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The Relative Performance of Single Index versus Multifactor Models in Determining the Efficient Frontier in Kenya

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Declaration

I declare that this work has not been previously submitted and approved for the award of a degree by this or any other University. To the best of my knowledge and belief, the Research Proposal contains no material previously published or written by another person except where due reference is made in the Research Proposal itself.

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Abstract

Since the early 1950s models have been developed to aid wealth allocation in order to optimize returns. This study seeks to compare the relative performance of the single-index models and multifactor models in determining the optimal portfolio wealth allocation. The efficient frontier is determined through minimizing risk as measured by standard deviation while taking into account historical factor betas between 2001 and 2012. The study establishes that the single index model outperforms the multifactor model as it yields the highest Sharpe ratios. These findings can be attributed to the fact that the market model contains the characteristics of the macroeconomic variables in the single index model.

Key words: single index model, multifactor model, cut off rate, Sharpe ratio and efficient frontier

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List of abbreviations

NSE	-	Nairobi Stock Exchange
I	-	Inflation
GDP	-	Gross Domestic Product
FX	-	Foreign Exchange
M2	-	Money supply, aggregate 2
C	-	Consumption
T-Bill	-	Treasury Bill
MVO	-	Mean Variance Optimization

1 Introduction

1.1 Background to the study

Starting with the mean-variance work of Markowitz (1952), scholars have relentlessly attempted to establish means of optimal wealth allocation. Under the mean-variance portfolio theorem framework, optimal portfolio allocation is achieved by minimize variance while holding expected return constant or maximizing expected return while holding variance constant. Therefore, the maximum return given the different levels of risk will give one the efficient portfolio, which plot on the efficient frontier. The efficient frontier would then enable investors to choose their preferred portfolio depending on their risk preferences.

Mean-variance portfolio theory raised questions regarding the importance of input estimation which further required the estimation of correlation coefficients using index models. The first index model was developed and popularized by Sharpe (1963), who used a single-index model with the market model being the explanatory variable. The single-index model can be seen as where the movement in stocks is due to a single common influence or index (Elton, Gruber, Brown and Goetzmann, 2007). The single-index model was accepted for its advantages such as: it reduces the number of estimates required, it is easy to understand and it increases the accuracy of the portfolio optimization.

However, the single-index model in particular the capital asset pricing model has been challenged by authors such as Roll (1977) and Fama & French (1996) for its shortcomings, mainly relating to the mean-variance efficiency of the market portfolio. In addition, the validity of the priced risk factor has been questioned.

Researchers resorted to the use of multifactor models to capture some of the nonmarket influences that cause securities to move together (Elton et.al, 2007). The first multifactor model, the Arbitrage Pricing Theory was introduced by Ross (1976). Subsequent inquiry has yielded increasing number of facts such as the Burmeister, McElroy and Wall (1986, 1988) five factor model, Fama and French (1995) three factor model and Fama and French

(2015) five factor model. Different scholars have different views about the factors that are to be included, which makes one wonder which one to use when conducting an empirical study.

In Kenya, and indeed in Africa, the use of index models methodology has been minimal. Some portfolio managers are using for example scenario analysis in determining which equities to invest in and forecasting of the yield curve so as to know which bonds to invest in. They then compute the justified price of each asset and long those ones which are underpriced and short those which are overpriced. Other portfolio managers evaluate the asset classes first to see which ones have a better performance so as to weight them which mostly results in 50% investment in equities, 35% investment in bonds, 10% investment in real estate and 5% cash holding. Therefore, we want to see whether the application of the single index and multifactor models would be better at determining the optimal portfolios.

1.2 Problem statement

Different scholars have different opinions about which index model is best suited for determining the efficient frontier. Scholars like Cohen (1967) and Chen and Mayers (1987), conclude that single index models are better whereas others such as Fama and French (1995) prefer using the multifactor model. Supporters of the single factor model argue that the model tends to have relatively more power for forecasting conditional returns for poorly diversified portfolios. The proponents of the multifactor model on the other hand posit that it tends to have relatively more power for forecasting conditional returns for well-diversified portfolios.

