Strathmore
UNIVERSITY

SCHOOL OF COMPUTING AND ENGINEERING SCIENCES

BACHELOR OF SCIENCE IN COMPUTER NETWORK AND SECURITY

END OF SEMESTER EXAMINATION

CNS 1103: DIFFERENTIAL CALCULUS

DATE: 16th November 2022 Time: 2 Hours

Instructions

1. This examination consists of **FIVE** questions.
2. Answer **Question ONE (COMPULSORY)** and any other **TWO** questions.

QUESTION ONE [30 MARKS]

(a) Evaluate the following limits:

(i) $\lim_{x \to 3} \frac{2x^2 - 6x}{x-3}$ **[2 Marks]**

(ii) $\lim_{x \to 25} \frac{-\sqrt{x}+5}{25-x}$ **[3 Marks]**

(iii) $\lim_{x \to \infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$ **[3 Marks]**

(b) Differentiate the following:

(i) $y = e^{2x} \ln 5x$ **[3 Marks]**

(ii) $y = \frac{\cos 2x}{x^2}$ **[3 Marks]**

(c) Differentiate $f(x) = \sqrt{x}$ from first principles. **[4 Marks]**

(d) If $x^2 + y^2 - 2x + 5 = 0$, find $\frac{dy}{dx}$ at the point $(4, 2)$.

[4 Marks]

(e) Prove that the function is continuous at $x = a$.

$$f(x) = \begin{cases} \frac{a^2}{x} - a, & for \quad 0 < x < a \\ 0, & for \quad x = a \\ a - \frac{a^3}{x^2}, & for \quad x > a \end{cases}$$

[4 Marks]

(f) A total cost function is given by the equation $TC = 120\ln(Q+10)$. Show that TC and MC do not have turning points.

[4 Marks]

QUESTION TWO [20 MARKS]

(a) Apply De L 'Hospitals' rule to evaluate

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^2}$$

[4 Marks]

(b) Find $\frac{dy}{dx}$, given that

(i) $y = 80e^{-\frac{1}{8}x}$ **[2 Marks]**

(ii) $y = 4 + x^{4.5} + \frac{2x}{\sqrt{x}}$ **[2 Marks]**

(iii) $y = (3+x)e^{-x}$ **[3 Marks]**

(c) A box with a square base and open top is to have a volume of $4{,}000\,cm^3$. Neglect the thickness of the material used, and find the dimensions that will minimize the amount of material used. **[9 Marks]**

QUESTION THREE [20 MARKS]

(a) Find the derivative of $y = \sqrt{\frac{2+t}{2-t}}$ with respect to t. **[6 Marks]**

(b) Show that $y = e^{-x}\sin x$ satisfies the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$.

[7 Marks]

(c) The distance q that an image is from a certain lens in terms of p, the distance of the object from the lens, is given by

$$q = \frac{10p}{p-10}$$

If the object distance is increasing at the rate of $0.2\, cm/s$, how fast is the image distance changing when $p = 15.0\, cm$? **[7 Marks]**

QUESTION FOUR [20 MARKS]

(a) Determine the turning points of the function $y = -x^3 + 9x^2 - 24x + 26$. Hence distinguish them.

[10 Marks]

(b) The curve whose equation is $y = 2x^3 + ax^2 + bx + 3$ has stationary values at $x = 1$ and $x = 2$. Find the values of the constants a and b. Find also the equation of the normal to the curve at the point where $x = \frac{1}{2}$. **[10 Marks]**

QUESTION FIVE [20 MARKS]

(a) The demand function for a good is given by the equation $P = 50 - 2Q$ while total cost is given by $TC = 160 + 2Q$.

(i) Write down the equations for total revenue and profit. **[2 Marks]**

(ii) Determine the maximum profit and the value of Q at which profit is maximum.

[6 Marks]

(b) An oil refinery has four distillation towers and operates them as they are needed to process available raw materials. Each tower has fixed cost of £300 per week whether operating or not . in addition each tower, if in operation , will incur additional fixed costs of £500 per week. The raw material cost is fixed at £0.50 per gallon of refined oil, and each tower can process at most $10{,}000$ gallons of refined oil each week.

i. Find the cost function $C(x)$; $x =$ number of gallons of refined oil for the refinery.

[4 Marks]

ii. Find the points of discontinuity of the function.

[8 Marks]

END

DEFINITIONS AND FORMULAE

Area and volume formulae

Volume of a cone or pyramid $= \frac{1}{3}Ah$, where A = base area, h = height of vertex.
Area of curved surface of a cone $= \pi rl$, where l = slant height, r = base radius.
Volume of a sphere $= \frac{4}{3}\pi r^3$.
Surface area of a sphere $= 4\pi r^2$.
Area of a spherical zone
(between planes distance h apart) $= 2\pi rh$.

