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Abstract

We consider scattering of a time-harmonic acoustic incident plane wave by a sound soft smooth object with Lipschitz boundary. The application of conventional boundary or finite element methods, have computational cost that grows linearly respect to the frequency of the incident wave. Recent research has been devoted in finding methods which does not loose robustness as frequency of the incident wave increases. *Arden, Chandler-Wilde and Langdon* proposed a collocation method to solve a high frequency scattering by convex polygons. They use a boundary element method, and incorporating products of plane wave basis functions with piecewise polynomials supported on a graded mesh into approximation space. They demonstrated via numerical experiments the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to frequency. Here we proposed a collocation method for high frequency scattering by smooth objects (objects with no corners, e.g. a circle). We applied same approximations as theirs, but employing uniform mesh. We demonstrate through numerical experiments the logarithmical grow of the solutions as frequency increases, with much reduced computational cost.

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Abstract. We consider scattering of a time-harmonic acoustic incident plane wave by a sound soft smooth object with Lipschitz boundary. The application of conventional boundary or finite element methods, have computational cost that grows linearly respect to the frequency of the incident wave. Recent research has been devoted in finding methods which does not loose robustness as frequency of the incident wave increases. *Arden, Chandler-Wilde and Langdon* proposed a collocation method to solve a high frequency scattering by convex polygons. They use a boundary element method, and incorporating products of plane wave basis functions with piecewise polynomials supported on a graded mesh into approximation space. They demonstrated via numerical experiments the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to frequency. Here we proposed a collocation method for high frequency scattering by smooth objects (objects with no corners, e.g. a circle). We applied same approximations as theirs, but employing uniform mesh. We demonstrate through numerical experiments the logarithmical grow of the solutions as frequency increases, with much reduced computational cost.

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1. Introduction

We consider the two-dimensional problem of scattering of a time-harmonic acoustic incident plane wave

$$u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}, \text{ in } D := \mathbb{R}^2\backslash\bar{\Omega},$$

by a smooth convex sound-soft obstacle Ω, with Lipschitz boundary Γ. Here $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{d} = (\sin\theta, -\cos\theta) \in \mathbb{R}^2$ is a unit vector representing the direction of the incident field, and the frequency of the incident wave is proportional to the wavenumber $k > 0$. The scattered field $u^s := u - u^i \in C^2(\bar{D})$ (where u and u^i denote the total and incident field respectively) satisfies the *Helmholtz equation*

$$\Delta u^s + k^2 u^s = 0, \text{ in } D, \tag{1}$$

We consider a sound-soft boundary condition,

$$u = 0, \text{ or } u^s = -u^i, \text{ on } \Gamma \tag{2}$$

we also need the Sommerfeld radiation condition,

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$$\lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0, \tag{3}$$

where $r := |\mathbf{x}|$ and the limit holds uniformly in $\mathbf{x}/|\mathbf{x}|$. The Sommerfeld radiation condition is essential to scattering problems because it ensures that the scattered field is not reflected back from infinity.

We can reformulate (1 - 3) using Green's theorems [12] and following the usual coupling procedure to obtain a Fredholm integral equation of the second kind for $\frac{\partial u}{\partial \mathbf{n}}$

$$\frac{1}{2}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) + \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{x})} + \mathrm{i}\eta\Phi(\mathbf{x},\mathbf{y})\right)\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = f(\mathbf{x}), \quad \mathbf{x}\in\Gamma, \tag{4}$$

where $\frac{\partial u}{\partial \mathbf{n}} \in L^2(\Gamma)$ is the unknown boundary data, $\Phi(\mathbf{x},\mathbf{y}) := \frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)$ is the fundamental solution of the Helmholtz equation in $2D$, $H_0^{(1)}$ is the Hankel function of first kind of order zero, $\mathbf{n}$ is the normal direction directed out of Ω, $f(\mathbf{x}) := \frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}} + i\eta u^i(\mathbf{x})$, and $\eta \in \mathbb{R}\backslash\{0\}$ is the coupling parameter. The total field throughout D is determined by

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x}\in D.$$

In order to achieve an accurate solution of (1) for large k with reasonable computational cost, the oscillatory nature of the solution must be taken into account when designing the computational method. Recent research has focused on enriching the approximation space with highly oscillating functions, such as plane waves or Bessel functions, in order to accurately represent the scattered field. To examine this work very closely, we survey some existing literature where some schemes were proposed. A common approach is to include in the approximation space a set of plane waves propagating in multiple directions. In the boundary element context, this approach was attempted by de La Bourdonnaye [7] under the microlocal discretization method and by Perrey-Debain et al. [13, 14].The approximation space can also be enriched by including a single solution of the Helmholtz equation, the known incident field, to capture some of the oscillatory behavior of the solution. The idea behind this approach is to represent the solution of the integral equation by a linear combination of basis functions, these basis functions being the product of a slowly oscillating amplitude and an oscillatory phase factor. The slowly oscillating amplitude can be viewed as the ratio of the total field to the incident field and the oscillatory phase factor could be the known incident field. Physical optics predicts that this ratio approaches a constant in the illuminated region and zero in the shadow region as $k\to\infty$ [6]. This ratio can be approximated by conventional methods, employing piecewise polynomials. This idea was first considered by Abboud et al. [1] for two-dimensional smooth convex obstacles, where they used a Galerkin boundary element method on a uniform mesh to approximate the ratio of the scattered field to the incident field. Their numerical experiments suggest that the number of degrees of freedom needed to maintain accuracy needs to grow like $k^{1/3}$ in the high frequency limit. This is a dramatic improvement over the usual $\mathcal{O}(k)$ for conventional methods, though a rigorous analysis is not available in [1] to support the numerical experiments.

