A Comparison of Mean-Variance and Mean-Semi Variance Optimization on the Nairobi Stock Exchange.

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DECLARATION

I declare that this work has not been previously submitted and approved for the award of a degree by this or any other University. To the best of my knowledge and belief, the Research Proposal contains no material previously published or written by another person except where due reference is made in the Research Proposal itself.

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20/11/2015

This Research Proposal has been submitted for examination with my approval as the Supervisor.

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Abstract.

The main objectives of this paper were to measure and compare the portfolio performance of portfolios weighted through the mean variance and semi variance approach in the Kenyan context and to compare portfolio performance in terms of return between portfolios weighted using a Geometric Mean Variance Frontier Approach Vs Semi Variance Approach. Equities used were from broad sectors of the NSE: Agricultural, Financial, Commercial and Services and Industrial. Based on the empirical results, there is no significant advantage in using semi variance as a risk measure as compared to variance in optimization. This is because the equity returns in the Nairobi Stock Exchange follow a normal probability distribution. The geometric mean variance returns are also compared to the semi variance optimization methods and results show they statistically approximate each other. Key limitations for the study that NSE duration used may have been a unique case for a normal distribution. Further research needs to be done to ascertain whether the geometric mean variance optimization approximates the semi variance approach.

Key words: NSE, Semi variance, Mean Variance Optimization

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Contents

List of Abbreviations.

E- Expected return

V- Variance

SVm- Below-mean semi variance

SVt- Below-target semi variance

SV- Semi variance

Vwretd-Value-weighted-return of the market

GMV-Geometric Mean Variance

SVP-Semi Variance Portfolio

MVP- Mean Variance Portfolio.

NSE-Nairobi Securities Exchange

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1 Introduction

1.1 Background

Portfolio Optimization (K. Liagkouras, 2013) is the process of choosing the assets and their proportions, so that it attains the maximum profitability for the risk undertaken .There are many theories that attempt to solve the optimization problem ,popular of this is the mean variance optimization theory (Markowitz H. , 1952).

The theory attempts to maximize portfolio expected return for a given amount of portfolio risk, or equivalently minimize risk for a given level of expected return by selectively choosing the proportions of various assets. The concept uses diversification in investing so that we obtain a collection of assets with collectively lower risk than any individual asset. (Sullivan & Steven.M., 2003).

In the theory (Markowitz, 1952) makes many assumptions about the investor and the market. The assumptions made are not entirely true and each has the mean variance theory to some extent.

These assumptions include:

- investors are interested in maximizing return for a given variance;
- asset returns are normally distributed; correlation between assets are fixed and constant forever;
- all investors are rational and risk averse;
- all investors have the same access to information at the same time; there are no taxes or
- transaction costs;
- Risk of an asset is known in advance and any investor can lend and borrow an unlimited
- Amount at the risk free rate.

Mean Variance has been a method of choice for many financial modelers in the 21'st Century (Estrada J. , 2006).Unknown to many Markowitz favored another measure of risk: the Semi-variance of returns. In fact, Markowitz (1959) allocates the entire chapter IX to discuss semi-variance, where he argues that "analyses based on S [semi variance] tend to produce better portfolios than those based on V [variance]" (see Markowitz, 1991, page 194). In the revised edition of his book (Markowitz H. , 1991), he goes further and claims that "semi variance is the more plausible measure of risk" (page 374). Later he claims that because "an investor worries about underperformance rather than over performance, semi deviation is a more appropriate measure of investor's risk than variance" (Markowitz, Todd, Xu, and Yamane, 1993, page 307).

However Markowitz gives preference to the mean variance optimization theory due to what he defines as the convenience and cost effective use of variance for analysis in comparison to semi variance as it is less costly to use and more familiar to practitioners. The foundations of this theory however, are based on a set of strict assumptions with the result that the majority of models fail to capture reality perfectly and exhibit significant model risk. Downside risk optimization, which models the efficient frontier using semi variance, has exhibited potential for providing better risk metrics.

In recent time however with improved technology and computing power in computers, downside risk has been gaining increasing attention, and the many magnitudes that capture downside risk are by now well-known and widely used. (Estrada J. , 2006). Over the last decade, researchers, individual and institution investors have come with various methods to maximize yields and minimize risk to suit the different types of investors. On the backdrop of the financial crisis in the U.S and the Euro Crisis over the last 10 years ,there has been an increased interest by the players in the financial markets to the risk portfolios and maximize returns over the long term.

Semi Variance optimization as a down side measure of risk offers such an alternative as it incorporates a number of real world constraints such as cardinality constraints, floor and ceiling constraints, non-negativity constraint and budget constraint and analyzes their effects on the efficient frontier formulation. (Markowitz, 2010), also proposes the use of a mean variance approach using geometric mean return as an alternative to avoid the use of the use of a semi variance approach.

While empirical tests has been investigated on foreign markets, studies relating to Kenya in particular are limited. This study aims investigate effectiveness of asset allocation and portfolio optimization using the mean-semi variance framework in the Kenyan context.

1.2 Motivation for Study

A study by (Mutuku, 2012) noted that there is high volatility in the Kenyan stocks over the past years. This has been partly attributed to the post-election violence of 2007/2008 following the disputed presidential elections; the global financial crisis of 2008/2009 and the steep depreciation of the Kenya shilling in 2011and 2015, which affected the financial asset prices significantly. Pension schemes which are significant investors in the financial assets have been greatly affected by this volatility.

1.3 Problem Statement

In Kenya financial performance of pension funds has been critical to their sustainability, enabling them to meet their obligation to members. A key aspect has been how the fund's assets are managed in order to achieve the desired returns. According to (Mutuku, 2012) Pension funds in Kenya have been significantly affected by market volatility over the years, with good periods showing significant positive growth and bad periods of negative performance. These swings are exacerbated by a significant negative correlation between the NSE prices and interest rates on government securities which together constitute 70% of pension scheme assets (Mutuku, 2012) .Diversification reduces risk without compromising on expected return (Reilley & Brown, 2012)The number of securities to invest in, their combination in a portfolio and the risk involved are equally important considerations (Reilley & Brown, 2012).

(Treynor & Black, 1973) , showed that portfolio performance can be improved by optimally weighting a fund manager's stocks selection. (Mwangangi, 2006) , surveyed the application of Markowitz's mean variance portfolio optimization model in overall asset allocation decisions by pension fund managers in Kenya. He used a questionnaire and secondary data from Retirement Benefit Authority on funds allocation for three years from 2003 to 2005. The results of the study showed that 60% of the fund managers applied the Markowitz's mean variance optimization model in their allocation criteria.

Markowitz (2010) gives preference to the semi variance measure as compared to the mean variance method in producing optimized portfolios .There has however been no research done on the application of mean semi variance optimization in Kenya. Should Kenyan fund managers switch to the mean semi-variance portfolio optimization approach?

This paper aims to investigate the viability of asset allocations and portfolio optimization in a mean-semi variance framework in Kenya"

1.4 Research Objectives.

1. To measure and compare the portfolio performance of portfolios weighted through the mean variance and semi variance approach in the Kenyan Context.
2. To compare portfolio performance in terms of return between portfolios weighted using a Geometric Mean Variance Frontier Approach Vs Semi Variance Approach.

1.5 Research Questions

1. Does the semi-variance portfolio optimization approach create portfolios that yield a higher return than mean variance optimized portfolios in Kenya?
2. Can the Geometric Mean Variance Frontier Approach estimate the Semi Variance Approach in Kenya?

1.6 Significance of Study.

This study contributes to the empirical evidence on the determination of the optimal portfolio for investors within the Nairobi Securities exchange. This study has important implications for investors in making portfolio securities selection and funds allocation decisions in Kenya. This study can inform future review of policy and regulatory guidelines for regulated institutional investors in Kenya. The study is of interest to researchers and financial analysts of the Kenyan economy. The study is of interest to portfolio and investment managers of insurance companies and retirement benefits schemes.

2 Literature Review

2.1 Introduction

Risk has been with defined differently over the decades. Frank H. Knight (1921) argues that there is a difference between uncertainty and risk. According to Knight, risk is a combination of the likelihood of an occurrence of a hazardous event, meaning an event that could cause harm in terms of losses or undesirable outcome, and its magnitude.

(Knight, 1921) , also proposes that it is possible to calculate the probability a risk, which makes it measureable. Uncertainty on the other hand is characterized as the existence of more than one possibility in the future, but unlike risk, uncertainty is not measureable.

(Hubbard, 2007) further affirms Knights position by defining uncertainty as The lack of complete certainty, that is, the existence of more than one possibility i.e. the true outcome value is not known. He also defines risk as a state of uncertainty where some of the possibilities involve loss, catastrophe or other undesirable outcome.

This section will focus on risk and various methods of calculating risk that have been there over time and the empirical applications of the methods in solving the optimization problem. I will then focus on the mean variance vs semi variance approach as proposed by (Markowitz, 2010)

2.2 Mean Variance Approach

Modern Portfolio Theory, a finance theory pioneered by (Markowitz, 1952) is a tool that attempts maximize portfolio expected return for a given amount of portfolio risk by carefully allocating the proportions of various assets. Markowitz used mean returns, variances and covariance's to derive an efficient frontier where every portfolio on the frontier maximizes the expected return for a given variance or minimizes the variance for a given expected return. This called the EV (Expected Return and Variance) criterion.