Despite the benefits of index models, investors in Kenya use the mean variance model in determining optimal portfolios. In his analysis of the Kenyan market, Muendo (2006) finds that 60% of the fund managers use the mean-variance method. Other portfolio managers use scenario analysis while others use asset evaluation to determine the asset that they would invest.

This study has compared and contrasted the performance of the single index and multifactor models in determining the efficient frontier in Kenya.

1.3 Research objective

The objective of this study is to compare the performance of the single-index and multifactor models for determining the efficient frontier in Kenya

1.4 Research question

Is the single-index better than the multifactor model in determining the efficient frontier in Kenya?

1.5 Significance of the study

This study contributes to the debate on index models and how they are a better determinant of the efficient frontier. This way the study will aid informed decision making by Kenyan investors in optimum portfolio selection thus increasing risk-adjusted expected returns.

2 Literature review

2.1 The efficient frontier

Under the Markowitz (1952), an investor would hold an asset that offers a higher return from the same risk and a lower risk for the same return. Such an asset would plot on the efficient frontier which is a set of portfolios that lie on the line between the attainable point with minimum variance and the point of maximum attainable expected return.

In order to derive the efficient set with no short sales, one should solve the problem for a given level of expected rerurn E^*:

$$Minimize\ Var\left(R_p\right) = X'CX$$

$$Subject\ to\ E\left(R_p\right) = X'E(R) = E^*$$

$$X'K = 1$$

$$X \geq 0 \tag{1}$$

Where, X is an N by 1 vector representing the proportion of the investor's funds that are to be placed in each of N securities, C is an N by N matrix representing the covariance of returns between N securities and $E(R)$ is the vector representing the expected returns of the N securities and K is an N by 1 vector with all its elements equal to one.

In constructing the efficient frontier assumptions arose when selecting the optimum portfolio. Tobin (1958) puts a restriction in that the investors are not allowed to borrow, whereas Lintner (1965) relaxes Tobin's restriction by allowing both short sales and borrowing. Merton (1972) shows that given certain conditions, the classic graphical technique for derivation of the efficient portfolio frontier is incorrect. He shows the efficient portfolio set first, when all securities are risky, the equation being,

$$E = \bar{E} + \frac{1}{C}\sqrt{DC(\sigma^2 - \bar{\sigma}^2)} \tag{2}$$

Where, E is the expected return on the portfolio, σ^2 is the variance of the portfolio,

given $A = \sum_1^m \sum_1^m v_{kj}E_j$; $B = \sum_1^m \sum_1^m v_{kj}E_jE_k$; $C = \sum_1^m \sum_1^m v_{kj}$ (v_{ij} - elements of the inverse of the variance-covariance matrix and m - the number of assets) and $D \equiv BC - A^2 > 0$, Second, in the case of a mutual fund and thirdly when there is introduction of a riskless asset with the approach being finding the efficient frontier for risky assets only and then draw a line from the intercept (R_f) tangent to the efficient frontier.

Buser (1977) concludes that while the fundamental portfolio problem is multidimensional, reflecting the number of available assets and liabilities, the set of frontier portfolios is essentially two dimensional. In contrast, Cochrane gives an example of multifactor efficient frontier which is three dimensional based on the mean, variance and recession sensitivity.

2.2 Index models

The mean-variance portfolio theory as shown by Merton, Sharpe, and Markowitz among other scholars has been used in determining the efficient portfolio, from which investors can determine optimum portfolio. The implementation of this theory is difficult in practice due to the amount and type of input data needed to perform the portfolio analysis and the computational difficulty in calculating the optimal portfolios. Simplification of the implementation process led to the development of index models. These index models facilitate the determination of the efficient frontier which will need less information as compared to the mean-variance theory. The index models also show how security returns are sensitive to various factors.

The index models can be classified according to whether returns are assumed to depend on one index, single index models or on a group of indices, multifactor models (Elton & Gruber, 1973).