Trigonometry

$$\sec\theta = \frac{1}{\cos\theta};\quad \csc\theta = \frac{1}{\sin\theta};\quad \tan\theta = \frac{\sin\theta}{\cos\theta};\quad \cot\theta = \frac{1}{\tan\theta}.$$

$$\cos^2\theta + \sin^2\theta = 1;\quad 1+\tan^2\theta = \sec^2\theta;\quad \cot^2\theta + 1 = \csc^2\theta.$$

$$\sin(\theta\pm\phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi;\quad \cos(\theta\pm\phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi;$$

$$\tan(\theta\pm\phi) = \frac{\tan\theta\pm\tan\phi}{1\mp\tan\theta\tan\phi}\quad [\theta\pm\phi \neq (k+\tfrac{1}{2})\pi].$$

$$\sin 2\theta = 2\sin\theta\cos\theta;\quad \cos 2\theta = \cos^2\theta - \sin^2\theta;\quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\quad [\theta \neq (\tfrac{1}{2}k+\tfrac{1}{4})\pi].$$

$$2\cos^2\theta = 1+\cos 2\theta;\quad 2\sin^2\theta = 1-\cos 2\theta.$$

If $t = \tan\frac{1}{2}\theta$, then $\sin\theta = \dfrac{2t}{1+t^2}$; $\cos\theta = \dfrac{1-t^2}{1+t^2}$; $\tan\theta = \dfrac{2t}{1-t^2}$; $\dfrac{d\theta}{dt} = \dfrac{2}{1+t^2}$.

$$2\sin\theta\cos\phi = \sin(\theta+\phi)+\sin(\theta-\phi);$$

$$2\cos\theta\cos\phi = \cos(\theta+\phi)+\cos(\theta-\phi);$$

$$2\sin\theta\sin\phi = \cos(\theta-\phi)-\cos(\theta+\phi).$$

$$\sin\alpha+\sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta);\quad \sin\alpha-\sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta);$$

$$\cos\alpha+\cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta);\quad \cos\alpha-\cos\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\beta-\alpha).$$

In the triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R;$$

$$a^2 = b^2+c^2-2bc\cos A, \text{ etc.};$$

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \text{ etc.};\quad \text{area} = \sqrt{\{s(s-a)(s-b)(s-c)\}};$$

where $s = \frac{1}{2}(a+b+c)$.

Ranges of the inverse functions:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1}x \leqslant \tfrac{1}{2}\pi;\quad 0 \leqslant \cos^{-1}x \leqslant \pi;\quad -\tfrac{1}{2}\pi < \tan^{-1}x < \tfrac{1}{2}\pi.$$

DEFINITIONS AND FORMULAE

Differentiation

If $\frac{f(x)-f(a)}{x-a}$ tends to a limit as x tends to a, then f is said to be differentiable at a. The limit is called the derivative of f at a and is usually written $f'(a)$. The function f' is called the derived function of f.

Derivatives of common functions

$f(x)$	$f'(x)$
x^m	mx^{m-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x \cot x$
$e^x (= \exp x)$	$e^x (= \exp x)$
a^x	$a^x \ln a$
$\ln x (= \log_e x)$	$1/x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

If $y = uv$, then
$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

If $y = u/v$, then
$$\frac{dy}{dx} = \left(\frac{du}{dx}v - u\frac{dv}{dx}\right)\Big/ v^2.$$

Approximations to derivatives

If the derivatives exist, then for small h

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h};$$

$$f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

Polynomial approximations to functions

Taylor's approximation. For small h,

$$f(a+h) \approx f(a)+f'(a).h+\frac{1}{2!}f''(a).h^2+\ldots+\frac{1}{n!}f^{(n)}(a).h^n$$

provided that the derivatives exist and are continuous at a.

Applications to particular functions:

$$(1+h)^m \approx 1+mh+\frac{m(m-1)}{2!}h^2+\ldots+\binom{m}{n}h^n.$$

$$\ln(1+h) \approx h-\frac{h^2}{2}+\frac{h^3}{3}-\ldots+(-1)^{n+1}\frac{h^n}{n}.$$

$$e^h \approx 1+h+\frac{h^2}{2!}+\ldots+\frac{h^n}{n!}.$$

$$\cosh h \approx 1+\frac{h^2}{2!}+\frac{h^4}{4!}+\ldots+\frac{h^{2k}}{(2k)!}.$$

$$\sinh h \approx h+\frac{h^3}{3!}+\ldots+\frac{h^{2k+1}}{(2k+1)!}.$$

$$\cos h \approx 1-\frac{h^2}{2!}+\frac{h^4}{4!}+\ldots+(-1)^k\frac{h^{2k}}{(2k)!}.$$

$$\sin h \approx h-\frac{h^3}{3!}+\ldots+(-1)^k\frac{h^{2k+1}}{(2k+1)!}.$$