A closely related method to that in [1] was recently considered by Dominguez et al. [8] for a two-dimensional smooth obstacle. In [8] a p-version boundary element method is proposed, that is, different degree polynomials are used to approximate the slowly oscillating amplitude in the illuminated and transition zones, and the slowly oscillating amplitude is approximated by zero in the shadow region. A rigorous error analysis is made, with error bounds showing that the polynomial degree needs to increase proportionally to $k^{1/9}$ to maintain a prescribe level of accuracy as $k\to\infty$. This was confirmed by numerical experiments using a Galerkin scheme. In fact the numerical experiments of [8] demonstrate that a reasonable accuracy can be achieved by keeping the number of degrees of freedom fixed as $k \to \infty$.

All of the schemes in [1, 8] can be expected to perform poorly for obstacles which have corners. For obstacles with corners there is, as well as the reflected field, a diffracted field due to the corners. These are not well represented in the ansatz proposed in [1, 8]. A scheme related to those in [1, 8], for obstacles with corners, has been proposed by Chandler-Wilde et al. [4, 5], Langdon et al. [9, 10] and Mokgolele et al. [11]. Chandler-Wilde and Langdon [4] proposed a novel Galerkin boundary element method to solve the problem of acoustic scattering by a sound soft straight line convex polygon.

In this paper we applied a collocation scheme on a uniform mesh to approximate the ratio of the scattered field to the incident field. We begin §2 by defining our boundary element method, here we explain the implementation of the collocation method on a smooth object, taking a circle as an example. In §3 we present some numerical results, the relative errors seem remain constant as wavenumber increases. Finally in §4 we present the conclusion.

2. Boundary Element Method

We begin by parametrising (4) on the boundary of a circle, Figure 1. We employ a uniform

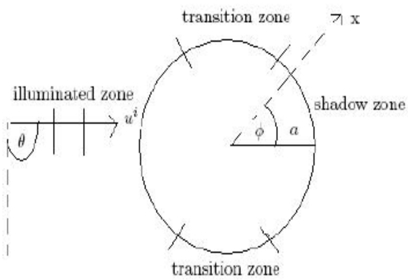

Figure 1. Scattering by a circle

mesh on the boundary of circle, and use a collocation method for computation of a numerical solution. The approximation space is a set of piecewise constants. As our starting point we re-write (4) in arc-length parametrised form as

$$(I + \mathcal{K})\psi(s) = F(s), \quad s \in [0, L], \tag{5}$$

where $\psi(s) := \frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s))$, $L := 2\pi a$, $F(s) := \frac{2}{k} f(\mathbf{x}(s))$ and, for $v \in L^2[0, L]$,

$$\mathcal{K}v(s) := 2\int_0^L K(s,t)v(t)\,\mathrm{d}t,$$

and $K(s,t) = \left(\frac{\partial\Phi(\mathbf{x}(s),\mathbf{x}(t))}{\partial\mathbf{n}(\mathbf{x}(s))} + \mathrm{i}\eta\Phi(\mathbf{x}(s),\mathbf{x}(t))\right)|\mathbf{x}'(t)|$ and for a circle of radius a

$$\mathbf{x}(t) = [a\cos t, a\sin t] \quad \text{and} \quad \mathbf{n}(\tau) = [\cos\tau, \sin\tau]. \tag{6}$$

Writing ψ_N as a linear combination of the basis functions gives

$$\psi(s) \approx \psi_N(s) = \sum_{j=1}^{M_N} v_j\rho_j(s), \tag{7}$$

where

$$\rho_j(s) := \begin{cases} 1, & s \in [0, L], \\ 0, & \text{elsewhere}, \end{cases}$$

are piecewise constant basis functions, M_N is the number of basis functions and v_j is the unknown to be determined. Substituting (7) into (5) gives

$$\sum_{j=1}^{M_N} \left[\rho_j(s) + \int_0^{2\pi} K(s,t)\rho_j(t)\mathrm{d}t \right] v_j = F(s). \quad (8)$$

The collocation method is to choose the sets of collocation points, $s_1, ..., s_{M_N} \in [0, 2\pi]$, and force (8) to hold at each collocation point, this leads to

$$\sum_{j=1}^{M_N} \left[\rho_j(s_m) + \int_0^{2\pi} K(s_m,t)\rho_j(t)\,\mathrm{d}t \right] v_j = F(s_m) \quad m = 1, ..., M_N, \quad (9)$$