2.2.1 Single-Index models

From Elton & Goetzmann (2007), a single index model explains the returns of a security using one factor, usually a market model. It is of the form

$$\hat{R}_i = \hat{\alpha}_i + \hat{\beta}_i R_m \tag{3}$$

The a_i can be broken into two components, α_i which represents the expected value of a_i and e_i[1] which denotes the random element of a_i. Thus, (3) can be rewritten as:

$$\hat{R}_i = \hat{\alpha}_i + \hat{\beta}_i R_m + e_i \quad (4)$$

where, $Cov(e_i, R_m) = 0$[2], $Cov(e_i, e_j) = 0$[3] and $E(e_i) = 0$

Equation (4) has also been shown by Blume (1970), where the return of the security is a linear function of a market factor.

These assumptions follow the ordinary least square regression. A violation of $Cov(e_i, R_m) = 0$ implies heteroskedasticity. Heteroskedasticity has been proven by Praetz (1969) although he tested heteroskedasticity on R_i. His findings can be related to e_i as the distribution of e_i is similar to the distribution of R_i adjusted to have a 0 mean. His conclusions were based on the Sidney Stock Exchange. In contrast, Fama & Roll (1969) and Martin & Klemkosky (1975) found that there is no serious violation of the homoscedasticity assumption of the market model.

The Black-Litterman is a single-factor model that was used by companies such as Goldman Sachs as an approach for asset allocation (Asl & Etula, 2012).

When using the single-index model, it has been derived from Elton & Gruber (1973), where the weights are equal; the risk of the portfolio can be shown to be

$$\sigma_p = \beta_p \sigma_m = \sigma_m \left[\sum_{i=1}^{N} X_i \beta_i \right] \quad (5)$$

As defined by Sharpe (1963), β_i is the slope term in the simple linear regression function where the rate of return on a market index is the independent variable and the security's rate of return is the independent variable as shown in equation (4). As shown in equation (5), since σ_m does not change, despite the stock being examined, the measure of its contribution to the risk of the portfolio is still β_i. β_i is the measure of a security's systematic risk. Since unsystematic risk can be eliminated through diversifying by holding a large

[1] $e_i \sim N(0, \sigma_2)$

[2] which implies that how well equation (5) describes the return on any security is independent of what the return on the market happens to be.

[3] which implies that there are no effects beyond

portfolio, β_i is used as the measure of the security's risk in selecting optimal portfolios (Jacob, 1971).

It has been shown by Jacob (1971) that the average return of the portfolio will not, in general, be highly consistent with its degree of systematic risk. Instead, the consistency is seen to be dependent on first, the length of the time horizon used to generate the holding period return distributions, second, market average return over this period, third, the length of the intended holding period, fourth, the number of securities in the investor's portfolio and fifth, the method used to select portfolios.

The importance of accurate estimation of the beta coefficient as given by Menachem & Smidt (1977) are first, they are important for understanding risk-return relationships in capital market theory and second, they are useful in making investment decisions. The intertemporal stability of the beta coefficient as shown by Porter & Ezzel (1975) is sensitive to the process used to select the portfolio. Alexander & Chervany (1980) suggest that the magnitude of intertemporal changes in portfolio beta coefficients is inversely related to the number of securities in the portfolio regardless of how the portfolio is formed.

In the conventional wisdom sector of the Blume (1975) paper, it showed that beta coefficients are not strictly stationary over time and tend to regress towards one over time. In Blume's empirical analysis, it showed that part of this observed regressions tendency represented non-stationarity in the betas of individual securities.

In this paper, estimates of the beta are arrived at by estimating the beta from past data and use the historic beta as an estimate of the future beta. The single index model will be used as the market model in this paper, using the NSE-20 as the market index. We will be able to see if what the scholars are saying is at par with the empirical study of Kenya.

