STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES

BACHELOR OF BUSINESS SCIENCE (ACTUARIAL SCIENCE,FINANCIAL ECONOMICS AND FINANCIAL ENGINEERING)

REMEDIAL/SPECIAL EXAMINATION

BSF 3108: STOCHASTIC MODELS FOR FINANCIAL AND ACTUARIAL APPLICATIONS

DATE: 8th April 2026 TIME: 2.5 HRS

Instructions:

a. *This examination consists of FIVE Questions. Answer Question ONE (COMPULSORY) and any TWO OTHER QUESTIONS.*

b. *Mark allocations are shown in brackets.*

c. *Use of a non-programmable electronic Calculator and actuarial table is allowed for this examination.*

d. *Do not write anything on this question paper.*

Question One (30 marks)

1. Define the term Stochastic Processes as used in stochastic modeling. [1 mark]

2. State FIVE defining properties of a standard Brownian motion W_t. [3 marks]

3. A stochastic process S_t has the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a standard Brownian motion. Another stochastic process F_t is given by the following equation: $F_t = S_t e^{(R-r)(T-t)}$

Apply Ito's formula to derive an SDE satisfied by F_t. [4 marks]

4. In the Vasicek model, the spot rate of interest is governed by the stochastic differential equation below. Find the solution for r_t. [5 marks]

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

Hint: A stochastic Process $\{U_t : t > 0\}$ can be defined as $U_t = r_t e^{at}$.

5. Apply the Itô Lemma formula to prove that:
   (a) $E[(W_t - W_s)^6] = 15(t-s)^3$ [6 marks]
   (b) $\int_0^t \cos W_s \, dW_s = \sin W_t + \frac{1}{2}\int_0^t \sin W_s \, ds$ [4 marks]

6. Let M_t be a stochastic process and W_t be a standard Brownian motion.
   (a) What necessary and sufficient conditions must be satisfied by M_t in order to be considered a Martingale? [2 marks]
   (b) By use of properties of conditional Expectations, show that $M_t = e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ is a martingale. [3 marks]
   (c) By use of Ito's Lemma, show that $M_t = e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ is a martingale. [2 marks]

Question Two (20 marks)

1. What is the importance of the concept of filtration in the study of a stochastic process? [1 mark]

2. Explain why both Generalized Wiener process and Standard Brownian Motion are unrealistic models for share price. [2 marks]

3. The data below is for closing stock price: 20.00, 19.90, 20.75, 21.75, 22.20
   (a) Estimate the GBM parameters, assuming 252 trading days. [5 marks]
   (b) State the SDE of this particular stock. [2 marks]
   (c) State also the solution for the stock. [2 marks]

4. Karambu, a first year BBS Student plans to travel in four years' time and hopes to celebrate graduation with a round the world cruise costing KES. 3 M. Karambu chooses to save by buying non-dividend paying shares with price S governed by:

$$dS = S(\mu dt + \sigma dZ)$$

   Where Z is a standard Brownian Motion, $\mu = 10\%$, $\sigma = 20\%$, and $S_0 = 1$. The risk-free rate is 4% p.a.
   (a) Derive the distribution of S. [5 marks]
   (b) Calculate the amount, A, that Karambu will need to invest to give a 40% probability of having at least KES 3,000,000 in four years' time. [3 marks]

Question Three (20 marks)

1. Let W_t be a standard Brownian motion. Consider the following stochastic differential equation (SDE):

$$dX_t = (\alpha - \beta X_t)dt + \sigma dW_t$$

   Let α, β and σ be constant parameters and $X_0 = x$. The solution is a mean-reverting process called the Ornstein-Uhlenbeck process. Given that $U_t = X_t e^{\beta t}$:

(a) Verify that the solution to the SDE can be expressed in the following form: [5marks]

$$X_t = e^{-\beta t}\left[x + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma \int_0^t e^{\beta s} dW_s\right]$$

(b) Deduce that: [2 marks]

$$Var(X_t) = \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})$$

2. Consider the Vasicek spot rate model $dr_t = \alpha(\mu - r_t)dt + \sigma dW_t$ where the parameters μ, α and σ are known positive constants. W_t is a standard Brownian motion and $U_t = r_t e^{\alpha t}$. Show for $s < t$ that:

   (a) $E[r_t | r_s] = \mu + [r_s - \mu]e^{-\alpha(t-s)}$ [4 marks]

   (b) $Var[r_t | r_s] = \frac{\sigma^2}{2\alpha}[1 - e^{-2\alpha(t-s)}]$ [3 marks]

   (c) Describe what these two equations imply for the conditional mean and variance of spot rates as $t \to \infty$. [2 marks]

   (d) Discuss the suitability of Vasicek as a model for interest rates. [4 marks]

Question Four (20 marks)

1. Describe six assumptions of the Black-Scholes Model. [6 marks]

2. Assume that a non-dividend paying share with price process S_t follows the Black Scholes model. Using the risk-neutral valuation, show that:

$$C_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

Where C_0 is the price at time 0 of a European call option with strike price K and time to maturity T, and where:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

[14 marks]

Question Five (20 marks)

1. When are Probability measures P and Q said to be equivalent? [2 marks]

2. State and explain the Risk-Neutral Valuation formula. Define all the symbols used. [3 marks]

3. Show, using the risk-neutral valuation formula, that the value at time $t = 0$ of a forward contract with delivery price K, maturity T, and underlying asset price S_t is:

$$V_0 = S_0 - Ke^{-rT}$$

[5 marks]

4. Discuss the Willkie Model highlighting its salient features. [10 marks]