Due to oscillatory of the kernel $K(s_m, t)$ and $F(s_m)$, the solution $\psi(s)$ will be oscillating everywhere between $[0, 2\pi]$ with larger amplitude in the illuminated region as wavenumber increases. In order to reduce these oscillations, some authors [3, 8] introduced the following ansatz:

$$\psi(s) := \frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s)) = \psi_{\text{slow}}(s)\mathrm{e}^{\mathrm{i}k\mathbf{x}(s)\cdot\mathbf{d}}. \quad (10)$$

Substitute (10) into (5) and dividing throughout by $\mathrm{e}^{\mathrm{i}k\mathbf{x}(s)\cdot\mathbf{d}}$ gives

$$(I^{'} + \mathcal{K}^{'})\psi'(s) = F^{'}(s), \quad s \in [0, L], \quad (11)$$

where $\psi^{'}(s) := \psi_{\text{slow}}(s)$, $L := 2\pi a$, $F^{'}(s) := F(s)\mathrm{e}^{-\mathrm{i}k\mathbf{x}(s)\cdot\mathbf{d}}$, $I^{'}$ is an identity operator and, for $v \in L^2[0, L]$,

$$\mathcal{K}^{'}v(s) := 2\int_0^L K^{'}(s,t)v(t)\,\mathrm{d}t,$$

where $K^{'}(s,t) = K(s,t)\mathrm{e}^{\mathrm{i}k(\mathbf{x}(t)-\mathbf{x}(s))\cdot\mathbf{d}}$. Using (6) it is easy to evaluate the formulas for $F^{'}(s)$ and $K^{'}(s,t)$. Starting from (11), we can follow the same procedure as before to arrive to a system of the form (9). Precisely, we can define a boundary element approximation $\psi_{\text{slow},N}(s)$, to $\psi_{\text{slow}}(s)$, by

$$\psi_{\text{slow}}(s) \approx \psi_{\text{slow},N}(s) = \sum_{j=1}^{M_N} v_j\rho_j(s), \quad (12)$$

where the coefficients $v_1, \ldots, v_{M_N}$ are defined as the solution to a version of (9) with the functions K and F replaced by $K^{'}$ and $F^{'}$.

3. Numerical results

For our numerical experiment, we code (11) on the boundary of circle, length $2\pi a$, where $a = 1$ is the radius of the circle. We evaluate relative L^2 errors $(\frac{\|\psi(s)-\psi_N(s)\|_2)}{\|\psi(s)\|_2})$ for fix $N = 8$. Here we take $\psi(s)$ to be our exact value, evaluated at $N = 16384$, (Note: is possible to derive the analytical solution of circle, for this particular problem). The relative errors seem to remain constant with increasing wavenumber k. These initial results demonstrate the robustness of our numerical scheme. In Figure 2, we show the total scattering field by a circle. This picture is infact the numerical results of Figure 1.

Table 1. Relative errors, Scattering by a circle

k	$\|\psi - \psi_N\|_2 / \|\psi\|_2$
5	$2.523x10^{-1}$
10	$2.770x10^{-1}$
20	$2.581x10^{-1}$
40	$1.9441x10^{-1}$
80	$2.025x10^{-1}$
160	$1.3970x10^{-1}$

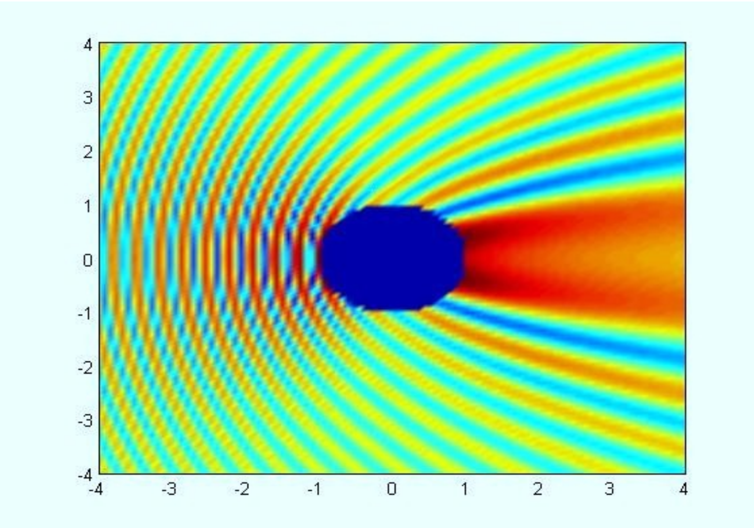

Figure 2. Total scattering field by a circle

4. Conclusion

In this paper we have proposed a collocation method for solving high frequency scattering by smooth objects. We demonstrated via numerical experiment that the relative errors remain constant as wavenumber increases, which in turn means, the number of degrees of freedom grows logarithmically with respect to frequency.

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