2.2.2 Multifactor models

The multifactor models attempt to capture some of the nonmarket influences that cause the co-movement of securities. Referring to Elton & Goetzmann (2007), a multifactor model of security returns attempts to explain the observed historical. The final representation is as shown in equation (6) below,

$$R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + \cdots + b_{iL}^* I_L^* + c_i \quad (6)$$

In order to simplify computation and the selection of the optimal portfolio, the indexes are manipulated so that they are uncorrelated i.e. orthogonal. The methodology used to convert the indexes from correlated to uncorrelated is shown in the Appendix A with the end result being,

$$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \cdots + b_{iL}I_L + c_i \tag{7}$$

The usual assumption of least square regression assumptions would be, $E(c_i) = 0$, $Var(c_i) = \sigma_{ic}^2$, $Cov(c_i, c_j) = 0$, $Cov(c_i, I_j) = 0$ and $Cov(I_j, I_k) = 0$

The expected return, variance and covariance of the multifactor model is respectively:

$$\bar{R}_i = a_i + b_{i1}\bar{I}_1 + b_{i2}\bar{I}_2 + \cdots + b_{iL}\bar{I}_L \tag{8}$$

$$\sigma_i^2 = b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \cdots + b_{iL}^2\sigma_{IL}^2 + \sigma_{ic}^2 \tag{9}$$

$$\sigma_{ij} = b_{i1}b_{j1}\sigma_{I1}^2 + b_{i2}b_{j2}\sigma_{I2}^2 + \cdots + b_{iL}b_{jL}\sigma_{IL}^2 \tag{10}$$

The builders of the model seek to find a set of factors which explain as much as possible of the observed historical variation, without introducing too much "noise" into predictions of future security returns CT (2011). Multifactor models are categorized into three depending on the factors used in the regression equation, i.e., macroeconomic factor models, fundamental factor models and statistical factor models.

Macroeconomics factor models use observable economic time series as the factors. Examples include annual rates of inflation and economic growth, short-term interest rates. The rationale is that the security's returns should reflect the future cash flows of the security. The size and discount rate for the cash flows are affected by the economic variables mentioned above (Chen, Roll & Ross, 1986). A related class of model uses a market index plus a set of industry indices as the factors. Burmeister & McElroy (1992) give two advantages of using macroeconomic factor models, first, the factors and their APT prices in principal can be given economic interpretations and second, rather than using asset prices to explain asset prices, measured macroeconomic factors introduce additional macroeconomic information. A related class of model uses a market index plus a set of industry indices as the factors.

In the fundamental factor models, the factors used are company specific. Examples include the level of gearing, the price earnings ratio, the level of R&D spending and the industry group to which the company belongs. Fama and French (1992) used fundamental factors which included size and value. When looking at statistical factor models, they do not rely on specifying the factors independently of the historical data.

An important question remains as to how many factors to include in the regression model. Roll & Ross (1980) find that at least three indexes are needed. Gibbons (1982) reported that six or seven indexes were needed in his case where he was analyzing stock and bond data. In 1984, Phoebus, Dhyrymes & Gultekin show that the number of indexes needed is dependent on the number of firms being analyzed. Fama and French (1993) proposed three indexes.

Fundamental multifactor models have stemmed prominently from Chen, Roll and Ross (1986) and Fama and French (1993). Even though the main focus of the Chen, Roll and Ross (1986) paper was to explain equilibrium returns the analysis laid groundwork for many models. The authors identified a set of variables that affect stock price and argue that because current beliefs about the variables are reflected in the price, it is only the unexpected changes in these variables that can affect return.

Burmeister and Wall (1986) find that five variables are sufficient to describe the security returns. The variables include unexpected difference in return between 20-year government bonds and 20-year corporate bonds, the shape of the interest rate relationship with maturity, measure of unexpected deflation, unexpected change in the growth rate in real final sales as a proxy for the unexpected changes in the long run profits for the economy and the difference (spread) between the excess return on the market for any month and the excess return predicted.

Sorensen & Fiore (1989) use a multifactor model with seven variables to explain the return on the securities. The variables are: economic growth, business cycle, long-term interest rates, short-term interest rates, inflation stock, U.S. dollar and that part of the market index that is uncorrelated with the six indexes above.

Fama & French (1992) based their multifactor model on firm characteristics. The characteristics were first, size and second, book to market equity value. Their results were that there is an inverse relationship between size and returns and a positive relationship between book to market equity value and returns.

