gamma_4] &= 684\beta_5, [\gamma_4, \gamma_4] = -36\beta_7 + 684\alpha_7, [\gamma_2, \gamma_4] = -36\beta_5, \\ [\theta_2, \theta_2, \theta_2] &= -4380\beta_5, [\theta_2, \gamma_4, \gamma_4] = -60\beta'_9 + 660\alpha_9, \\ [\theta_2, \gamma_2, \gamma_2] &= -36\beta_5, [\theta_2, \gamma_4, \gamma_2] = -36\beta_7 + 660\alpha_7, \\ [\theta_2, \theta_2, \gamma_4] &= -660\beta_7 - 4380\alpha_7, [\theta_2, \theta_2, \gamma_2] = -660\beta_5, \\ [\gamma_2, \gamma_2, \gamma_2] &= 24\beta_5, [\gamma_2, \gamma_2, \gamma_4] = -34\alpha_7 + 24\beta_7, \\ [\gamma_4, \gamma_4, \gamma_4] &= 24\beta_{11} + 24\alpha_{11} - 60\alpha'_{11}, [\gamma_4, \gamma_4, \gamma_2] = 24\beta'_9 - 34\alpha_9, \\ [\theta_2, \theta_2, \theta_2, \gamma_4] &= -60\beta_9 + 720\beta'_9 - 4320\alpha_9, \\ [\theta_2, \theta_2, \theta_2, \gamma_2] &= 660\beta_7 - 4320\alpha_7, \\ [\theta_2, \theta_2, \gamma_4, \gamma_4] &= -60\beta_{11} - 60\alpha_{11} + 720\alpha'_{11}, \\ [\theta_2, \theta_2, \gamma_2, \gamma_2] &= -60\beta_7 + 660\alpha_7, [\theta_2, \theta_2, \gamma_2, \gamma_4] = -60\beta'_9 + 660\alpha_9, \\ [\theta_2, \theta_2, \theta_2, \theta_2] &= -4320\beta_7 + 36000\alpha_7, [\theta_2, \gamma_2, \gamma_2, \gamma_2] = 24\beta_7 - 60\alpha_7, \\ [\theta_2, \gamma_4, \gamma_4, \gamma_4] &= 24\beta_{13} - 60\alpha_{13}, [\theta_2, \gamma_2, \gamma_2, \gamma_4] = 24\beta'_9 - 60\alpha_9, \\ [\theta_2, \gamma_4, \gamma_4, \gamma_2] &= 24\beta_{11} - 60\alpha'_{11}, [\gamma_4, \gamma_4, \gamma_4, \gamma_4] = 24\alpha_{15}, \\ [\gamma_2, \gamma_2, \gamma_2, \gamma_2] &= 24\alpha_7, [\gamma_2, \gamma_2, \gamma_4, \gamma_4] = 24\alpha'_{11}, \\ [\gamma_2, \gamma_2, \gamma_2, \gamma_4] &= 24\alpha'_9, [\gamma_2, \gamma_4, \gamma_4, \gamma_4] = 24\alpha_{13}, \\ [\theta_2, \theta_2, \gamma_4, \gamma_4, \gamma_4] &= -60\alpha_{15}, [\theta_2, \theta_2, \theta_2, \gamma_4, \gamma_4] = -60\beta_{13} + 720\alpha_{13}, \\ [\theta_2, \theta_2, \theta_2, \gamma_2, \gamma_2] &= -60\beta_9 + 720\alpha_9, \\ [\theta_2, \theta_2, \theta_2, \gamma_2, \gamma_4] &= -60\beta_{11} + 720\alpha'_{11}, [\theta_2, \theta_2, \gamma_4, \gamma_4, \gamma_2] = -60\alpha_{13}, \\ [\theta_2, \theta_2, \gamma_2, \gamma_2, \gamma_2] &= -60\alpha_9, [\theta_2, \theta_2, \gamma_2, \gamma_2, \gamma_4] = -60\alpha_{11}, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= -120\beta_9 - 5040\beta'_9 + 35280\alpha_9, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \gamma_2] &= 7200\beta'_9 - 4320\alpha_9, \\ [\theta_2, \theta_2, \theta_2, \theta_2, \gamma_4] &= 720\beta_{11} + 720\alpha_{11} - 5040\alpha'_{11}, \end{aligned}$$

$$\begin{aligned}
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= -5040\beta_{11} - 5040\alpha_{11} + 40320\alpha'_{11}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \gamma_2] &= 720\beta_{11} - 5040\alpha'_{11}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \gamma_4] &= 7200\beta_{13} - 50400\alpha_{13}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \gamma_4, \gamma_4] &= 720\alpha'_{15}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \gamma_2, \gamma_2] &= 720\alpha'_{11}, [\theta_2, \theta_2, \theta_2, \theta_2, \gamma_4, \gamma_2] = 720\alpha_{13}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= -5040\beta_{13}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \gamma_4] &= -5040\alpha_{15}, \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \gamma_2] &= -5040\alpha_{13} \text{ and} \\
[\theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2, \theta_2] &= 40320\alpha_{15}.
\end{aligned}$$

with $\delta\gamma_2 = \delta\gamma_4 = \delta\theta_2 = 0$ and the other brackets being zero as well. Note that $\delta = 0$ hence $Der(\wedge V, B, \phi)$ is a minimal L_∞ algebra. Put $x_i = \alpha_i^\sharp$, $y_j = \beta_j^\sharp$, $w_j = \gamma_j^\sharp$ and $z_2 = \theta_2^\sharp$. Hence

$$C^\infty(L) = \left(\wedge(z_2, w_2, w_4, y_5, y_7, x_7, y_9, y_9', x_9, y_{11}, x_{11}, x_{11}', y_{13}, x_{13}, x_{15}), d \right),$$

where; $d(z_2) = d(w_2) = d(w_4) = 0$,

$$\begin{aligned}
d(y_5) &= \frac{1}{684}z_2w_4 - \frac{1}{36}w_2w_4 - \frac{1}{660}z_2^2w_2 - \frac{1}{36}z_2w_2^2 + \frac{1}{24}w_2^3 \\
&\quad - \frac{1}{4380}z_2^3, \\
d(y_7) &= -\frac{1}{36}w_4^2 - \frac{1}{660}w_2^2w_4 + \frac{1}{24}z_2w_2w_4 + \frac{1}{660}z_2^3w_2 - \frac{1}{60}z_2^2w_2^2 \\
&\quad - \frac{1}{4320}z_2^4 + \frac{1}{24}z_2w_2^3, \\
d(x_7) &= \frac{1}{684}w_4^2 - \frac{1}{4380}z_2^2w_4 - \frac{1}{34}w_2^2w_4 + \frac{1}{660}z_2w_2w_4 - \frac{1}{4320}z_2^3w_2 \\
&\quad + \frac{1}{660}z_2^2w_2^2 + \frac{1}{36000}z_2^4 - \frac{1}{60}z_2w_2^3, \\
d(y_9) &= -\frac{1}{60}z_2^3w_4 - \frac{1}{60}z_2^3w_2^2 + \frac{1}{120}z_2^5, \\
d(y_9') &= -\frac{1}{60}z_2w_4^2 + \frac{1}{24}w_2w_4^2 + \frac{1}{720}z_2^3w_4 - \frac{1}{5040}z_2^5 + \frac{1}{720}z_2^4w_2 \\
&\quad + \frac{1}{24}z_2w_2^2w_4, \\
d(x_9) &= \frac{1}{660}z_2w_4^2 - \frac{1}{34}w_2w_4^2 - \frac{1}{4320}z_2^3w_4 + \frac{1}{660}z_2^2w_2w_4 - \frac{1}{60}z_2w_2^2w_4 \\
&\quad + \frac{1}{35280}z_2^5 + \frac{1}{720}z_2^3w_2^2 - \frac{1}{4320}z_2^4w_2 - \frac{1}{60}z_2^3w_2^3,
\end{aligned}$$