In selecting a portfolio of assets to invest in, an investor needs to make a tradeoff between risk and return. The investor's sensitivity to changing wealth and risk is known as a utility function. The utility function have been subjective so far and scholars have differed on the methods used to model utility.

(Boasson, Boasson, & Zhou, 2011), notes that, Markowitz's mean-variance approach, has two drawbacks. It assumes that the distribution of investment returns is jointly elliptically distributed. If the underlying return data is not normally distributed, the variance is likely to give misleading results. Studies have demonstrated that investment returns are not normally distributed (Fama, 1968). In the real world, security returns tend to be asymmetrically distributed, e.g. log normally distributed. The skewed distribution of investment returns makes the variance as an inefficient risk measure, because variance treats the favorable upside dispersion of investment return over the mean value of return as a part of risk and penalizes it as much as the unfavorable downside deviation from the mean returns.

If the returns are not normally distributed, investors using variance or standard deviation to measure risk are more often than not likely to reach wrong asset allocation decisions. Skewness and kurtosis in real rates of return data with non-normal distributions can cause variance or standard deviation to underestimate risk. The mean-variance approach also overlooks the investor's risk aversion. Because variance is only a measure of dispersion of returns around a mean, it cannot be customized for individual investors' aversion ((Boasson, Boasson, & Zhou, 2011).

Moreover, the real-world application of Markowitz' mean-variance optimal portfolio allocation has many pitfalls. The optimal portfolio constructed in a mean-variance framework may not lead to an optimal portfolio that optimizes expected returns while minimizing risk as required. (Michaud, 1989) , indicates that these real world portfolio optimizers are essentially "error maximizers" because "optimizers" tend to treat the inputs as if they were exact quantities, while in reality they can only be estimated with error. The optimal portfolios constructed based on this framework tend to suggest large bets on stocks with large estimation error in expected returns, often leading to poor-out of- sample performance. (Markowitz, 1970) , realized the limitations of variance. He showed that both the downside risk measurement and the variance measurement can produce the same correct results when return distributions are normal.

However, in situations where return distributions are not normal, the downside risk measurement is more likely to produce a better solution. Because of this limitation in using variance as a risk measurement, various downside-risk measurements have been proposed and developed. One of the downside risk measurements is semi variance. By definition, a downside-risk measurement measures only the returns below a certain threshold. This threshold captures the risk perspectives from investors to investors. Unlike standard deviation, downside risk accommodates different views of risk.

Markowitz however affirms that variance as a measure of risk has an edge over other downside risk measures "with respect to cost, convenience, and familiarity. The difference in cost, (Markowitz, 1959) is given by the fact that efficient sets based on down side risk took, back then, two to four times as much computing time as those based on variance. The difference in convenience, in turn, is given by the fact that efficient sets based on variance require as inputs only means, variances, and covariance, where as those based on downside risk require the entire joint distribution of returns (Estrada J. , 2006).With the improvement of technology and computing power of computers there has been increased interest in downside risk measures (Estrada, 2007)

2.3 Downside Risk Measures

The (Markowitz, 1959) defense of mean-variance optimization criterion was not unconditional. It asserts the existence of situations in which returns on the portfolio-as-whole are mostly confined to a range in which the investor's utility function can be approximated sufficiently well by a quadratic, and occasional departures from this range are "not too serious." As (Levy & Markowitz, 1979) and others confirm, for many utility functions and for distributions of returns, such as historical returns of investment companies, functions of mean-variance supply robust approximations to expected utility.

(Markowitz, 2010) , assess the other alternatives available in the situations where the mean-variance approach is not applicable. Major alternatives include the following :(1). Use other measures of risk or return in a risk-return analysis (2.)Determine the investor's utility function explicitly and maximize its expected value (3.) Do not optimize; instead, use constraints and guidelines.

We look at the first alternative that opts for the use of other risk measures in this case the downside risk measures. Roy (1952) was the first to look at other measures of risk. He proposed the safety first ratio. Roy states that an investor will prefer safety of principal first and will set some minimum acceptable return that will conserve the principal. Roy called it the minimum acceptable return the disaster level and the resulting technique is the Roy safety first technique. Roy stated that the investor would prefer the investment with the smallest probability of going below the disaster level or target return. By maximizing a reward to variability ratio, **(r - d)/s**, the investor will choose the portfolio with the lowest probability of going below the disaster level, (d), given an expected mean return, (r), and a standard deviation (s).

(Markowitz, 2010) , advances Roy's safety principal first by recognizing Roy's important concept of downside risk measure. Markowitz observes that investors are interested in minimizing downside risk for two reasons: (1) only downside risk or safety first is relevant to an investor and (2) security distributions may not be normally distributed. Therefore a downside risk measure would help investors make proper decisions when faced with non-normal security return distributions.

(Markowitz, 1959), provided two suggestions for measuring downside risk: a semi variance computed from the mean return or below-mean semi variance (SVm) and a semi variance computed from a target return or below-target semi variance (SVt). The two measures compute a variance using only the returns below the mean return (SVm) or below a target return (SVt). Since only a subset of the return distribution is used, Markowitz called these measures partial or semi-variances. Markowitz however notes that the computations of semi variances is rather tedious and would not add any major benefits at the time.

Fishburn (1877) and Harlow and Rao(1989) developed the (α-t) model where α denotes the investors risk aversion while t represents the target return of investment or disaster level or proposed by Roy(1952).Fish burn(1997)also proposed the Mean-Lower Partial Moment model of which Harlow(1991) applied to portfolio selection.

Harlow (1991) defined Lower Partial Model as: $LPM_n = \sum_{R_P}^{T} P_P(\tau - R_P)^n$, where P_P is the probability that the return,R_P occurs. The type of moment, unspecified in the LPM equation captures an investors preference. For n = 0, the risk measure becomes a 0th-order moment (LPM0) which measures the probability of falling below the target rate.

However, for n = 1, LPM1 becomes the expected deviation of returns below the target. For n = 2, LPM2 is analogous to variance, in that it is a probability weighting of squared deviations. Thus, LPM2 can be referred to as a target semi variance. Harlow (1991) further explained that many popular notions of risk are special cases of the generalized LPM,n, measure. For example, with n = 0 and a target rate = 0%, LPM0 is simply the probability of a loss.

For n = 2 and a target rate = mean return, LPM2 becomes the traditional semi variance. Overall, LPM1 (target shortfall) and LPM2 (target semi variance) provide an intuitive set of risk definitions that are more useful than traditional approaches (Harlow, 1991).

However, Harlow and Rao (1989) failed to consider the correlation of asset returns which is an important consideration for diversification of risk. This renders the low partial model only effective for those assets whose returns are perfectly or highly correlated. Lower Partial Model was however regarded as having more complexity in computation than does the variance measurement.

Foo& Eng (2000) made adjustments to Harlow & Rao (1989) by incorporating the model with downside covariance of correlated asset returns. Hogan & Warren (1974) introduced the concept of co-lower-partial-variance, which measures risky asset and market portfolio. Bawa & Lindenberg (1977) further developed this co-lower-partial variance measure to an n-degree framework called generalized asymmetric co-LPM. However their methods were still more complex and computationally burdened and the variance measure was still regarded as the best method due to its simplicity.

(Estrada, 2007) , proposes the use of a heuristic approach in semi variance optimization. His method however does takes more time to calculate even using advance laptops to calculate.

(Markowitz, 2010) proposes a new a simple formula to calculating the optimal portfolio, He proposes that to ease the calculation of downside risk using semi variance and finding the optimized portfolio we should combine the semi deviation as a measure of risk with the geometric mean as the measure of return, R since ,

$$\log(1 + GM) = Elog(1 + R)$$

(Markowitz, 2010) , reckons that it is much easier to compute a mean-variance efficient frontier than a GM-Semi variance efficient frontier. "So long as E log(1 + R) can be estimated sufficiently well from E and V, the economical way to generate a GM-V frontier is to generate a mean-variance efficient frontier and then plot GM on the return axis. However, when distributions are too spread out for mean-variance approximations to be adequate, the extra expense is justified for deriving efficient portfolios on the GM-Sb frontier. This expense is not only computational, but also includes additional estimation requirements because E log (1 + R) is not a function of first and second moments only."

3 Methodology

3.1 Introduction

To achieve the objective of the study the research adopted the methodology proposed by (Markowitz, 2010) and (Boasson, Boasson, & Zhou, 2011) in the calculation of the mean semi variance optimization. The research also employed the equations proposed by Markowitz 1952 in the calculation of the mean variance optimization frontier.

3.2 Research Design

This study is exploratory in nature as it seeks to assess the performance of portfolios created using the mean variance approach in comparison to the ones created using the mean semi variance approach. The exploratory design was also selected as previous researchers' ((Estrada, 2007), (Boasson, Boasson, & Zhou, 2011), similar topics also used the same approach.