The models are many thus concluding whether multifactor models are better than single index may not have a definite answer. Some scholars like Cohen conclude that single index models are better whereas Fama and French prefer using the multifactor model. Chen & Mayers (1987) conclude that the single factor models tends to have relatively more power for forecasting conditional returns for poorly diversified portfolios, whereas the multifactor model tends to have relatively more power for forecasting conditional returns for well-diversified portfolios With the empirical studies that will be carried out based in Kenya, we will find out which model is favorable with Kenya and under what circumstances are the models good to determine the efficient frontier.

The variables used in this study include, consumption which was used by Chen, Roll and Ross (1986) where changes in real consumption will influence pricing and such effects should also show up as unanticipated changes in risk premia. Another variable is Treasury bill which will influence the demand for stocks and therefore is expected to have a negative relationship with stock. Inflation is negatively related to stock returns as shown by Kaul (1987) as an increase in inflation will reduce the real investment return from the stock. Economic growth is also used because, as suggested by Ritter (2005), an increase in economic growth will influence investors to bid up stock prices thus lowering the dividend yield, thus for the investors to receive the same yield they have to put in more capital which will lower the return. I will also use exchange rate (Ksh/USD), where the domestic currency depreciation improves the competitiveness of local firms, which in turn leads to increase in exports and future cash flows thus stock prices will move up as a result of increase in cash flows. Money supply has been shown by, Kumar (2014) where he finds that money supply can opperate in two opposite ways. In one way, money supply has a positive relationship with inflation that would reduce the real returns and on the other hand increase in money supply would boost the economic stimulus thus an increase in corporate earnings and stock prices.

2.3 Literature review summary

This study seeks to compare the single-index model and the multifactor model in determining the efficient frontier. The efficient frontier is defined as the set of portfolios that minimize risk holding expected return constant and maximize expected return holding risk constant. Markowitz (1956) pioneered the use of the mean-variance theorem to determine the expected return and the standard deviation. The mean-variance theorem was criticized as it required too many parameters. Index models were then developed to determine the expected return and the standard deviation. Sharpe (1963) developed the single index model where the market was the index used to determine the return on the stocks. Ross (1977) developed the multifactor model followed by scholars such as Fama and French (1996). Looking at the multifactor model, the return was determined using different factors, and the factors were either fundamental, macroeconomic or statistical.

In this study, the NSE-20 index return is used as the market index in the single-index model. In the case for multifactor model, macroeconomic factors are used in which include; GDP growth, 91-day T-bill, USD/KES exchange rate return, consumption growth, inflation and M2 as a percentage of GDP.

3 Methodology

The study seeks to compare single index versus multifactor models. This comparison was done by constructing an efficient frontier using a single-index model and a multifactor model and looking at which model will yield a higher efficient frontier. The efficient frontier will be plotted using portfolio returns and risks of the portfolio so as to plot a line that would join the optimal returns given a level of risk.

3.1 Research design

The study takes on a positivist view with the approach being predominantly quantitative. This is because the study was concerned with calculating the expected return and risk which were represented in a quantitative way so as to draw inferences from the result of the efficient frontier plotted.

3.1.1 Population and sampling

For simplification, the asset classes that were put into consideration in this study were domestic common equities. Asset classes such as domestic fixed income, non-domestic common equity, non-domestic fixed income and alternative investment were not considered because of the accessibility of the data required. A disadvantage of this is that the portfolios may not be well diversified as one asset class is being considered. The different portfolios were generated from a weighted combination of assets included in the domestic common equities. The assets were randomly selected and the assets that were included in the portfolio were those ones which their excess return to beta ratio exceeded the cut-off rate that was computed. This is discussed in section 3.2.

3.1.2 Data and data sources

The study covers a fourteen-year period between 2000-2012. This period is used to calculate the historic beta using the single index and the multifactor model regressions. The data used in this study are quantitative secondary data. Data are obtained from World Bank, Central Bank of Kenya, Kenya National Bureau of Statistics and Nairobi Stock Exchange.