$$\begin{aligned}
 d(y_{11}) &= \frac{1}{24}w_4^3 - \frac{1}{60}z_2^2w_4^2 + \frac{1}{24}z_2w_4^2w_2 - \frac{1}{60}z_2^3w_2w_4 + \frac{1}{720}z_2^4w_4 \\
 &\quad - \frac{1}{5040}z_2^6 + \frac{1}{720}z_2^5w_2, \\
 d(x_{11}) &= \frac{1}{24}w_4^3 - \frac{1}{60}z_2^2w_4 + \frac{1}{720}z_2^4w_4 - \frac{1}{60}z_2^2w_2^2w_4 - \frac{1}{5040}z_2^6, \\
 d(x'_{11}) &= -\frac{1}{60}w_4^3 + \frac{1}{720}z_2^2w_4^2 - \frac{1}{60}z_2w_4^2w_2 + \frac{1}{24}w_2^2w_4^2 + \frac{1}{720}z_2^3w_2w_4 \\
 &\quad - \frac{1}{5040}z_2^4w_4 + \frac{1}{40320}z_2^6 - \frac{1}{5040}z_2^5w_2 + \frac{1}{720}z_2^4w_2^2, \\
 d(y_{13}) &= -\frac{1}{24}z_2w_4^3 - \frac{1}{60}z_2^3w_4^2 - \frac{1}{5040}z_2^7, \\
 d(x_{13}) &= -\frac{1}{60}z_2w_4^3 + \frac{1}{24}w_2w_4^3 + \frac{1}{720}z_2^3w_4^2 - \frac{1}{60}z_2^2w_4^2w_2 - \frac{1}{5040}z_2^5w_4 \\
 &\quad + \frac{1}{720}z_2^4w_4w_2 - \frac{1}{5040}z_2^6w_2, \\
 d(x_{15}) &= \frac{1}{24}w_4^4 - \frac{1}{60}z_2^2w_4^3 + \frac{1}{720}z_2^4w_4^2 - \frac{1}{5040}z_2^6w_4 + \frac{1}{40320}z_2^8.
 \end{aligned}$$

Note that $[w_2]$, $[w_2^2]$, $[w_2^3]$ are non zero cohomology classes in $H^*(C^\infty(L))$. We can generalise as follows.

Theorem 2. *If $r > n-1$, then the cohomology of map $(Gr(2, n), Gr(2, n+r), i_{n,r})$ contains a truncated algebra $\mathbb{Q}[x]/(x^{r-n+4})$, where $|x| = 2$.*

Proof. Recall that the cohomology rings of $Gr(2, n)$ and $Gr(2, n+r)$ are respectively:

$$\begin{aligned}
 B &= H^*(Gr(2, n), \mathbb{Q}) = \wedge(x_2, x_4)/(h_{n-1}, h_n) \text{ and} \\
 H^*(Gr(2, n+r), \mathbb{Q}) &= \wedge(x_2, x_4)/(h_{n+r-1}, h_{n+r}).
 \end{aligned}$$

The minimal model of $Gr(2, n+r)$ is

$(\wedge V, d) = (\wedge(x_2, x_4, x_{2n+2r-3}, x_{2n+2r-1}), d)$ with $d(x_2) = d(x_4) = 0$, $d(x_{2n+2r-3}) = h_{n+r-1}$, $d(x_{2n+2r-1}) = h_{n+r}$. A model of the inclusion is then given by

$$\phi : \wedge(x_2, x_4, x_{2n+2r-3}, x_{2n+2r-1}) \longrightarrow \wedge(y_2, y_4)/(h_{n-1}, h_n),$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$ and $\phi(x_{2n+2r-3}) = \phi(x_{2n+2r-1}) = 0$. The vector space $Der(\wedge V, B, \phi)$ is spanned by:

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$$\begin{aligned} \alpha_{2n+2r-1} &= (x_{2n+2r-1}, 1), \alpha_{2n+2r-3} = (x_{2n+2r-1}, y_2), \alpha_{2n+2r-5} = \\ &= (x_{2n+2r-1}, y_4), \dots, \alpha_{2n+2r-9} = (x_{2n+2r-1}, y_2^{2n-4}), \\ \beta_{2n+2r-3} &= (x_{2n+2r-3}, 1), \beta_{2n+2r-5} = (x_{2n+2r-3}, y_2), \beta_{2n+2r-7} = \\ &= (x_{2n+2r-3}, y_4), \dots, \beta_{2n+2r-11} = (x_{2n+2r-3}, y_2^{2n-4}), \\ \gamma_4 &= (x_4, 1), \gamma_2 = (x_4, y_2) \text{ and} \\ \theta_2 &= (x_2, 1). \end{aligned}$$

As $p_1 = (\delta\gamma_4)(x_{2n+2r-3})$ and $p_2 = (\delta\gamma_4)(x_{2n+2r-1})$ are polynomials of degree at least $2n + 2r - 6$. If $r > n - 1$, then $2n + 2r - 6 > 4n - 8$. Therefore p_1 and p_2 are zero in $\wedge(x_2, x_4) / (h_{n-1}, h_n)$. Hence, $(\delta\gamma_4) = 0$. Similarly, $\delta\gamma_2 = \delta\theta_2 = 0$.

On the other hand $q_1 = l_k(\gamma_2, \dots, \gamma_2)(dx_{2n+2r-3})$ and $q_2 = l_k(\gamma_2, \dots, \gamma_2)(dx_{2n+2r-1})$ are polynomials of degree $\geq 2n + 2r - 2 - 2k$. If $k < r - n + 3$, then q_1 and q_2 are of degree $> 4n - 8$. Therefore, $l_k(\gamma_2, \dots, \gamma_2) = 0$ for $k = 2, \dots, r - n + 3$. In $C^\infty(L)$, put $\gamma_2^\# = w_2$. Then $[w_2^i] \neq 0$ for $i = 1, \dots, r - n + 3$. $\square$

Theorem 3. Consider the inclusion $i_{n,r} : Gr(2, n) \hookrightarrow Gr(2, n + r)$. If $r > 3n - 6$, then the cohomology of map $(Gr(2, n), Gr(2, n + r); i_{n,r})$ contains a polynomial algebra over a generator of degree 2.

Proof. Let $\phi : (\wedge(x_2, x_4, x_{2n+2r-3}, x_{2n+2r-1}), d) \rightarrow \wedge(y_2, y_4) / (h_{n-1}, h_n)$ be a model of $i_{n,r}$. Assume $n \geq 4$, in particular $r \geq 3n - 8$ then any odd derivation is of degree at least $(2n + 2r - 3) - (4n - 8) = 2r - 2n + 5 \geq 5$. Hence $L_3 = L_1 = 0$. Moreover, $\gamma_2 = (x_4, y_2)$ is automatically a cycle (as $L_1 = 0$) which is not a boundary (as $L_3 = 0$). It is sufficient to show that $l_m(\gamma_2, \dots, \gamma_2) = 0$ for $m \geq 2$, where l_m denotes the bracket length m .

We consider the case when $n + r$ is even, the other case is dealt with in a similar way.