3.3 Sampling Design

This study analyses portfolio returns for various asset allocations in the Nairobi Securities Exchange from between January 2009 to November 2014.This duration was chosen as it avoids the effect of the post-election violence that occurred in 2007-2008.

3.4 Data

For the purposes of constructing efficient frontiers, we used data obtained from the Nairobi Stock Exchange on the daily closing prices of stocks over the period. This data was sourced from the Nairobi Stock Exchange.

3.5 Population and Sampling

The population for the study comprised firms listed in the Nairobi Securities Exchange. The study used a census of all securities in the population which had complete information on prices for all the months over the study period January 2009 to December 2014. Any firm that was delisted or suspended over the period was not considered.

Portfolio sizes will the between 8 to 20 stocks as recommended by (Mbithi, 2013) in his research of the optimal sizes of portfolios he found that portfolio risk reduced by 40% with 8-securities portfolio, 46% with 20-securities portfolio and 47% with 30-ssecurities portfolio. The risk reduction achieved with 8-securities portfolio represents 85% of the risk reduction achievable with a 30-security portfolio.

3.6 Conceptual Model.

Calculate the Expected Return for Stocks and Standard Deviations and Correlations between Stocks

To minimize the variance and semi variance of the portfolio for a target return, we need to first calculate the expected return and variance for the risky portfolio from expected returns and variance of the securities comprising the portfolio.

Calculating Expected return of the risky portfolio

Arithmetic Mean Method

The expected return of the risky portfolio is simply the weighted average of expected returns of the securities within the portfolio.

$$E[r_p] = \sum_{i=1}^{n} w_i E[r_i] \quad (\ equation\ 1)$$

Or using matrices

$$E[r_p] = W^T R \text{ where } W = \begin{pmatrix} W_1 \\ \vdots \\ W_n \end{pmatrix}, R = \begin{pmatrix} E[r_1] \\ \vdots \\ E[r_n] \end{pmatrix} \quad (equation\ 2)$$

Calculation of Geometric Mean:

In an investor's wealth perspective, the growth of the asset over the entire period [0, T] should be expressed as a geometric mean. The use of geometric mean has far better properties in terms of the interpretation of asset returns as compared to the arithmetic mean. In analyzing wealth over a longer period of time, the geometric mean conveys what the average financial rate of return would have been over the whole duration of the investment period.

Then for n single periods returns, a sequence of asset returns defined as {Rt}nt=1 based on the sequence of asset prices {St}nt=0, the geometric mean return or the so-called "time-weighted rate of return," r(0,n), can be expressed as follows:

$$r_{(0,n)} = \left(\prod_{t=1}^{n}(1+R_t)\right)^{\frac{1}{n}} - 1 \qquad (3)$$

3.7 Selection of the securities to be used in constructing portfolios.

For the purposes of the research stocks will be chosen on the basis of their average daily mean returns. The top 13 were selected to create n asset portfolios while giving considerations of the correlation between the assets.

These assets were then to be used to create efficient frontier for the, n, asset portfolio for both the mean variance and semi variance optimization method.

3.8 Mean Variance Optimization

Calculating Variance of the Risky Portfolio

Defining w_i as the portfolio weight for security i and $E[r_i]$ as the expected return for security i .

The variance of the risky portfolio can be derived as follows:

$$Var[E(r_p)] = Var\ \sum_{i=1}^{n} h_i E[r_i] = \sum_{i=1}^{n}\sum_{i=1}^{n} w_i w_j \sigma_{ij} \qquad \textbf{(4)}$$

Since $\sigma_{ii} = Var(r_i)$ and $\sigma_{ij} = Covar(r_i, r_j) = \sigma_{ji}$ because $Covar\ (r_i, r_j) = Covar(r_j, r_i)$

We can make use of a variance-covariance matrix to simplify the calculations.

$$Z = \begin{pmatrix} \sigma_{11} \cdots \sigma_{1n} \\ \vdots \ \ddots \ \vdots \\ \sigma_{n1} \cdots \sigma_{nn} \end{pmatrix}$$

Thus Z is a symmetric matrix.

The variance of the portfolio using matrices can be written as follows:

$$Var(E[r_p]) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = (w_1 \cdots w_n) \begin{pmatrix} \sigma_{11} \cdots \sigma_{1n} \\ \vdots \ \ddots \ \vdots \\ \sigma_{1n} \cdots \sigma_{nn} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = W^T Z W$$

Constraints

To minimize the variance of the portfolio we need two constraints:

I. The expected return for the optimized portfolio should be equal to the one obtained by getting the sum of the weighted returns in the portfolio

$$\sum_{i=1}^{n} w_i\, E[r_i] = r^* \rightarrow \sum_{i=1}^{N} w_i\, E[r_i] - r^* = 0$$

Or written using matrices:

$$w.R - r^*$$

II. The portfolio weights should sum up to one (the full investment constraint).So as to meet the assumption that the investor wants to invest all his money in the risky portfolio.

$$\sum_{i=1}^{N} w_i = 1 \rightarrow \sum_{i=1}^{N} w_i - 1 = 0$$

Constraint optimization

Using the Lagrangian method of optimization we can minimize the variance of the portfolio with respect to the two constraints and solve using the tools for the linear algebra.

$$\mathcal{L}(w_1, \dots, w_N, \lambda_1, \lambda_{2)} = \sum_{i=1}^{N}\sum_{j=1}^{N} Cov(x_m, x_n) + \lambda_1\left(\sum_{i=1}^{N} w_i\, E[r_i] - r^*\right) + \lambda_2\left(\sum_{i=1}^{N} w_i - 1\right)$$

This equation can be simplified using matrix notation as follows:

$$\mathcal{L}(W, \lambda_1, \lambda_2) = W^T VW + \lambda_1(W.R - r^*) + \lambda_2(W.\underset{1}{\rightarrow} - 1)$$

Taking the first order condition of this lagrangian equation we find the partial derivative of a scalar with respect to a vector as follows:

$$\frac{\partial \mathcal{L}}{\partial W} = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial W_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial W_n} \end{pmatrix}$$

$$\mathcal{L} = (w_1 \dots w_{n)} \begin{pmatrix} \sigma_{11} & \dots & \sigma_{n1} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \dots & \sigma_{nn} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} + \lambda_1\left(\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} . \begin{pmatrix} E[r_1] \\ \vdots \\ E[r_n] \end{pmatrix} - r^*\right) + \lambda_2\left(\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} . \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - 1\right)$$

$$\mathcal{L} = (w_1 \quad \dots \quad w_n) \begin{pmatrix} w_1 + & \dots + & w_n\sigma_{1n} \\ & \vdots & \\ w_n\sigma_{n1} & \dots & w_n\sigma_{nn} \end{pmatrix} + \lambda_1(w_1 E[r_1] + \dots + w_n E[r_n] - r^*) + \lambda_2(w_1 + \dots$$
$$+ w_n - 1)$$

$$\mathcal{L} = w_1^2\sigma_{11} + \dots + w_n w_1 \sigma_{1n} + \dots + w_n w_1 \sigma_{n1} + \dots + w_n^2 \sigma_{nn} + \lambda_1(w_1 E[r_1] + \dots + w_n E[r_n])$$
$$+ \lambda_2(w_1 + \dots + w_n - 1)$$

$$\mathcal{L} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} + \lambda_1(w_1 E[r_1] + \dots + w_n E[r_n]) + \lambda_2(w_1 + \dots + w_n - 1)$$

$$\frac{\partial \mathcal{L}}{\partial W} = \begin{pmatrix} 2w_1\sigma_{11} + \dots + 2w_n\sigma_{1n} \\ \vdots \\ 2w_1\sigma_{n1} + \dots + 2w_n\sigma_{nn} \end{pmatrix} + \lambda_1 \begin{pmatrix} E[r_1] \\ \vdots \\ E[r_n] \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial W} = 2.VW + \lambda_1 R + \lambda_2.1$$

The partial derivative with respect to the other variables, λ_1, λ_2, will be:

$$\frac{\partial L}{\partial \lambda_1} = W.R - r^*$$

$$\frac{\partial L}{\partial \lambda_2} = W.1 - 1$$

Equating the partial derivatives for to zero will enable as to obtain the values of the λ_1, λ_2 i.e

$$\frac{\partial L}{\partial \lambda_1} = W.R - r^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial W} = 2.VW + \lambda_1 R + \lambda_2.1 = 0$$

The optimization process using the Lagrangian optimization method is however tedious, time consuming and prone to mistakes if done manually. This can however be simplified and done much faster using the solver add in on Microsoft excel or in Mat lab. It involves inputting the data required in excel. You then use the data to find the mean returns of the securities and the variance covariance matrix. Setting up the weights, you can then find the expected return and variance of the portfolio. Finally using the Solver add-in you can minimize the variance of the expected return subject to the constraints that the weights of securities in the portfolio should be equal to one. The other constraint, that the expected return should be equal to the weighted sum of the expected returns of the securities, is already satisfied by the method used in obtaining the expected returns before optimization in excel.

