Strathmore
UNIVERSITY

STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES

MASTER OF SCIENCE IN DATA SCIENCE AND ANALYTICS

END OF SEMESTER EXAMINATION

DSA 8405 PROBABILITY AND STOCHASTIC PROCESSES

DATE: 13th December 2023 Time: 3 Hours

Instructions

1. This examination consists of **Five** questions.
2. Answer **Question ONE (COMPULSORY)** and any other **TWO** questions.

Question 1 (20 Marks)

(a) Explain the following terms as used in *Probability and Stochastic Processes*

(i) A Stochastic process (1 Mark)

(ii) A Markov property (1 Mark)

(b) Let the probability density function (p.d.f) a binomial random variable X be defined as

$$P(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, x = 0,1,2,\ldots \\ o, elsewhere \end{cases}$$

(i) Find the probability generating function of X (2 Marks)

(ii) Hence find its mean and variance (3 Marks)

(c) Classify the states of the following Markov chains.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.3 & 0.0 & 0.7 & 0.0 \\ 0.1 & 0.2 & 0.0 & 0.7 \end{pmatrix} \end{array}$$

(3 Marks)

(d) A housewife buys three kinds of cereals; A, B, and C. She never buys the same cereals on successive weeks. If she buys cereals A, then the next week she buys B. However, if she buys either B or C, then the next week she is three times as likely to buy A as the other brands. In the long run, how often does she buy each of the three brands**?** (4 Marks)

(e)(i) Find the generating function of the sequence defined by Let $\{a_k\} = \left\{\frac{1}{k!}\right\}$ (2 Marks)

(ii) Show that the following is a Poisson distribution with a parameter λt

$$f(x)=\begin{cases}\dfrac{e^{-\lambda t}(\lambda t)^x}{x!}, x=0,1,2,.... \\ 0, otherwise\end{cases}$$ (4 Marks)

Question 2 (20 Marks)

(a) Define the term *stationary stochastic process* for arbitrary random variables $t_1, t_2, t_3, ..t_n$ (2 Marks)

(b) Consider the process $X(t) = A\cos\lambda t + B\sin\lambda t$ where A and B are uncorrelated random variables with mean 0 and variance 1, whereas λ is a positive constant. Show that $X(t)$ is a weakly stationary process (8 Marks)

(c) Consider the process $\{X(t): t \in T\}$ with

$$\Pr\{X(t)=n\}=\begin{cases}\dfrac{e^{-at}(at)^n}{n!}, a>0, n=0,1,2,.. \\ o \quad , otherwise\end{cases}$$

Show that $X(t)$ is not stationary. (10 Marks)

Question 3 (20 Marks)

(a) Explain **Four** characteristics of a Markov Chain. (4 Marks)

(b) Explain the following terms as used in Markov chains

(i) A transient state (1 Mark)

(ii) Steady state (1 Mark)

(iii) Absorbing state (1 Mark)

(c) A manufacturing company has a certain piece of equipment that is inspected at the end of each day and classified as just overhauled, good, fair or inoperative. If the item is inoperative it is overhauled, a procedure that takes one day. Suppose the four classifications can be denoted by 1, 2, 3 and 4 respectively. Assume that the working condition of the equipment follows a Markov chain with a transition matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \end{bmatrix}\end{array}$$

(i) Determine the equilibrium states of the Markov chain (10 Marks)

(ii)If it costs US$ 205 to overhaul a machine (including lost time) on average and US$ 110 as production lost if a machine is found inoperative. Using steady state probabilities, compute the expected per day cost of maintenance. (3 Marks)

Question 4 (20 Marks)

Let Z(t) represent the population size at a time t and $P_n(t)$ be the probability that a population is of size n at a time t. Further let $P_n(t) = \Pr[Z(t) = n]$. Let Δt represent a small interval of time over which this population is being studied, $\lambda_n(t) + 0\Delta t$ is the probability that a birth occurs within the time interval Δt and $\mu_n(t) + 0\Delta t$ is the probability that from a population of size n a death occurs. Let the difference differential equations for the birth death process are given by

$$P_n^{/}(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t) \text{, for } n \geq 1$$

and $P_n^{/}(t) = -\mu_0 P_0(t) - \lambda_0 P_0(t) + \mu_1 P_1(t)$ for $n = 0$,

(a) Write down the difference differential equations given that $\lambda_n = \lambda$ and $\mu_n = 0$

(2 Marks)

(b) Hence using initial conditions

$$P_n(0) = \begin{cases} 1, n = 1 \\ 0, otherwise \end{cases}$$

(i) show that the probability generating function $G(s,t) = e^{-(1-s)\lambda t} = e^{-\lambda t} e^{\lambda ts}$

(11 Marks)

(ii) Determine $P_n(t)$ from $G(s,t)$ (2 Marks)

*(c)*Find the mean and variance of $P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$, $n = 0,1,2,\ldots$, (5 Marks)

Question 5 (20 Marks)

Let the difference differential equations for the birth-death process for a population size n at a time t be given by

$$P_n^{/}(t) = -\left(\lambda_n\left[\frac{1+an}{1+\lambda at}\right]\right)P_n(t) + \lambda\left[\frac{1+an}{1+\lambda at}\right]P_{n-1}(t) \text{, for } n \geq 1$$

and

$$P_n^{/}(t) = -\left[\frac{\lambda}{1+\lambda at}\right]P_0(t) \text{ , for } n = 0,$$

Where a is an arbitrary parameter

(a) Show that the generating function $G(s,t)$ is given by

$$G(s,t) = (1+\lambda at)^{\frac{-1}{a}}\left[1 - \frac{\lambda at}{1+\lambda at}s\right]^{\frac{-1}{a}}$$ (16 Marks)

(c) Find the probability that the population is of size n at time t, (4 Marks)