For $m > \frac{n+r}{2}$, $l_m = 0$. Assume $2 \leq m \leq \frac{n+r}{2}$,

$$l_m(\gamma_2, \dots, \gamma_2)(x_{2n+2r-1}) = \sum_{i \geq m} \alpha_i y_4^{i-m} \cdot y_2^m y_2^{n+r-2i}, \quad (1)$$

$$l_m(\gamma_2, \dots, \gamma_2)(x_{2n+2r-3}) = \sum_{i \geq m} \beta_i y_4^{i-m} \cdot y_2^m \dots y_2^{n+r-1-2i}. \quad (2)$$

The right hand side of (1) and (2) are polynomials of total degree $\geq 2n + 2r - 2m - 2$. Assume now that $r > 3n - 6$, then

$$\begin{aligned} n + r - 2 > 4n - 8 &\Rightarrow 2n + 2r - 2 - (n + r) > 4n - 8 \\ &\Rightarrow 2n + 2r - 2m - 2 > 4n - 8. \end{aligned}$$

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Therefore, $l_m(\gamma_2, \dots, \gamma_2) = 0$ for all $m \geq 0$. This shows that $H^*(C^\infty(L))$ contains a polynomial algebra over a generator of degree 2.

□

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Rationalized evaluation subgroups of mapping spaces between complex Grassmannians.

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RATIONALIZED EVALUATION SUBGROUPS OF MAPPING SPACES BETWEEN COMPLEX GRASSMANNIANS.

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AND VITALIS ONYANGO-OTIENO

ABSTRACT. We determine evaluation subgroups of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ between complex Grassmannians.

1. INTRODUCTION

Given a pointed topological space (X, x_0) , the n^{th} Gottlieb group of X , also called the evaluation subgroup of $\pi_n(X)$ and denoted by $G_n(X)$, consists of those $\alpha \in \pi_n(X)$ for which there is a map $F : X \times S^n \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times S^n & \xrightarrow{F} & X \\ J \uparrow & & \nabla \uparrow \\ X \vee S^n & \xrightarrow{1_X \vee f} & X \vee X, \end{array}$$

where $f : S^n \rightarrow X$ is a representative of α and ∇ is the folding map. Thus for every $\alpha \in G_n(X, x_0)$, there exists at least one map $F : X \times S^n \rightarrow X$ such that $F(x_0, s) = f(s)$. We say that F is an affiliated map to α [3]. If X has a base point x_0 and $\text{aut } X$ denotes the monoid of self homotopy equivalences of X with $ev : \text{aut } X \rightarrow X$ the evaluation map at x_0 , then it follows from the definition that

$$G_n(X) = \text{im} \left(ev_{\#} : \pi_n(\text{aut } X, 1_X) \rightarrow \pi_n(X, x_0) \right).$$

Similarly, if $f : X \rightarrow Y$ is a based map between simply connected CW complexes and $\text{map}(X, Y; f)$, the space of maps from X to Y which are homotopic to f , then $G_n(Y, X; f) = \text{im} \left(ev_{\#} : \text{map}(X, Y; f) \rightarrow \pi_n(Y) \right)$ is the n^{th} evaluation subgroup of f [9]. In [10], Woo and Lee

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defined relative evaluation subgroups $G_n^{rel}(X, Y; f)$ and showed that they fit in a sequence

$$\cdots \longrightarrow G_{n+1}^{rel}(Y, X; f) \longrightarrow G_n(X) \longrightarrow G_n(Y, X; f) \longrightarrow \cdots$$

called the G -sequence of f .

We use Sullivan models to compute rational relative Gottlieb groups of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$. We refer to [4] for details and work over a field of characteristic zero in this case $\mathbb{Q}$.

Definition 1. A differential graded algebra (dga) is a graded algebra $A = \bigoplus_{n \geq 0} A^n$ together with a derivation d , $d = d_n : A^n \longrightarrow A^{n+1}$ such that $d \circ d = 0$. Then (A, d) is called a cochain algebra. A graded algebra A is commutative if $a \cdot b = (-1)^{deg\ a \cdot deg\ b} b \cdot a$ for $a, b \in A$ [2, Chap. 3].

Definition 2. A Sullivan algebra is a commutative cochain algebra of the form $(\wedge V, d)$ where $V = \{V^p\}_{p \geq 2}$ and $\wedge V$ denotes the graded free commutative algebra on V . A Sullivan model for a commutative cochain algebra (A, d) is a quasi-isomorphism $m : (\wedge V, d) \longrightarrow (A, d)$ from a Sullivan algebra $(\wedge V, d)$. A Sullivan algebra is said to be minimal if the differential is decomposable, that is, $Im\ d \subset \wedge^+ V \cdot \wedge^+ V$. Moreover, if $H^0(A) = \mathbb{Q}$ then (A, d) has a minimal model which is unique up to isomorphism. If X is a nilpotent space and $A_{PL}(X)$ the commutative differential graded algebra ($cdga$) of piecewise linear forms on X , then a Sullivan model of X is a Sullivan model of $A_{PL}(X)$ [2, Chap.12].

2. DERIVATION SPACES AND THE RATIONALIZED G -SEQUENCE

Given commutative differential graded algebras (A, d_A) and (B, d_B) and a map $\phi : A \longrightarrow B$, define a ϕ -derivation of degree n to be a linear map $\theta : A^* \longrightarrow B^{*-n}$ which satisfies $\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|}\phi(x)\theta(y)$. We only consider derivations of positive degree. Let $Der_n(A, B; \phi)$ denote the vector space of all ϕ -derivations of degree n for $n > 0$. Define a linear map $D : Der_n(A, B; \phi) \longrightarrow Der_{n-1}(A, B; \phi)$ by $D(\theta) = d_B \circ \theta - (-1)^{|\theta|}\theta \circ d_A$. Then, $(Der_*(A, B; \phi), D)$ is a chain complex. In case $A = B$ and $\phi = 1_B$, the chain complex of derivations $Der_*(B, B; 1)$ is just the usual complex of derivations on the commutative differential graded algebra B [6]. If $\phi : (\wedge V, d) \longrightarrow (\wedge W, d)$ is a Sullivan minimal model of $f : X \longrightarrow Y$, then $H_n(Der(\wedge V, \wedge W; \phi), D) \cong \pi_n(map(X, Y; f)) \otimes \mathbb{Q}; n \geq 2$ [6], [1].

We note that $Der(\wedge V, B; \phi) \cong Hom(V, B)$. If $\{v_i\}$ is a basis of V , we

denote by (v_i, b) , the unique ϕ -derivation θ such that

$$\begin{cases} \theta(v_i) = b_i & b_i \in B, \\ \theta(v_j) = 0 & i \neq j. \end{cases}$$

Pre-composition with ϕ , respectively post-composition with the augmentation $\varepsilon : B \rightarrow \mathbb{Q}$, gives a map of chain complexes

$$\phi^* : Der_*(B, B; 1) \rightarrow Der_*(A, B; \phi),$$

respectively

$$\varepsilon_* : Der_*(A, B; \phi) \rightarrow Der_*(A, \mathbb{Q}; \varepsilon).$$

Definition 3. Let $\phi : V \rightarrow W$ be a map of differential graded vector spaces. Define a differential graded vector space, $Rel_*(\phi)$, called the mapping cone as follows. $Rel_n(\phi) = sV_{n-1} \oplus W_n$ with differential δ of degree -1 given by $\delta(sv, w) = (-sd_V(v), \phi(v) + d_W(w))$ [7]. There are chain maps $J : W_n \rightarrow Rel_n(\phi)$ and $P : Rel_n(\phi) \rightarrow V_{n-1}$ defined by $J(w) = (0, w)$ and $P(sv, w) = v$. These give a short exact sequence of chain complexes

$$0 \longrightarrow W_* \xrightarrow{J} Rel_*(\phi) \xrightarrow{P} V_{*-1} \longrightarrow 0,$$

which leads to a long exact sequence in homology

$$\cdots \rightarrow H_{n+1}(Rel(\phi)) \xrightarrow{H(P)} H_n(V) \xrightarrow{H(\phi)} H_n(W) \xrightarrow{H(J)} H_n(Rel(\phi)) \rightarrow \cdots,$$

whose connecting homomorphism is $H(\phi)$. We refer to this sequence as the long exact homology sequence of ϕ .