3.9 Mean Semi Variance Approach

The underlying principle for semi variance model is the same as the variance model, in that investors are willing to bring downside risk as low as possible while keeping the rate of return above a certain level.

The definition of semi variance below the mean value can be expressed by the following formula:

$$Semi - Variance = \frac{1}{n} \sum_{r_i < Target}^{n} (\ddot{r} - r_t)^2 \ (4)$$

$$SV \ = \frac{1}{N} \sum (r_t - b)^2) = (r_t - E) + (E - b) \quad (5)$$

$$SV(<) = \sum (\sum r_{jt} w_j - \sum r_j w_j)^2) P_{(t)}) \quad (6)$$

Where for each observation $\check{r}_j$ satisfying

$$\sum r_{jt} w_j \leq \sum_{j=1}^{n} r_j w_j \quad (7)$$

Similarly semi variance above the mean can be expressed as:

$$SV(>) = \sum r_{jt} w_j \leq \sum_{j=1}^{n} r_j w_j \quad (8)$$

$$SV(>) = [(\sum_{j=1}^{n} r_{jt} w_j - \sum_{j=1}^{n} r_j w_j)^2 \, P_{(t)}] \quad (9)$$

Where P_t denotes the probability. If we assign the same probability for all observations then we have $P_t = 1/T$

To further derive the simplified formula for semi variance model, we have to make an important assumption, which is the validity of Sharpe (1964) beta regression equation:

$$\tilde{r}_j = \alpha_j + \beta_j . \tilde{r}_M + \varepsilon_J \quad (10)$$

It states that the random variable of the jth asset's return is related to the market portfolio return, return is related to the market portfolio return, whereα_J and β_j are constant, and $\breve{\varepsilon}_J$ is a random error with zero covariance for ($\breve{\varepsilon}_J$, $\widetilde{\varepsilon_n}$) zero covariance for ($\breve{\varepsilon}_j$, $\tilde{r}_M$). The market portfolio is the weighted sum of asset returns.

In addition, we can obtain β_J by:

$$\beta_J = \frac{cov(\check{r}_j, \tilde{r}_{m.})}{\sigma_M^2} \quad (11)$$

Where $cov(\check{r}_j, \tilde{r}_{m.})$ is the covariance between the return of the jth asset and the return of the market portfolio, σ_m is the variance of the market return.

Based on equation (7), we can get:

$$\check{r}_j - r_j = \theta_j + \beta_j.(\tilde{r}_m - r_m) \; (12)$$

Where $r_j$and r_m are expected return of the jth asset and market portfolio respectively.$\breve{\theta}_j$ has zero mean value.

Considering all the assets in the portfolio and adding them up based on equation (13),

$$\sum_{j}^{n} (\tilde{r}_j{}_- r_j) w_j = \tilde{\theta} + (\widetilde{\tilde{r}_m} - r_m) \sum \beta_j w_j \; (13)$$

Where $\tilde{\theta} = \sum_{j=1}^{n} \tilde{\theta}_j . w_j$

Based on (13), we can rewrite the equation of *SV* (>) as:

$$SV(>) = \sum [\left(\breve{\theta} + (\tilde{r}_m - r_m) \sum_{j=1}^{n} \beta_j w_j \right)^2 P_{(t)}] \; (14)$$

When the level of diversification goes to infinity, we can prove that:

$$\lim_{n \to \infty} SV(>) = (\sum_{j,h} \beta_j \, \beta_h w_j w_h).SV(\tilde{r}_m > r_m) \qquad (15)$$

The definition of *SV* (>) and SV (<) implies that:

$$SV(<) + SV(>) = V = \sum_{t=1}^{t} [\sum_{j=1}^{n} r_{jt} w_j - \sum_{j=1}^{n} r_j \, w_j]^2 \, P_{(t)} \quad (16)$$

Hence, we can get the expression of SV (<) by subtracting SV (>) from V:

$$\lim_{L\to\infty} SV(<) = V - \lim_{n\to\infty} SV(>) \quad (17)$$

$$\lim_{L\to\infty} SV(<) = \sum_{j,h} V_{j,h} - \beta_j\beta_h.SV(\tilde{r}_m > r_m)w_jw_h \quad (18)$$

Note that the definition of the level of diversification is: a portfolio is considered to reach a level L of diversification if: $\max_{j=1,,,n} w_j = 1/L$ and $n/L = q\ Q > q \geq 1\ \ and\ \sum_{j=1}^{n} w_j = 1, w_j \geq 0$ for all j. Q is a constant that defines the bound of q. Then the higher the level of diversification is reached, i.e., the higher the value of L, the lower is the greatest weight.

From equation (17), we can see the whole calculation is much simplified, and thus, all the data or parameter can thus be obtained.

Hogan & Warren (1972) presents the essential mathematical properties of mean-semi variance models, where they prove the convexity and differentiability of this model. Their contributions make the theoretical and computational viability of mean-semi variance model guaranteed.

1. Calculation of Semi Variance Matrix

The first step is to derive the beta coefficients of various securities. We calculate the variance and covariance matrix of different securities.

The second step is to compute the market mean and variance values as follows:

$$E_m = \frac{1}{N}\sum_{t=1}^{N} vwretd_t$$

$$\sigma_m = \frac{1}{N-1}\sum_{t=1}^{N}(vwretd_t - E_m)^2$$

Where *vwretd* is the value-weighted-return of the market. The covariance of different assets with then market return is calculated as follows:

$$Cov(X_j, X_M) = \frac{1}{N}\sum_{t=1}^{n}(\breve{R}_{jt} - R_j)(vwretd_t - E_m)$$

Then Beta is calculated according to equation (18):

$$\beta_J = \frac{cov(\breve{r}_j, \tilde{r}_m)}{\sigma_M^2}$$

The third step is to calculate the market portfolio's semi variance above the mean return:

$$V_m(>)\frac{\sum_{t=1}^{T}\max(\tilde{R}_{m,t} - E_m, 0)^2}{T}$$

The fourth step is to calculate the required semi variance matrix. Alternatively, the semi variance matrix can also be calculated as a result of the following matrix:

$$V_{nxn}(<) = Cov_{nxn}(\,) - V_m(>).\beta_{nx1}.\beta_{n*1}{}^T$$

$$Cov_{n*n}(\,) = \begin{pmatrix} \sigma_1{}^2 & \cdots & cov_{1,n} \\ \vdots & \ddots & \vdots \\ cov_{n1} & \cdots & \sigma_n^2 \end{pmatrix}$$

$$\beta_{n*1}^T = (\beta_1, \beta_2, \ldots., \beta_n)$$

Optimization is then applied using mat lab, R, excel or a combination of all three.

3.10 Efficient Frontiers

Efficient frontiers were drawn graphically after a simulation on the optimal weights for the risk measures.

Once the various weights are obtained an analysis for the respective model will be applied. We compare the return using the performance of the securities over the 7-year period and come up with a conclusion to meet the research's first objective.

To meet the second objective, the research then uses the geometric mean to compute the mean variance optimized portfolio and compare the results and variances of the same using a statistical model.

To compare the effectiveness of the two optimization methods the research will first determine if there is a difference in the risk-adjusted return metrics produced by both optimization methods and secondly to statistically test the differences. Two tests will be used to analyze the output data, an F-test, to test whether the output had equal variances or not.

4 Results and Analysis

4.1 Portfolio Selection

The portfolio to be used in the research was derived from the Nairobi Securities Exchange data which contained 50 stocks. To select 13 portfolios. All stocks that had been delisted between 2008 and 2015 were filtered out. The research assumed that 2011 investment date any stock and data used to create the efficient frontier would therefore be a stock that had existed for at least two years any stock that was not in existence as from 2008 was also filtered out.

For the remaining stocks the mean returns of the stocks over the two-year period was found and the loss making stocks were filtered out, since rational investor would not invest in a stock that has been losing value over a long period of time. The stocks that had positive returns on average were then ranked. From the top 20 stocks ,13 stocks were selected. The selection criteria being the top two stocks from each available industry. The 13th stock was chosen on the basis of which had the highest return over the time period from the remaining 8 stocks. This was to ensure that the portfolio universe was well diversified.

4.2 Portfolio Analysis

In order to achieve the first objective of the research i.e. comparing the mean variance approach to optimization with the semi variance approach, the mean monthly returns for the individual stocks were computed together with their respective risk measures i.e. variance, semi variance. Semi variance was computed using the mean returns for each stock as its benchmark return.

The results were as follows:

Mean and Variance of the selected stocks

Table 1

Stock	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6	Stock 7	Stock 8	Stock 9	Stock 10	Stock 11	Stock 12	Stock 13
MMR	1.65%	3.08%	1.79%	2.46%	6.85%	6.11%	4.81%	3.05%	3.45%	2.98%	4.07%	2.37%	0.31%
Variance	1.39%	0.89%	1.14%	4.02%	7.06%	2.34%	1.97%	0.23%	1.78%	2.18%	0.75%	1.26%	0.74%

Mean and Semi variance of selected stocks:

Stock	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6	Stock 7	Stock 8	Stock 9	Stock 10	Stock 11	Stock 12	Stock 13
MMR	1.65%	3.08%	1.79%	2.46%	6.85%	6.11%	4.81%	3.05%	3.45%	2.98%	4.07%	2.37%	0.31%
SV	0.19%	0.43%	0.40%	1.05%	0.59%	0.37%	0.67%	0.05%	0.39%	0.74%	0.27%	0.47%	0.23%

Table 2

The second objective was to compare the geometric mean variance portfolio of returns and that of the created using the semi variance optimization methodology.

