



Strathmore Institute of Mathematical Sciences(SIMS)
Bachelor of Business Science (Financial Engineering)
End of Semester Examination
BSM 3114 - Differential Equations with Financial Applications

Date: Wednesday 24th July 2024

Time: 2 Hours

Instruction

1. Answer **QUESTION ONE** and any other **TWO QUESTIONS**

QUESTION ONE [30 Marks]

a) Solve the following differential equations by direct integration:

i. $\frac{d^2y}{dx^2} = x^2 + 2$ given that $y = 1$ when $x = 0$ and $y' = 2$ when $x = 0$. [3 Marks]

ii. $\frac{\partial^2 u}{\partial x^2} = 2xe^t$ given that $u = t$ when $x = 0$ and $\frac{\partial u}{\partial t} = e^t$ when $x = 0$. [3 Marks]

iii. $\frac{\partial^2 u}{\partial x^2} = \text{Sin}(x)\text{Cos}(y)$ given that $\frac{\partial u}{\partial x} = 2x$ at $y = \frac{\pi}{2}$ and $u = 2\text{Sin}(y)$ at $x = \pi$. [3 Marks]

b) Give the order and degree of the following differential equations:

i. $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$. [2 Marks]

ii. $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{0.5}$. [2 Marks]

c) Given that $g_1 = x^{-2}$ is a solution to the differential equation $x^2y'' + 6xy' + 6y = 0$, use reduction of order to find the other solution g_2 . [3 Marks]

d) Give the integrating factor for the differential equation below:

$$\frac{dy}{dx} (x \log(x)) + y = 2 \log(x)$$

[2 Marks]

- e) Show that $x = \exp(2t)[t + 1]$ and $y = \exp(2t)[t - 1]$ is a solution to the system below: [4 Marks]

$$\begin{aligned}x' &= x + y + 3\exp(2t) \\y' &= 2x - 3\exp(2t)\end{aligned}$$

- f) Solve using the integrating factor method: [4 Marks]

$$\frac{dy}{dx} + y \tan(x) = 2 \sin(x) \cos(x)$$

- g) Solve by separation of variables the differential equation $y' = e^{2x+y}$ given that $x = 0$ when $y = 0$. [4 Marks]

QUESTION TWO [20 Marks]

- a) A Bernoulli differential equation (named after James Bernoulli) is of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$. Determine whether the equation is linear or nonlinear when $n = 0$ and $n = 1$. Further, show that the substitution $u = y^{n-1}$ transforms the Bernoulli equation into the linear equation; [5 Marks]

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x)$$

- b) Let $P(t)$ be the performance level of someone learning a skill as a function of training time t . The graph of P is called a learning curve and can be defined by

$$\frac{dP}{dt} = k(M - P(t))$$

for some positive constant k . Is the equation linear or non linear?, solve it using the integrating factor method. [5 Marks]

- c) Find the constant $\lambda \in \mathbb{R}$ such that $y(t) = x^\lambda$ is a solution to the differential equation $x^2 y'' + x y' = y$. [3 Marks]

- d) Two workers are hired for an assembly line. Pauline processed 25 units during the first hour and 45 units during the second hour. Grace processed 35 units during the first hour and 50 units during the second hour. Using the model $P(t) = M + Ce^{-kt}$ where $P(t)$ is the performance level of someone learning a new skill as a function of training time t . Determine the number of units that can be produced in one hour by each worker. [7 Marks]

QUESTION THREE [20 Marks]

a) Solve the following systems:

i. $\mathbf{x}' = A\mathbf{x}$ given that $A = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$ [4 Marks]

ii. $\mathbf{y}' = B\mathbf{y}$ given that $B = \begin{bmatrix} -2 & 6 \\ -3 & 4 \end{bmatrix}$ [5 Marks]

iii. $\mathbf{t}' = C\mathbf{t}$ given that $C = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$ if $\mathbf{t}(0) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ [5 marks]

b) Find the general solution of the following system of differential equations:

$$\begin{cases} y_1' = 3y_1 - 7y_2 - 3y_3 \\ y_2' = y_1 - 4y_2 - 2y_3 \\ y_3' = y_1 + 2y_2 + 2y_3 \end{cases}$$
 [6 marks]

$$\begin{aligned} y_1' &= 3y_1 - 7y_2 - 3y_3 \\ y_2' &= y_1 - 4y_2 - 2y_3 \\ y_3' &= y_1 + 2y_2 + 2y_3 \end{aligned}$$

QUESTION FOUR [20 Marks]

a) Use the method of integrating factor to solve

$$x \frac{dy}{dx} - 2y = x^4 \sin(x)$$

$$\cdot$$
 [5 Marks]

b) Find the general solution to $2t^2 y'' + ty' - 3y = 0$ given that $y_1 = t^{-1}$ is one solution. [5 Marks]

c) By dividing through by x^2 and making the substitution $y = vx$ where v is also a function of x , solve the IVP $xyy' + 4x^2 + y^2 = 0$ given that $y(2) = -7$. [5 Marks]

d) Given $y' - (4x - y + 1)^2 = 0$ with the condition $y(0) = 2$, make the substitution $v = 4x - y$ and find its solution. [5 Marks]

QUESTION FIVE [20 Marks]

a) Given

$$\begin{aligned}\frac{dp}{dS} &= q^2 - pq - q \\ \frac{dq}{dS} &= 2p^2 + pq - 7p.\end{aligned}$$

Find the equilibrium point(s), phase plane diagram and the relationship between the variables p and q . [8 Marks]

b) Consider the non-linear system

$$\begin{aligned}x' &= -x + xy \\ y' &= xy - 2y.\end{aligned}$$

Determine:

- i. The critical point(s) [2 Marks]
 - ii. The Jacobian matrix/matrices [2 Marks]
 - iii. The eigenvalues of *ii* above. [2 Marks]
 - iv. The corresponding eigenvectors. [2 Marks]
 - v. The trajectories of the system [2 Marks]
- c) List the two methods that can be used to analyse a non linear system of the form [2 Marks]

$$\begin{aligned}\frac{dp}{dR} &= F(p, q) \\ \frac{dq}{dR} &= G(p, q)\end{aligned}$$