We use Theorem 3.3 of [6]

Theorem. Let $f : X \rightarrow Y$ be a map between simply connected CW-complexes and $\phi : (\wedge V, d) \rightarrow (\wedge W, d)$ its Sullivan model. The long exact sequence induced by $f_* : map(X, X; 1) \rightarrow map(X, Y; f)$ on rational homotopy groups is equivalent to the long exact homology sequence of the map

$$\phi^* : Der_*(\wedge W, \wedge W; 1) \rightarrow Der_*(\wedge V, \wedge W; \phi)$$

induced by the minimal model $\phi : (\wedge V, d) \rightarrow (\wedge W, d)$ of the map $f : X \rightarrow Y$.

Definition 4. Given a commutative differential graded algebra map $\phi : A \rightarrow B$, we have the following commutative diagram of differential

graded vector spaces;

$$\begin{array}{ccc} \text{Der}_*(B, B; 1) & \xrightarrow{\phi^*} & \text{Der}_*(A, B; \phi) \\ \downarrow \varepsilon_* & & \downarrow \varepsilon_* \\ \text{Der}_*(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi}^*} & \text{Der}_*(A, \mathbb{Q}; \varepsilon). \end{array}$$

Here, ε denotes the augmentation of either A or B . On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder ($n \geq 2$),

$$\begin{array}{ccccccc} \cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\phi^*)) & \xrightarrow{H(P)} & H_n(\text{Der}(B, B; 1)) & \xrightarrow{H(\phi^*)} & H_n(\text{Der}(A, B; \phi)) \cdots \\ & & \downarrow H(\varepsilon_* \varepsilon_*) & & \downarrow H(\varepsilon_*) & & \downarrow H(\varepsilon_*) \\ \cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\widehat{\phi}^*)) & \xrightarrow{H(\widehat{P})} & H_n(\text{Der}(B, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\widehat{\phi}^*)} & H_n(\text{Der}(A, \mathbb{Q}; \varepsilon)) \cdots \end{array}$$

Definition 5. Suppose $\phi : A \longrightarrow B$ is a map of commutative differential graded algebras, we define the evaluation subgroups of ϕ by

$$G_n(A, B; \phi) = \text{im}\{H(\varepsilon_*) : H_n(\text{Der}(A, B; \phi)) \longrightarrow H_n(\text{Der}(A, \mathbb{Q}; \varepsilon))\}.$$

In the special case $A = B$ and $\phi = 1_B$, we refer to the Gottlieb group of B , and use the notation $G_n(B)$. If $B = (\wedge V, d)$ is a model of a simply connected space X , then an element $\alpha \in G_n(B)$ is represented by a linear mapping $f : V^n \longrightarrow \mathbb{Q}$ that extends into a derivation θ of $\wedge V$ such that $\delta\theta = 0$. Moreover, $G_n(B) \cong G_n(X_{\mathbb{Q}})$ [2, Proposition 29.8].

The n^{th} relative evaluation subgroup of ϕ is defined by;

$$G_n^{\text{rel}}(A, B; \phi) = \text{im}\{H(\varepsilon_*, \varepsilon_*) : H_n(\text{Rel}(\phi^*)) \longrightarrow H_n(\text{Rel}(\widehat{\phi}^*))\}.$$

Then the image of the upper long sequence in the lower, of the ladder above, gives a sequence

$$\cdots \xrightarrow{H(\widehat{J})} G_{n+1}^{\text{rel}}(A, B; \phi) \xrightarrow{H(\widehat{P})} G_n(B) \xrightarrow{H(\widehat{\phi}^*)} G_n(A, B; \phi) \xrightarrow{H(\widehat{J})} \cdots \quad (1)$$

that terminates in $G_2(A, B; \phi)$.

We refer to this sequence as the G -sequence of the map $\phi : A \longrightarrow B$. This can be applied to the minimal model $\phi : (\wedge V, d) \longrightarrow (\wedge W, d)$ of the map $f : X \longrightarrow Y$ as stated and proved in [6, Theorem 3.5].

3. THE INCLUSION $Gr(2, n) \hookrightarrow Gr(2, n+1)$

Let $Gr(k, n)$ be the Grassmann manifold of k -dimensional subspaces of $\mathbb{C}^n$. The cohomology ring $H^*(Gr(k, n), \mathbb{Q})$ is generated by the Chern classes $c_i \in H^{2i}(Gr(k, n), \mathbb{Q})$, for $1 \leq i \leq k$. Further, the cohomology ring has a presentation

$$H^*(Gr(k, n), \mathbb{Q}) = \wedge(c_1, c_2, \dots, c_k) / (h_{n-k+1}, \dots, h_n),$$

as the quotient of the polynomial ring generated by $c_1, c_2, \dots, c_k$, $|c_i| = 2i$, modulo the ideal generated by the elements h_j , $n - k + 1 \leq j \leq n$. Here, h_j is defined as the $2j^{th}$ degree term in the Taylor's expansion of $(1 + c_1 + c_2 + c_3 + \dots + c_k)^{-1}$ where $(1 + c_1 + c_2 + c_3 + \dots + c_k)$ is the total Chern class [4].

In particular, the cohomology rings of $Gr(2, n)$ and $Gr(2, n+1)$ are:

$$\begin{aligned} H^*(Gr(2, n), \mathbb{Q}) &= \wedge(y_2, y_4) / (h_{n-1}, h_n) \text{ and} \\ H^*(Gr(2, n+1), \mathbb{Q}) &= \wedge(x_2, x_4) / (h_n, h_{n+1}) \text{ respectively.} \end{aligned}$$

The minimal model of $Gr(2, n)$ is

$$\left(\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d \right) \text{ with } d(y_2) = d(y_4) = 0, d(y_{2n-3}) = h_{n-1}, d(y_{2n-1}) = h_n. \text{ In the same way, a model of } Gr(2, n+1) \text{ is given by}$$

$$\left(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d \right) \text{ with } dx_2 = dx_4 = 0, dx_{2n-1} = h_n \text{ and } dx_{2n+1} = h_{n+1}.$$

Lemma 1. $h_{n+1} = -x_2 h_n - x_4 h_{n-1}$.

Proof. Write the Taylor series $(1 + x_2 + x_4)^{-1} = 1 + h_1 + h_2 + \dots$ where, $|h_i| = 2i$. From $(1 + x_2 + x_4)(1 + x_2 + x_4)^{-1} = 1$, one gets the relation $h_{n+1} = -x_2 h_n - x_4 h_{n-1}$ $\square$

In particular, h_{n+1} is co-boundary in $\left(\wedge(x_2, x_4, x_{2n-3}, x_{2n-1}), d \right)$, that is, there exists α of degree $2n+1$ such that $d\alpha = h_{n+1}$.

Theorem 1. Let $B = \left(\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d \right)$. Then $G_n(B) = \langle [y_{2n-3}^*], [y_{2n-1}^*] \rangle$.