The resultant geometric means for the individual stock returns over the 2-year period are as tabulated below.

Stock	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6	Stock 7	Stock 8	Stock 9	Stock 10	Stock 11	Stock 12	Stock 13
MMR	1.65%	3.08%	1.79%	2.46%	6.85%	6.11%	4.81%	3.05%	3.45%	2.98%	4.07%	2.37%	0.31%
GMR	1.05%	2.62%	1.24%	0.79%	4.40%	5.16%	3.91%	2.94%	2.69%	1.93%	3.71%	1.76%	-0.04%

Table 3

4.3 Optimization

To obtain the set optimal portfolio returns per unit risk the portfolio had to be optimized to give the highest return for a given set of risk. For the purposes of the research and to eliminate bias from the research results optimization was done using three different scenarios.

1. An investor whose annualized rate of return is 55% with no restrictions on the weightings of the portfolio.
2. An investor whose annualized rate of return is 45% with restrictions on the weightings of the portfolio such that it must invest 50% on the first 6 stocks and 50% on the remaining 7 stocks.

The selection of scenarios was to see if the research results were consistent under a different set of investor objectives and to ensure that there was no bias in the research results.

No short selling of assets was allowed in any of the scenarios i.e. no negative weights.

4.4 Scenario 1

4.4.1 Mean Variance Optimization

As per the set conditions in scenario one the efficient frontier generated using the financial toolbox was as follows.

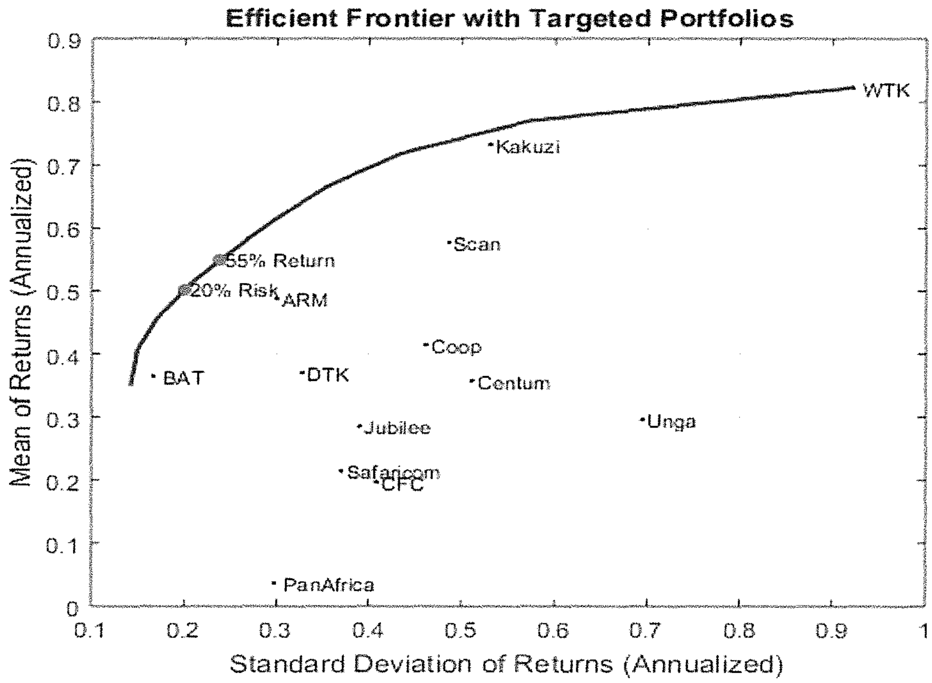

Figure 1

The optimal portfolio for an annualized return of 55% was as follows:

Stock	Weights
WTK	10%
Kakuzi	20%
Scan	16%
BAT	28%
ARM	26%

Table 4

BAT and ARM were accorded the highest weights as they had high return compared to the risk taken up as measured by standard deviation. The least weights were to WTK as it had a higher risk and higher return.

4.4.2 Mean Semi Variance optimization

As per the set conditions in scenario one the efficient frontier generated using the financial toolbox was as follows.

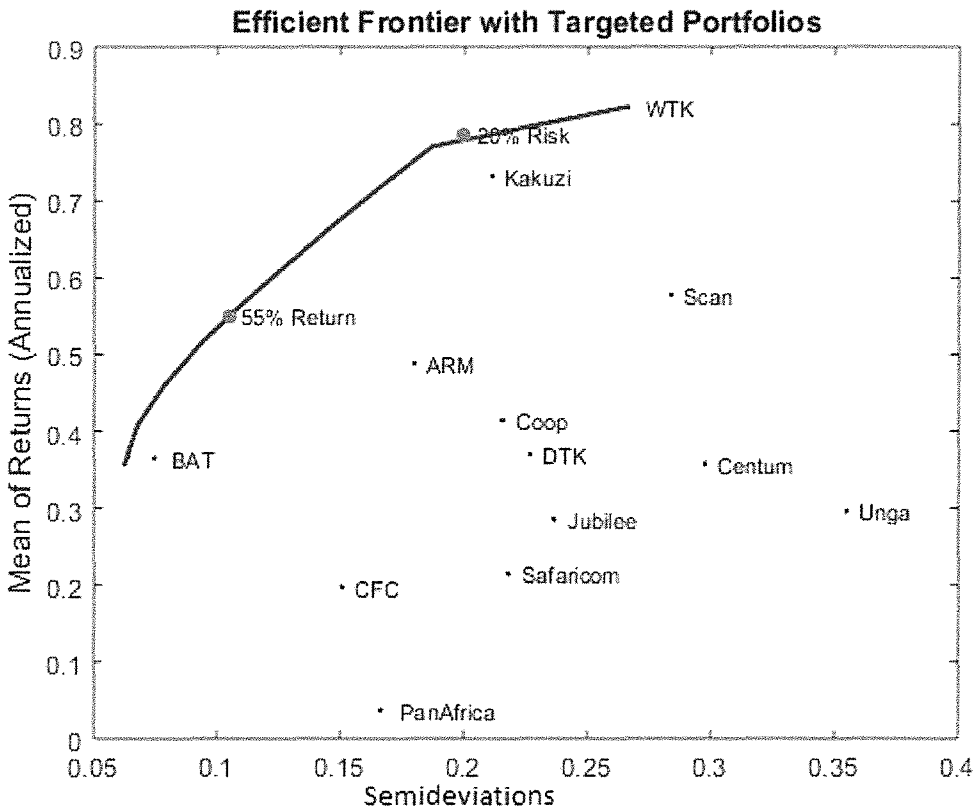

Figure 2

The optimal portfolio for a targeted return of 55% comprised of the following stocks:

Stock	Weights
WTK	21%
Kakuzi	20%
BAT	45%
ARM	15%

Table 5

BAT was accorded the highest weight as it had the lowest semi variance hence less risk. Scan group was eliminated as it had a higher semi deviance compared to standard deviation.

4.4.3 Geometric Mean Variance Optimization.

Using the geometric mean variance optimization method under scenario 1 yielded the following as the constituents of the optimal portfolio.

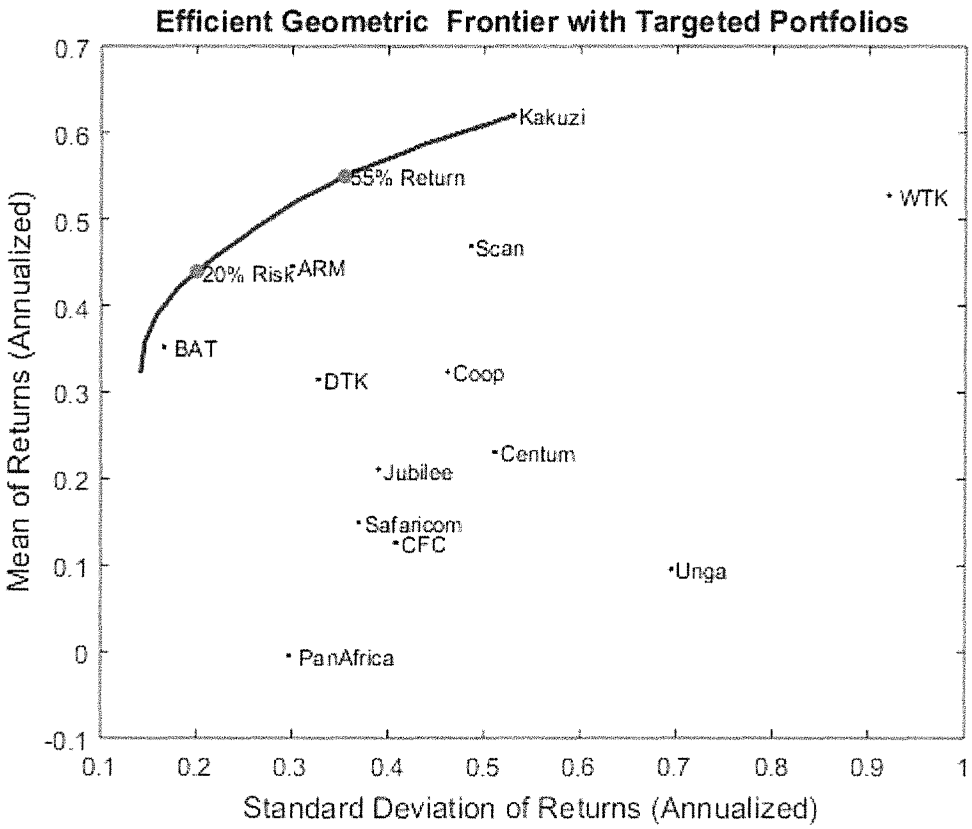

Figure 3

The portfolio weights were distributed as follows.