Proof. Let $\alpha_{2n-1} = (y_{2n-1}, 1)$ and $\alpha_{2n-3} = (y_{2n-3}, 1)$. Then $\delta\alpha_{n-1} = \delta\alpha_{2n-3} = 0$. Moreover, α_{2n-3} and α_{2n-1} can not be boundaries for degree reason. Therefore, $[\alpha_{2n-3}]$ and $[\alpha_{2n-1}]$ are non zero homology classes in $H_*(Der(B, B; 1))$. Further, $\varepsilon_*(\alpha_{2n-3}) = y_{2n-3}^*$ and $\varepsilon_*(\alpha_{2n-1}) = y_{2n-1}^*$.

As $Gr(2, n)$ is a finite CW -complex then $G_{even}(B) = 0$ [2, Pg.379]. Hence, $G_n(B) = \langle [y_{2n-3}^*], [y_{2n-1}^*] \rangle$. $\square$

The inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ has a model of the form

$$\phi : \wedge V = \left(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d \right) \longrightarrow \left(\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d \right) = B$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_{2n-1}) = y_{2n-1}$ and $\phi(x_{2n+1}) = \alpha$ where $d\alpha = h_{n+1}$ by Lemma 1.

Theorem 2. Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ and $\phi : (\wedge V, d) \longrightarrow (B, d)$ its Sullivan model, then $G_*(\wedge V, B; \phi) \cong \langle [x_{2n-1}^*], [x_{2n+1}^*] \rangle$.

Proof. As $Gr(2, n)$ is formal, $Der(\wedge V, B; \phi) \xrightarrow{\cong} Der(\wedge V, H(B); f \circ \phi)$ where $f : B \xrightarrow{\cong} H(B)$ is a quasi isomorphism. Similarly, since B is formal $Der(B, B; 1) \simeq Der(B, H(B); f)$. Define $\theta_{2n-1} = (x_{2n-1}, 1)$, $\theta_{2n+1} = (x_{2n+1}, 1)$ in $Der(\wedge V, H(B); f \circ \phi)$. Then $\delta\theta_{2n-1} = \delta\theta_{2n+1} = 0$. Moreover, $[\theta_{2n-1}]$ and $[\theta_{2n+1}]$ are nonzero cohomology classes in $H_*(Der(\wedge V, H^*(B); f \circ \phi))$.

We note that, $\theta_2 = (x_2, 1)$ and $\theta_4 = (x_4, 1)$ are not cycles in $Der(\wedge V, H(B); f \circ \phi)$ [8].

Further, $H(\varepsilon_*)([\theta_{2n-1}]) = [x_{2n-1}^*] \in G_{2n-1}(\wedge V, B; f \circ \phi)$. In a similar way, $H(\varepsilon_*)([\theta_{2n+1}]) = [x_{2n+1}^*] \in G_{2n+1}(\wedge V, B; f \circ \phi)$. It then follows that $G_*(\wedge V, B; f \circ \phi) = \langle [x_{2n-1}^*], [x_{2n+1}^*] \rangle$. $\square$

Theorem 3. Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ and

$$\phi : (\wedge V, d) = \left(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d \right) \longrightarrow \left(\wedge V(y_2, y_4, y_{2n-3}, y_{2n-1}), d \right) = B$$

its Sullivan model, then $G_*^{rel}(\wedge V, B; \phi) = \langle [(x_{2n-3}^*, 0)], [(0, y_{2n+1}^*)] \rangle$.

Proof. Consider the diagram below [6].

$$\begin{array}{ccccc} Der(B, H(B); f) & \xrightarrow{\phi^*} & Der(\wedge V, H(B); f \circ \phi) & \xrightarrow{J} & Rel(\phi^*) \\ \downarrow \varepsilon_* & & \downarrow \varepsilon_* & & \downarrow (\varepsilon_*, \varepsilon_*) \\ Der(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi}^*} & Der(\wedge V, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{J}} & Rel(\widehat{\phi}^*). \end{array}$$

Let $\alpha_{2n-1} = (y_{2n-1}, 1)$, $\alpha_{2n-3} = (y_{2n-3}, 1) \in Der(B, H(B); f)$ and $\theta_{2n-1}, \theta_{2n+1} \in Der(\wedge V, H^*(B); \phi)$ as defined above. Then $\phi^*(\alpha_{2n-1}) = \theta_{2n-1}$ and $\phi^*(\alpha_{2n-3}) = 0$.

Further, $D(\alpha_{2n-1}, 0) = (0, \theta_{2n-1})$, $D(\alpha_{2n-3}, 0) = (0, 0)$ and $D(0, \theta_{2n-1}, 0) = 0 = D(0, \theta_{2n+1})$. Therefore, $[(\alpha_{2n-3}, 0)]$ and $[(0, \theta_{2n+1})]$ are non zero homology classes in $H_*(\text{Rel}(\phi^*))$. Moreover, $H(\varepsilon_*, \varepsilon_*)([\alpha_{2n-3}, 0]) = [(x_{2n-3}^*, 0)]$ and $H(\varepsilon_*, \varepsilon_*)([(0, \theta_{2n+1})]) = [(0, y_{2n+1}^*)]$. A straightforward computation shows that $[(x_{2n-3}^*, 0)]$ and $[(0, y_{2n+1}^*)]$ span $H(\varepsilon_*, \varepsilon_*)$. $\square$

The G -sequence reduces to

$$\begin{aligned} 0 &\longrightarrow G_{2n+1}(\wedge V, B; \phi) \xrightarrow[\simeq]{H(J)} G_{2n+1}^{\text{rel}}(\wedge V, B; \phi) \longrightarrow 0 \cdots \\ \cdots 0 &\longrightarrow G_{2n-1}(B) \xrightarrow[\simeq]{H(\phi^*)} G_{2n-1}(\wedge V, B; \phi) \longrightarrow 0 \cdots \\ \cdots 0 &\longrightarrow G_{2n-2}^{\text{rel}}(\wedge V, B; \phi) \xrightarrow[\simeq]{H(P)} G_{2n-3}(B) \longrightarrow 0. \end{aligned}$$

and is exact.

Example 1. Consider $Gr(2, 4) \hookrightarrow Gr(2, 5)$. A model of the inclusion is given by

$$\phi : \wedge V = \left(\wedge (x_2, x_4, x_7, x_9), d \right) \longrightarrow \left(\wedge (y_2, y_4, y_5, y_7), d \right) = B,$$

where $dx_2 = dx_4 = 0$, $dx_7 = x_4^2 - 3x_2x_4 + x_2^4$, $dx_9 = 4x_2^3x_4 - 3x_2x_4^2 - x_2^5$, $dy_2 = dy_4 = 0$, $dy_5 = 2y_2y_4 - y_2^3$ and $dy_7 = y_4^2 - 3y_2^2y_4 + y_2^4$.

Moreover, $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_7) = y_7$ and $\phi(x_9) = -y_2y_7 - y_4y_4$.

We compute $G_*^{\text{rel}}(\wedge V, B; \phi)$. Let $\alpha_7 = (y_7, 1)$, $\alpha_5 = (y_5, 1) \in \text{Der}_7(B, H(B); f)$ where $f : B \xrightarrow{\cong} H(B)$ and $\theta_7 = (x_7, 1)$, $\theta_9 = (x_9, 1) \in \text{Der}(\wedge V, H^*(B); f \circ \phi)$ then $\phi^*(\alpha_7) = \theta_7$ and $\phi^*(\alpha_5) = 0$. Moreover, $D(s\alpha_7, 0) = (0, \theta_7)$, $D(s\alpha_5, 0) = (0, 0)$ and $D(0, \theta_7) = D(0, \theta_9) = (0, 0)$. Hence $[(s\alpha_5, 0)]$ and $[(0, \theta_9)]$ are non zero homology classes. Moreover, $(\varepsilon_*, \varepsilon_*)(s\alpha_5, 0) = (sy_5^*, 0)$, $(\varepsilon_*, \varepsilon_*)(0, \theta_9) = (0, y_9^*)$. Therefore

$$G_*^{\text{rel}}(\wedge V, B; \phi) = \langle [(0, x_9^*)], [(sy_5^*, 0)] \rangle.$$

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RELATIVE GOTTLIEB GROUPS OF THE PLÜCKER EMBEDDING OF SOME COMPLEX GRASSMANIANS

J.-B. GATSINZI, V. ONYANGO-OTIENO, AND P. A. OTIENO

ABSTRACT. Let $\text{Gr}(k, n)$ be the Grassmann manifold of k -linear subspaces in $\mathbb{C}^n$. We compute rational relative Gottlieb groups of the Plücker embedding $i : \text{Gr}(2, n) \rightarrow \mathbb{C}P^{N-1}$, where $N = n(n-1)/2$.

1. Introduction

All spaces are assumed to be simply connected CW complexes of finite type. We denote by $h : X \rightarrow X_{\mathbb{Q}}$ the rationalization of X [3, §9]. Let $f : X \rightarrow Y$ be a pointed continuous mapping and $\text{map}(X, Y; f)$ the component of f in the space of continuous mappings from X to Y . Consider the evaluation map $\text{ev} : \text{map}(X, Y; f) \rightarrow Y$. The n th evaluation subgroup of f , $G_n(Y, X; f)$, is the image of $\pi_n(\text{ev})$ in $\pi_n(Y)$ [17]. In the special case where $X = Y$ and $f = 1_X$, one obtains the Gottlieb group $G_n(X)$ of X [4]. It is also obtained as the image of the connecting map of the long homotopy exact sequence of the universal fibration of fibre X . Gottlieb groups play an important role in topology. For instance, if $G_n(X) = 0$, then any fibration $X \rightarrow E \rightarrow S^{n+1}$ admits a section [4, Corollary 2-7]. If X is a finite CW complex and Y a simply connected CW complex of finite type, then the rationalization $h : Y \rightarrow Y_{\mathbb{Q}}$ induces a rationalization $h_* : \text{map}(X, Y; f) \rightarrow \text{map}(X, Y; h \circ f)$ [5]. Therefore

$$\text{ev}_*(\pi_*(\text{map}(X, Y; f)) \otimes \mathbb{Q}) \cong \text{ev}_*(\pi_*(\text{map}(X, Y_{\mathbb{Q}}; h \circ f))).$$

In [18] Lee and Woo introduce relative evaluation groups $G_n^{\text{rel}}(Y, X; f)$ and obtain a long sequence

$$\cdots \rightarrow G_{n+1}^{\text{rel}}(Y, X; f) \rightarrow G_n(X) \rightarrow G_n(Y, X; f) \rightarrow G_n^{\text{rel}}(Y, X; f) \rightarrow \cdots,$$

called G -sequence [8]. This sequence is exact in some cases, for instance if f is a homotopy monomorphism [13].

In this paper we describe the rational relative Gottlieb subgroup of the Plücker embedding $i : \text{Gr}(2, n) \hookrightarrow \mathbb{C}P^{N-1}$ of the Grassmanian of 2-subspaces of $\mathbb{C}^n$ into the complex projective space of dimension $N-1$ where $N = n(n-1)/2$.

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1)/2. More specifically, we show that: $G_*^{rel}(\mathbb{C}P^{N-1}, Gr(2, n)_{\mathbb{Q}}, h \circ i)$ splits as the direct sum of the suspension of $G_*(Gr(2, n)_{\mathbb{Q}})$ and $G_{2N-1}(\mathbb{C}P^{N-1}, Gr(2, n)_{\mathbb{Q}}, h \circ i)$, where $h : \mathbb{C}P^{N-1} \rightarrow \mathbb{C}P_{\mathbb{Q}}^{N-1}$ is the rationalization.

2. Relative Gottlieb groups

We work with rational homotopy models of simply connected topological spaces which were introduced by Quillen and Sullivan [14, 16]. In this section we give relevant definitions and fix notation. Details can be found in [3]. All vector spaces and algebras are over the field of rational numbers $\mathbb{Q}$.

Definition 1. A differential graded algebra (A, d) is a graded algebra $A = \bigoplus_{n \geq 0} A^n$ together with a differential $d : A^n \rightarrow A^{n+1}$ which is a derivation. The degree of an element a is denoted by $|a|$. We assume that (A, d) is 1-connected, that is, $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$. A graded algebra A is called commutative if $ab = (-1)^{|a||b|}ba$, $a, b \in A$. A commutative differential graded algebra (cdga for short) (A, d) is called a Sullivan algebra if $A = S(V^{even}) \otimes E(V^{odd})$, where $V = \bigoplus_{k \geq 2} V^k$. It will be denoted by $(\wedge V, d)$.

Definition 2. A Sullivan algebra $(\wedge V, d)$ is called minimal if $dV \subset \wedge^{\geq 2} V$. A Sullivan model of (A, d) is given by a Sullivan algebra $(\wedge V, d)$ together with a quasi-isomorphism $f : (\wedge V, d) \rightarrow (A, d)$. It is unique up to isomorphism.

If X is a simply connected space of finite type, then the (minimal) Sullivan model of X is the (minimal) Sullivan model of the cdga $A_{PL}(X)$ of polynomial differential forms on X [3, §10]. A simply connected topological space X is called formal if there exists a quasi-isomorphism $(\wedge V, d) \rightarrow H^*(X, \mathbb{Q})$, where $(\wedge V, d)$ is a Sullivan model of X .

The complex Grassmann manifold $Gr(k, n)$ is the space of k dimensional subspaces of $\mathbb{C}^n$. As $Gr(k, n) = U(n)/(U(k) \times U(n-k))$ is a homogeneous space of equal rank as well as a Kähler manifold, hence it is formal [16, §12], [2]. A Sullivan model of $Gr(k, n)$ can be computed using Proposition 15.6 of [3] and it is explicitly given by Murillo [11, Theorem 2]. Moreover $Gr(k, n)$ is a complex manifold of dimension $k(n-k)$ for which the rational cohomology is the quotient algebra

$$\wedge(y_2, y_4, \dots, y_{2k}) / (h_{n-k+1}, \dots, h_{n-1}, h_n),$$

where h_j is the polynomial of degree $2j$ in the Taylor expansion of the expression $1/(1 + y_2 + \dots + y_{2k})$ [6]. A Sullivan model is given by

$$(\wedge(y_2, \dots, y_{2k}, y_{2n-2k+1}, \dots, y_{2n-1}), d),$$

where $dy_{2i} = 0$, $dy_{2n-2k+2j-1} = h_{n-k+j}$, $j = 1, \dots, k$. Moreover this model is minimal. In particular, the minimal Sullivan model of $Gr(2, n)$ is

$$\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), \quad dy_2 = dy_4 = 0, \quad dy_{2n-3} = h_{n-1}, \quad dy_{2n-1} = h_n.$$

It can be shown by induction that

$$N = \binom{n}{k} > k(n-k).$$

Therefore $H^{\geq 2N}(\mathrm{Gr}(k, n), \mathbb{Q}) = 0$. Hence y_2^N is a coboundary in

$$(\wedge(y_2, \dots, y_{2k}, y_{2n-2k+1}, \dots, y_{2n-1}), d).$$

We recall that the minimal Sullivan model of $\mathbb{C}P^{N-1}$ is given by

$$(\wedge(x_2, x_{2N-1}), d),$$

where $dx_2 = 0$ and $dx_{2N-1} = x_2^N$. A Sullivan model of the inclusion $i : \mathrm{Gr}(2, n) \rightarrow \mathbb{C}P^{N-1}$ is then

$$(1) \quad \phi : (\wedge(x_2, x_{2N-1}), d) \rightarrow (\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d),$$

where $\phi(x_2) = y_2$ and $\phi(x_{2N-1}) = y$, where $dy = y_2^N$.