Stock	Weights
WTK	2%
Kakuzi	57%
Scan	15%
ARM	26%

Table 6

Kakuzi was accorded most weight as it lied on the efficient frontier.BAT had a no allocation in the geometric frontier as there were more efficient stocks in comparison i.e. ARM.

4.5 Scenario 2

Scenario 2 involved pacing constraints to the optimization tool. The portfolio scenario was to invest a maximum 50% to stocks 1 to 6 and the rest to stocks 7 to 13.

4.5.1 Mean Variance Optimization

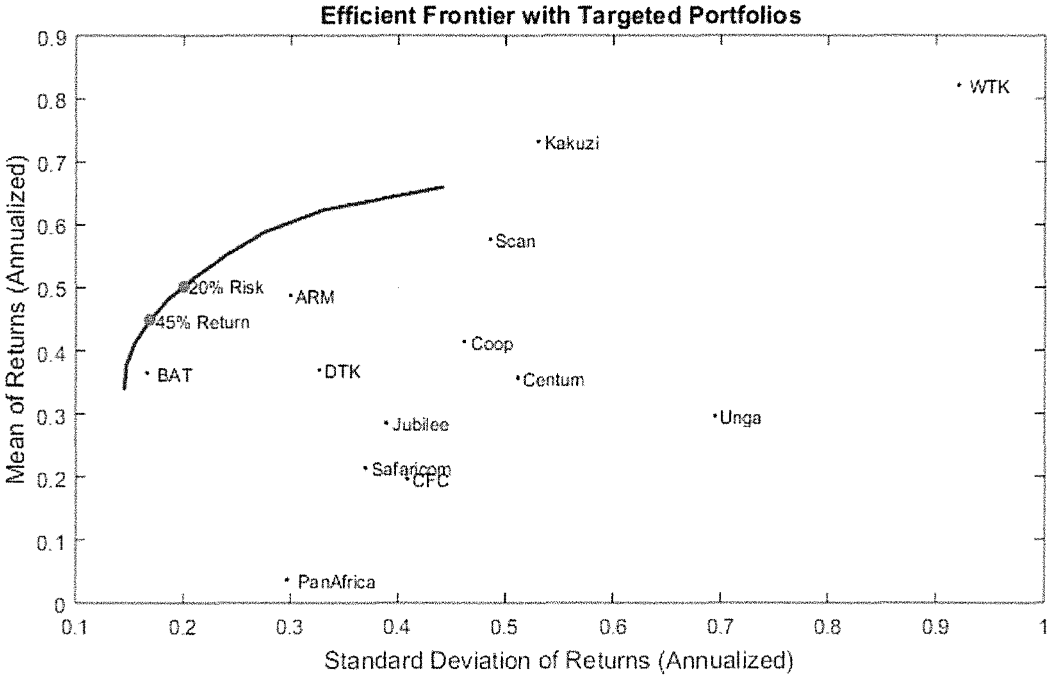

Figure 4

For a return of 45% the standard deviation was 0.18%. The optimal stocks were as follows:

Stock	Weights
Safaricom	3%
WTK	7%
Kakuzi	2%
Scan	9%
BAT	54%
ARM	24%

BAT, ARM and Scan group were accorded most of the weight as they had a low risk return ratio as compared to the other stocks.

4.5.2 Mean Semi variance optimization.

Using the Matlab quadratic programing tool and the financial tool box an efficient frontier was generated for semi variance. The output is shown on the next page.

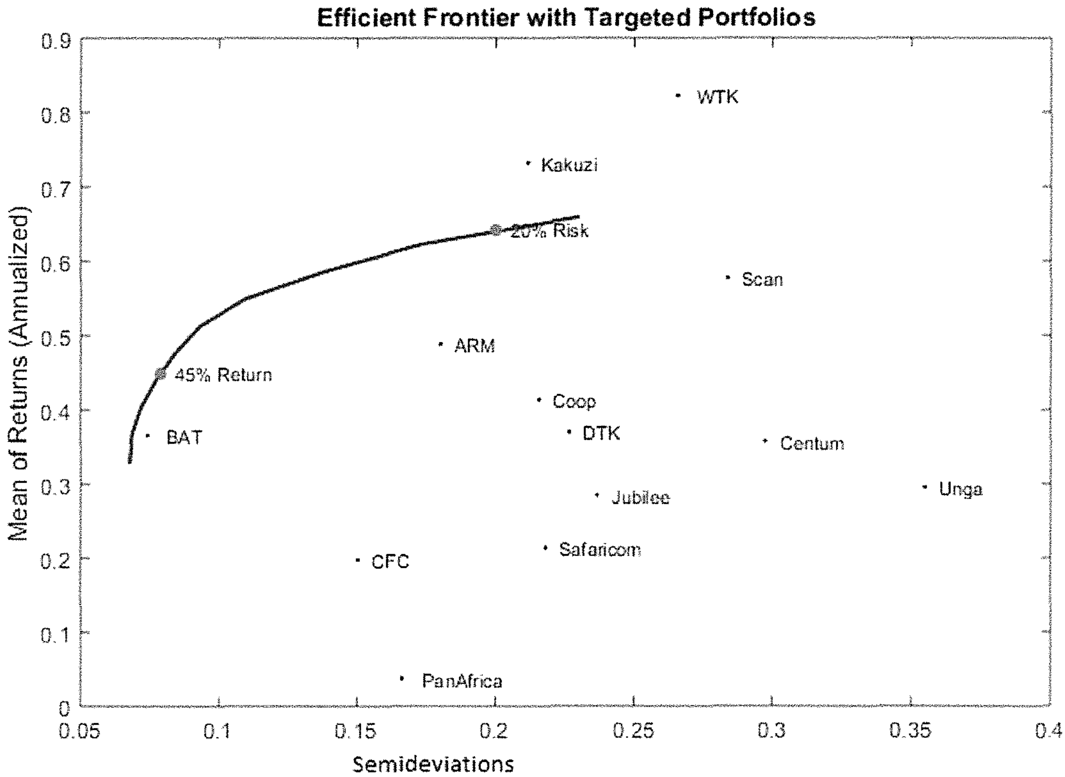

Figure 5

At a return of the 45% the portfolio generated consisted of the following stocks.

Stock	Weights
CFC	1%
Safaricom	3%
WTK	14%
BAT	59%
ARM	23%

Table 7

A stark difference between the two portfolios is the omission of scan group by the optimization tool which as shown in the graph has a higher semi variance compared to other stocks.

4.5.3 Geometric mean variance yielded the following efficient frontier.

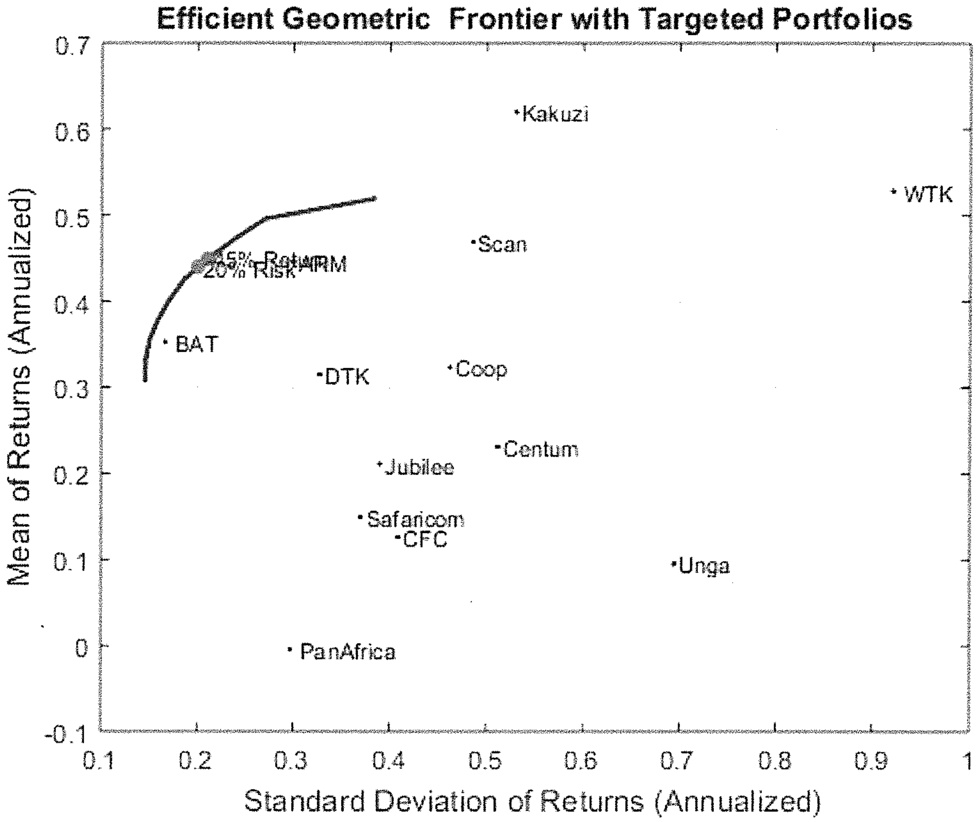

Figure 6

The resultant 45% portfolio had the following stocks comprising the portfolio:

Stock	Weights
WTK	4%
Kakuzi	19%
Scan	9%
BAT	38%
ARM	30%

Table 8

BAT and ARM and ARM were accorded the most weights as they had a high risk return ratio. WTK had the least weight given its had a low return but high risk.

4.6 Portfolio Performance Analysis.

Using an assumption of a base capital of a million Kenya shillings invested in the 3 different portfolios. Portfolio returns were measured on a monthly basis to check the performance of the three portfolios over time.

4.6.1 Scenario 1.

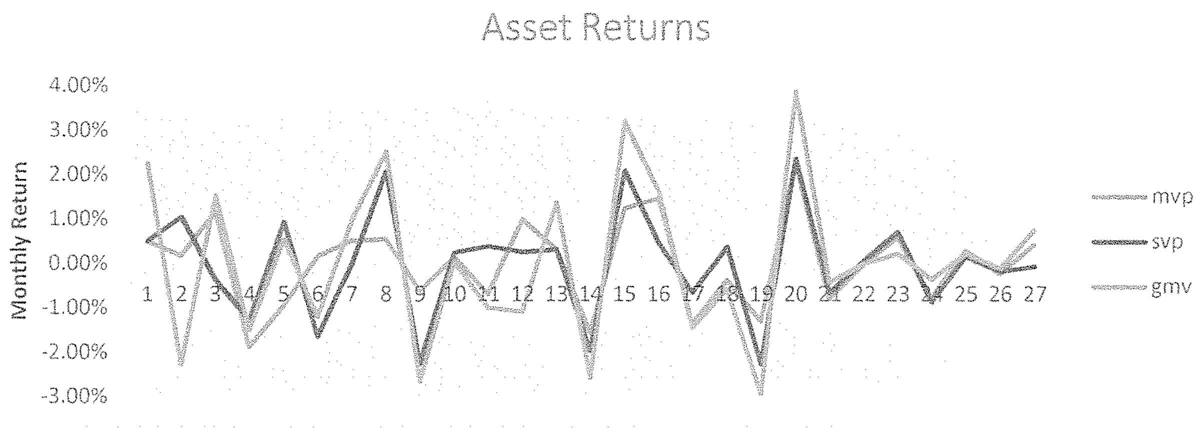

Figure 7

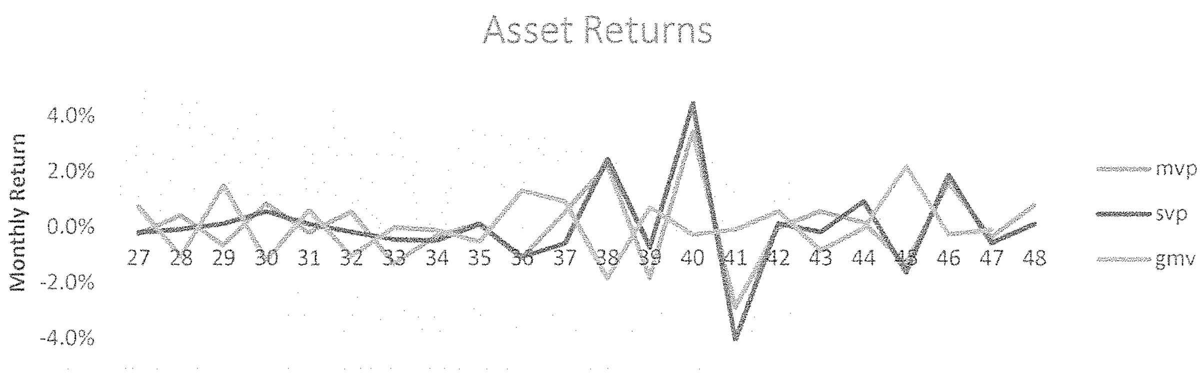

Figure 8

The first objective of the research is to compare the returns obtained using the mean variance approach and the mean semi variance approach. For the first months of investments in the stocks i.e. between month one to 11 there were differences between the portfolios under each strategy. Past the month eleven the results smoothen out and the mean variance and semi variance portfolio give roughly the same returns.

The variability of returns can be linked to the post-election violence where the stock exchange was still recovering from it.

The second objective was to compare the returns of the geometric mean variance optimization and that of the semi-variance. As seen from the above graphs the geometric mean portfolio tends to follow the geometric mean variance especially in the periods of great volatility in return

Past the recovery phase the portfolio exhibited more or less the same monthly returns with slight differences.

4.6.2 Scenario 2.

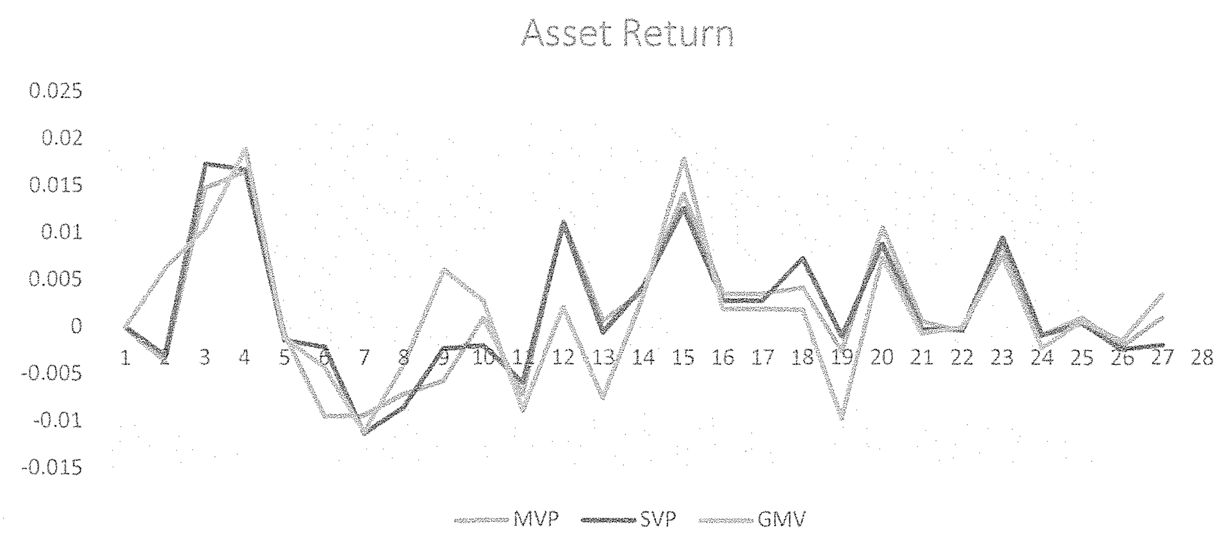

Figure 9

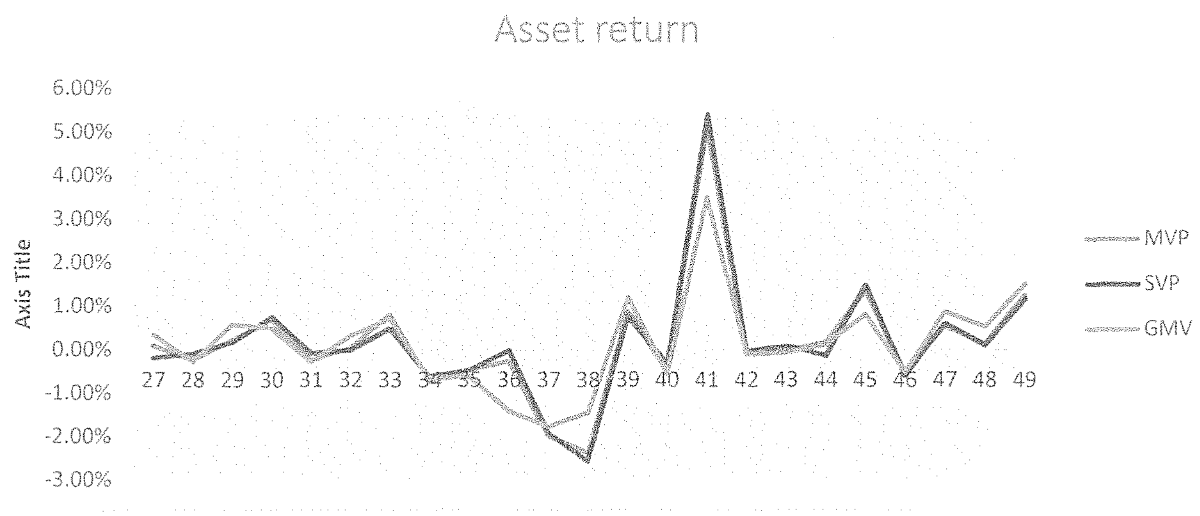

Figure 10

The mean variance and semi variance portfolios have the same returns generally as seen in the graph above. The sharp peaks and drops relate to ARM which had share splits over 2013 that led to volatility to in its price. The geometric mean variance portfolio had more of its assets invested in ARM hence the significant drop in value.