Recall that if $\phi : (A, d_A) \rightarrow (B, d_B)$ is a map of chain complexes, the mapping cone of ϕ , denoted by $\mathrm{Rel}(\phi)$ is given by

$$\mathrm{Rel}(\phi)_* = (sA_{*-1} \oplus B_*, D),$$

where the differential is defined by $D(sa, b) = (-sd_A(a), \phi(a) + d_B b)$ [9] or [10, p. 46]. Define chain maps $J : B_n \rightarrow \mathrm{Rel}_n(\phi)$ and $P : \mathrm{Rel}_n(\phi) \rightarrow A_{n-1}$ by $J(b) = (0, b)$ and $P(sa, b) = a$. There is an exact sequence of chain complexes

$$0 \rightarrow B_* \xrightarrow{J} \mathrm{Rel}_*(\phi) \xrightarrow{P} A_{*-1} \rightarrow 0$$

which induces a long exact sequence in homology [10, Proposition 4.3].

Definition 3. Let X be a topological space. We say $\alpha \in \pi_n(X)$ is a Gottlieb element if the map: $f \vee 1_X : S^n \vee X \rightarrow X$ extends to $S^n \times X$, where f represents the homotopy class α [4].

Gottlieb elements form a subgroup of $\pi_*(X)$ which will be denoted by $G_*(X)$. It comes from the definition that $G_*(X)$ is the image of $\pi_*(\mathrm{ev}) : \pi_*(\mathrm{aut}_1 X, 1_X) \rightarrow \pi_*(X, x_0)$, where ev is the evaluation map at x_0 .

Definition 4. Let $\phi : (A, d) \rightarrow (B, d)$ be a morphism of cdga's. A ϕ -derivation of degree k is a linear mapping $\theta : A^n \rightarrow B^{n-k}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$. We denote by $\mathrm{Der}_n(A, B; \phi)$ the vector space of ϕ -derivations of degree n and by $\mathrm{Der}(A, B; \phi) = \bigoplus_n \mathrm{Der}_n(A, B; \phi)$ the $\mathbb{Z}$ -graded vector space of all ϕ -derivations. The differential on $\mathrm{Der}(A, B; \phi)$ is defined by $\delta\theta = d\theta - (-1)^k\theta d$. We will restrict to derivations of positive degree, however in degree one, we only consider those derivations which are cycles.

If $\phi : A \rightarrow A$ is the identity mapping, we simply write $\mathrm{Der} A$ for $\mathrm{Der}(A, A; 1_A)$. Moreover if $A = \wedge V$ where $\{v_1, v_2, \dots\}$ is a basis of V and $\phi : (\wedge V, d) \rightarrow (B, d)$ is a morphism of cdga's, we denote by (v_i, b) the unique ϕ -derivation θ such that $\theta(v_i) = b$ and zero on other elements of the basis.

Define the Gottlieb group of $(\wedge V, d)$

$$G_n(\wedge V) = \{[\theta] \in H_n(\text{Der } \wedge V, \delta) : \theta(v) = 1, v \in V^n\}.$$

Hence $G_*(\wedge V) \cong \text{im } H_*(\epsilon_*)$, where $\epsilon_* : \text{Der } \wedge V \rightarrow \text{Der}(\wedge V, \mathbb{Q}; \epsilon)$ is the post composition with the augmentation map $\epsilon : \wedge V \rightarrow \mathbb{Q}$. If X is simply connected and $(\wedge V, d)$ is the minimal Sullivan model of X , then $G_n(\wedge V) \cong G_n(X_{\mathbb{Q}})$, where $h : X \rightarrow X_{\mathbb{Q}}$ is the rationalization [3, Propostion 29.8].

Let $\phi : (\wedge V, d) \rightarrow (\wedge W, d)$ be a Sullivan model of $f : X \rightarrow Y$. It induces a chain map $\phi^* : \text{Der}(\wedge W) \rightarrow \text{Der}(\wedge V, \wedge W; \phi)$ by pre-composition by ϕ . We get the following commutative diagram.

$$(2) \quad \begin{array}{ccccc} \text{Der } \wedge W & \xrightarrow{\phi^*} & \text{Der}(\wedge V, \wedge W; \phi) & \xrightarrow{J} & \text{Rel}(\phi^*) \\ \downarrow \epsilon_* & & \downarrow \epsilon_* & & \downarrow (\epsilon_*, \epsilon_*) \\ \text{Der}(\wedge W, \mathbb{Q}, \epsilon) & \xrightarrow{\widehat{\phi^*}} & \text{Der}(\wedge V, \mathbb{Q}, \epsilon) & \xrightarrow{\widehat{J}} & \text{Rel}(\widehat{\phi^*}). \end{array}$$

Then rational evaluation subgroups are corresponding images in the lower ladder induced in homology by vertical maps. Therefore there is a long sequence

$$\cdots \rightarrow G_n(\wedge W) \xrightarrow{H(\widehat{\phi^*})} G_n(\wedge V, \wedge W; \phi) \xrightarrow{H(\widehat{J})} G_n^{\text{rel}}(\wedge V, \wedge W; \phi) \xrightarrow{H(\widehat{P})} \cdots.$$

We will use the following result for our computations [9, Theorem 2.1], [1, Corollary 7].

Theorem 5. *Let $f : X \rightarrow Y$ be a map between simply connected CW complexes where X is of finite type and $\phi : (\wedge V, d) \rightarrow (\wedge W, d)$ its Sullivan model. The long exact sequence induced by the map $f_* : \text{map}(X, X; 1_X) \rightarrow \text{map}(X, Y; f)$ on rational homotopy groups is equivalent to the long exact sequence of*

$$\phi^* : \text{Der}(\wedge W, d) \rightarrow \text{Der}(\wedge V, \wedge W; \phi).$$

We consider the particular case where f is the inclusion $i : \text{Gr}(2, n) \rightarrow \mathbb{C}P^{n-1}$ for which the Sullivan model is given by (1). A direct computation shows $G_*(\wedge W) = \langle [y_{2n-3}^*], [y_{2n-1}^*] \rangle$. See [15] for a general result on rational Gottlieb groups of flag manifolds.