The other sharp drops were due to country risk which led to the NSE shares falling in prices. See Appendix.

4.7 Test for Normality

A test for normality for the NSE index was performed to check whether the returns tend towards the normal distribution.

The Shapiro-Wilk Normality Test for normality for the NSE monthly returns gave, W = 0.93502, p-value = 0.1928, we can therefore fail to reject the null hypothesis and conclude that NSE returns during the period under study tend to be normally distributed though not a perfect fit.

A normal QQ plot was performed for the index return and yielded the following plot

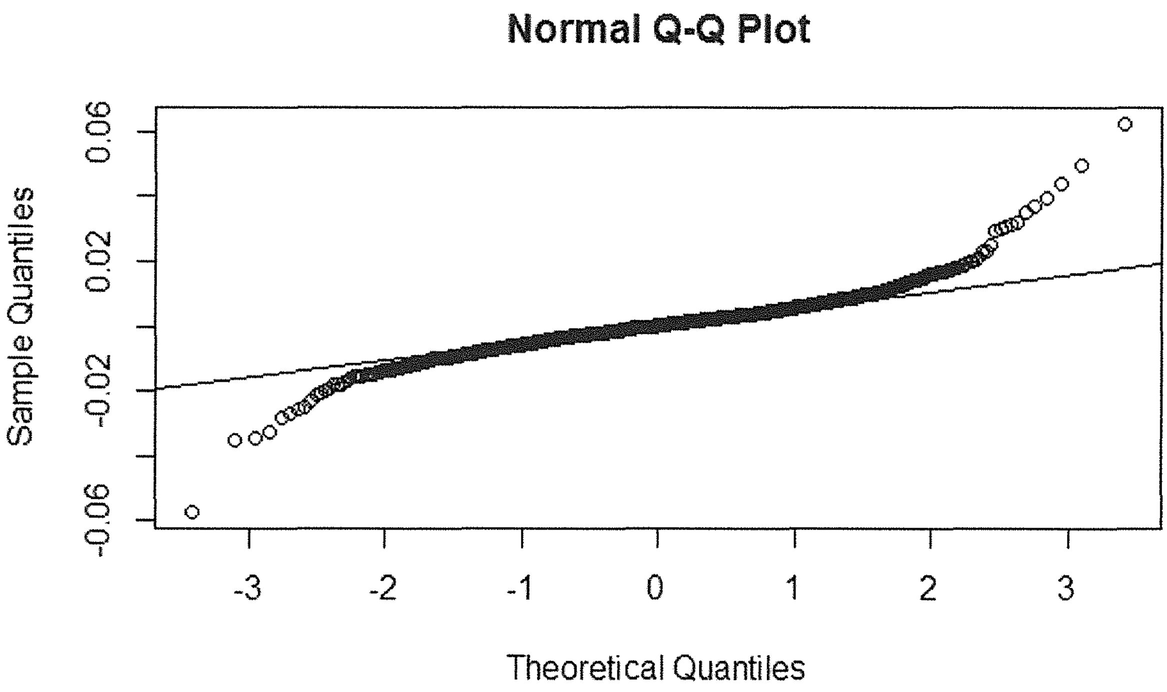

Figure 11

Most of the data fits into the normal QQ line and hence can be said to tend towards a normal distribution.

5 Discussions and Conclusion

The results on the performance of the portfolios show that there is no much difference using a semi variance approach as opposed to a mean variance approach to investing. This is due to the fact that the Nairobi Stock exchange monthly equity returns seem to follow the normal distribution. According to (Boasson, Boasson, & Zhou, 2011) variance can only prove to be an inefficient risk measure if the returns of securities are not normally distributed as is normally the case. According to (Hull, 2008) if returns on assets are symmetrically distributed semi-variance is proportional to variance. This may explain why there is no significant difference between the two strategies.

An F-test was conducted to check whether the results of the performance Geometric mean variance and Mean semi variance approach, at the 95% percent confidence interval a p-value of 0.98 was obtained meaning that they are not statistically different.

On the question of whether the geometric mean variance can approximate the mean semi variance approach, the portfolio performance and F-test indicate that it actually can approximate the semi variance approach. This may be however due to the fact that equity returns in Kenya are normally distributed. Further research should therefore be done to ascertain if it actually holds when the returns are from another distribution.

5.1 Limitations of the Study.

The study only considers two scenarios in its study of the various optimization techniques rather than using different sets of scenarios. Further research should be done in order to check if the results are consistent with different set of portfolio constraints.

The data set used from the Nairobi securities exchange was from 2009-2014, which emerged to be normally distributed. It may not be the case for a different time period as the economic environment keeps changing over time.

Historical data for security prices from the NSE are not readily available but have to be purchased. This limits the amount of data that can be obtained. There was time constraint in conducting the study.

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7 Appendix

1.NSE Returns over time.

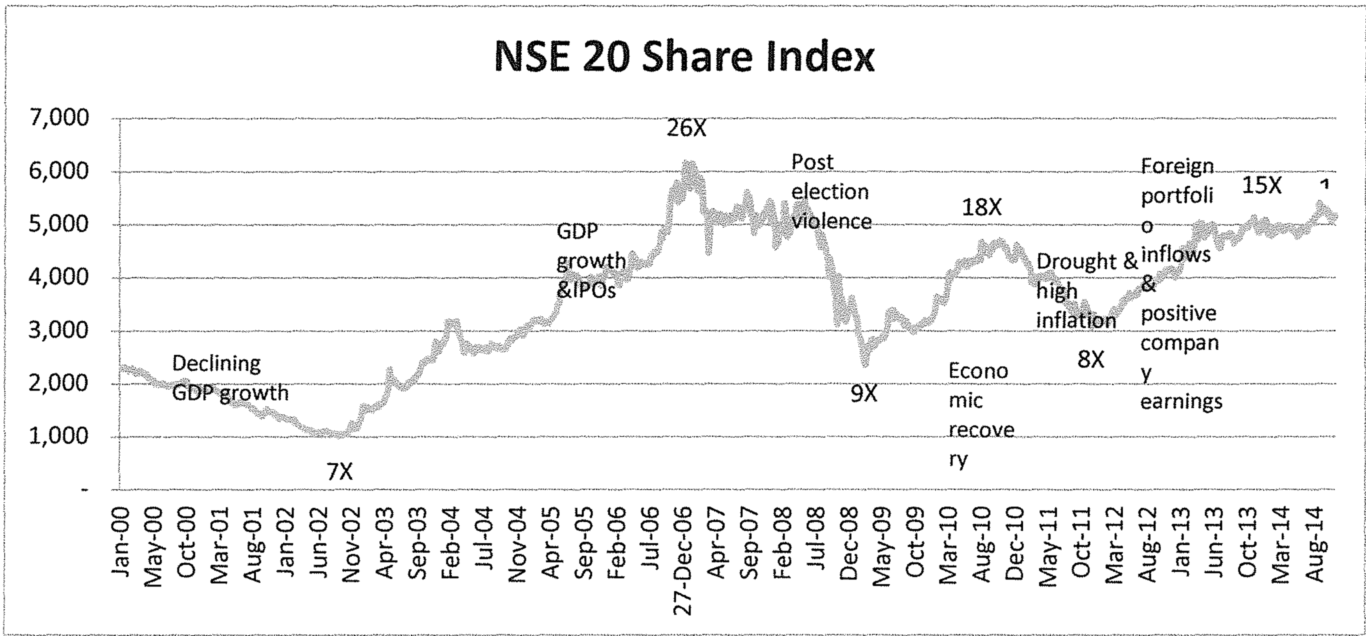

Figure 12

2.F-test results.

	3. F test to compare two variances
	4.
	5. data: SVP and GMV
	6. F = 1.0048, num df = 48, denom df = 48, p-value = 0.9869
	7. alternative hypothesis: true ratio of variances is not equal to 1
	8. 95 percent confidence interval:
	9. 0.5667627 1.7812574
	10. sample estimates:
	11. ratio of variances
	12. 1.004764

Figure 13