Lemma 6. *Let $N = n(n-1)/2$. Then $2(N-1) > 4n-8$ for $n \geq 4$.*

Proof. Here $2(N-1) = n(n-1) - 2$. The inequality $n(n-1) > 4n-8$ holds for $n = 4$. For $n \geq 5$, then $n(n-1) \geq 4n$, hence $n(n-1) - 2 > 4n-8$. $\square$

Lemma 7. *Let $\phi : (\wedge(x_2, x_{2N-1}), d) \rightarrow (\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d)$, where $\phi(x_2) = y_2$, $\phi(x_{2N-1}) = y$ and $dy = y_2^N$. There is a ϕ -derivation α_2 such that $\alpha_2(x_2) = 1$ and which is a cycle.*

Proof. We need to define α_2 on x_{2N-1} so that

$$d\alpha_2(x_{2N-1}) - \alpha_2(dx_{2N-1}) = 0.$$

Hence

$$\begin{aligned} d\alpha_2(x_{2N-1}) - \alpha_2(dx_{2N-1}) &= d\alpha_2(x_{2N-1}) - \alpha_2(x_2^N) \\ &= d\alpha_2(x_{2N-1}) - Ny_2^{N-1}. \end{aligned}$$

As the dimension of $\text{Gr}(2, n)$ is less than $2(N-1)$ by Lemma 6, then Ny_2^{N-1} is a coboundary, that is, $Ny_2^{N-1} = dz$. Define $\alpha_2(x_{2N-1}) = z$. Moreover $\delta\alpha_2 = 0$. As $\alpha_2(x_2) = 1$, then α_2 cannot be a boundary, hence it represents a non zero homology class in $H_*(\text{Der}(\wedge V, \wedge W; \phi), \delta)$. $\square$

It is easily seen that the ϕ -derivation $\alpha_{2N-1} = (x_{2N-1}, 1)$ represents a non zero homology class as well. As $H(\epsilon_*)([\alpha_2]) = [x_2^*]$ and $H(\epsilon_*)([\alpha_{2N-1}]) = [x_{2N-1}^*]$, we get the following result.

Proposition 8. $G_*(\wedge V, \wedge W; \phi) = \mathbb{Q}([x_2^*], [x_{2N-1}^*])$.

We decompose $\text{Rel}_*(\phi^*) = \text{Ker}(\epsilon_*, \epsilon_*) \oplus K$. As K is isomorphic to $\text{im}(\epsilon_*, \epsilon_*)$, a non zero element in $G_*^{\text{rel}}(\wedge V, \wedge W; \phi)$ is represented by an element in K , modulo the kernel of (ϵ_*, ϵ_*) . Moreover the vector space K is spanned by

$$\{(0, \alpha_2), (0, \alpha_{2N-1}), (s\beta_2, 0), (s\beta_4, 0), (s\beta_{2n-3}, 0), (s\beta_{2n-1}, 0)\},$$

where $\beta_j = (y_j, 1)$.

Lemma 9. $(0, \alpha_2) = J(\alpha_2)$ represents a non zero homology class in $H_2(\text{Rel}_*(\phi^*))$.

Proof. As $(0, \alpha_2)$ is a cycle, it remains to show that it is not a boundary. Otherwise there exist $\beta'_2 \in \text{Der}(\wedge W)$ and $\alpha_3 \in \text{Der}(\wedge V, \wedge W; \phi)$ such that $d(s\beta'_2, \alpha_3) = (-s\delta\beta'_2, \delta\alpha_3 + \phi^*(\beta'_2)) = (0, \alpha_2)$. Hence $\delta\beta'_2 = 0$ and $\phi^*(\beta'_2) = \alpha_2 - \delta\alpha_3$. For degree reasons $\alpha_3(x_2) = 0$, hence $(\delta\alpha_3)(x_2) = 0$. Therefore $\phi^*(\beta'_2)(x_2) = \alpha_2(x_2) = 1$. We deduce that $\beta'_2(y_2) = 1$. Moreover β'_2 cannot be a cycle, otherwise $H(\epsilon_*)([\beta'_2]) = [y_2^*] \neq 0$ would be a non zero Gottlieb element in $\pi_2(X_{\mathbb{Q}})$, which is impossible [3, Proposition 28.8]. We conclude that $[(0, \alpha_2)]$ is non zero homology class in $H_2(\text{Rel}(\phi^*))$. $\square$

We can now determine $G_*^{\text{rel}}(\wedge V; \wedge W; \phi)$.

Theorem 10. The relative rational Gottlieb group $G_*^{\text{rel}}(\wedge V; \wedge W; \phi)$ is isomorphic to the direct sum $sG_*(\wedge W) \oplus G_{2N-1}(\wedge V, \wedge W; \phi)$. Moreover the sequence

$$(3) \quad 0 = G_2(\wedge W) \longrightarrow G_2(\wedge V, \wedge W; \phi) \xrightarrow{H(\hat{J})} G_2^{\text{rel}}(\wedge V, \wedge W; \phi) \longrightarrow 0$$

is not exact.

Proof. A short calculation shows that $d\beta_2 \neq 0$ and $d\beta_4 \neq 0$ (see [12]). Moreover $(s\beta_{2n-3}, 0)$ and $(s\beta_{2n-1}, 0)$ represent non zero homology classes in $\text{Rel}_*(\phi^*)$. However, in $\text{Rel}_*(\hat{\phi}^*)$, one gets $d(sy_2^*, 0) = (0, \hat{\phi}^*(y_2^*)) = (0, x_2^*)$. Hence $H(\epsilon_*, \epsilon_*)([(0, \alpha_2)]) = [(0, x_2^*)]$ is zero in $H(\text{Rel}_*(\hat{\phi}^*))$. We conclude that

$$G_2^{\text{rel}}(\wedge V, \wedge W; \phi) = 0.$$

Therefore

$$\begin{aligned} G_*^{rel}(\wedge V, \wedge W; \phi^*) &= \text{im } H(\epsilon_*, \epsilon_*) \\ &= \langle [(sy_{2n-3}^*, 0)], [(sy_{2n-1}^*, 0)], [(0, x_{2N-1}^*)] \rangle \\ &= sG_*(\wedge W) \oplus G_{2N-1}(\wedge V, \wedge W; \phi). \end{aligned}$$

The sequence (3) is not exact as $H(\widehat{J})([x_2^*]) = 0$. Hence $H(\widehat{J})$ is not injective. $\square$

Remark 11. It was incorrectly stated that $G_*^{rel}(\wedge V, \wedge W; \phi^*) = 0$ in [7].

Example 12. Consider $\text{Gr}(2, 4)$ for which the minimal Sullivan model is

$$(\wedge(y_2, y_4, y_5, y_7), d),$$

where $dy_2 = dy_4 = 0$, $dy_5 = y_2^3 - 2y_2y_4$, $dy_7 = y_4^2 - y_2^2y_4$. The inclusion $\text{Gr}(2, 4) \rightarrow \mathbb{C}P^5$ is modelled by

$$\phi : (\wedge(x_2, x_{11}), d) \rightarrow (\wedge(y_2, y_4, y_5, y_7), d),$$

where $\phi(x_2) = y_2$. As $\text{Gr}(2, 4)$ is a smooth manifold of dimension 8, then the cohomology class $[y_2^6]$ is zero. Therefore there exists $y \in \wedge W$ such that $dy = y_2^6$. A short computation shows that $y_2^5 = d(y_2^2y_5 - 2y_4y_5 - 4y_2y_7)$, therefore $y_2^6 = d(y_2^3y_5 - 2y_2y_4y_5 - 4y_2^2y_7)$. Hence $\phi(x_{11}) = y_2^3y_5 - 2y_2y_4y_5 - 4y_2^2y_7$.

Let $\alpha_2 \in \text{Der}(\wedge V, \wedge W; \phi)$ be the ϕ -derivation defined by $\alpha_2(x_2) = 1$ and $\alpha_2(x_{11}) = 6(y_2^2y_5 - 2y_4y_5 - 4y_2y_7)$. Then α_2 is a cycle, hence $H(\epsilon_*)([\alpha_2]) = [x_2^*] \in G_2(\wedge V, \wedge W; \phi)$ is in the kernel of $H(\widehat{J})$.

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