In Kenya, there are two market indices, NSE-ALL SHARE and NSE-20 SHARE. For the single-index model the NSE-20 SHARE is used as the market index. This is because NSE-20, the oldest and most widely used of the NSE's performance barometers, is a geometrically-weighted average of the largest 20 listed companies, measured by market capitalization. The index is constructed from stock price data (excluding dividends), adjusted for corporate actions, such as stock splits, and changes in firm's market capitalization over time. The NSE-20 index is a good proxy for the whole market because its 20 companies represent over 80% of the market capitalization of the entire exchange. The 20 stocks constitute a fairly well diversified portfolio.

Variable	**Description**
Equity returns	The NSE-20 was used as the market index. The return is an annual average return
Economic growth	Gross Domestic Product (GDP) was used as a measure of economic growth. I looked at the percentage growth rate of the quarterly real GDP (Antti Ilmanen & Ross, 2014)
Inflation	A sustained increase in the general level of prices for goods and services. I looked at the quarterly percentage inflation (Antti Ilmanen & Ross, 2014).
Short-Term Treasury Bill	A short-term debt obligation backed by the Government with a maturity of less than one year. I looked at a 3-month treasury bill yield (Burmeister & Wall, 1986).
Exchange rate	The price of a nation's currency in terms of another currency. I looked at the return on exchange rate, which will be USD/KSh. (Asl & Etula, 2012).
Consumption	Final purchase of goods and services. I looked at consumption as a percentage of GDP (Nai-Fu Chen & Ross, 1986).
Money Supply	The entire stock of currency and other liquid instruments in a country's economy as of a particular time. In this study, M2 was used as the aggregate which includes currency +

	travelers cheques + demand deposits + other chequable deposits + time and savings deposit + certificates of deposits. It was measured as a percentage of GDP to measure the flow of income in the economy.

Table 1: Description and Measurement of the Variables used in the Study

3.2 Selection of Optimal Portfolios

The stocks used in this study were obtained from the Nairobi Stock Exchange All Share Index (NASI). A two-step process was used to narrow down the stocks from 63 stocks to 31 stocks. Initially, screening was conducted, which eliminated companies that were either delisted, suspended, or had an average volume of less than 15,000, given the latter stocks would be considered illiquid. Then, stocks that were included in NASI in 2012 were also not considered due to the inability to compute the excess return to beta ratio given the period under study is till 2013, thus one cannot calculate the beta.

This study considered 10 portfolios. The stocks were grouped according to their sectors and one stock from each sector chosen randomly to form part of the portfolio, this catered for diversification across sectors. After the stocks in each portfolio were chosen, stocks that were below the cut off rate are eliminated.

The desirability of any stocks is directly related to its excess return to Beta ratio. Excess return is the difference between the expected return on the stock and the riskless rate of interest, which will be taken as the T-Bill. The excess return to Beta ratio measures the additional return on a security per unit of nondiversifiable risk. This, as shown by Elton et al 2007, can be expressed as,

$$(\bar{R}_i - R_f)/\beta_i \quad (11)$$

The ranking represents the desirability of any stock's inclusion in a portfolio. If a stock with a particular ratio of $(\bar{R}_i - R_f)/\beta_i$ is included in an optimal portfolio, all stocks with a higher ratio will be included. On the other hand, if a stock is excluded from an optimal portfolio, all stocks with lower ratios will be excluded.

The excess return to beta ratio for the chosen stocks was calculated and ranked from the highest to lowest. Then, computation of the potential cut-off rates for each stock and the one with the highest C_i becomes the cut off rate C^*. Thus, the optimal portfolio consisted of investing in all stocks for which the excess return to beta is greater than C^*.

C_i is expressed as

$$C_i = \frac{\sigma_m^2 \sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma_{ej}^2}}{1 + \sigma_m^2 \sum_{j=1}^{i} \left(\frac{\beta_j^2}{\sigma_{ej}^2}\right)} \qquad (12)$$

This expression can also be stated in the form,

$$C_i = \frac{\beta_{iP}(\bar{R}_P - R_f)}{\beta_i} \qquad (13)$$

where β_{iP} is expected change in the rate of return on stock i associated with a 1% change in the return of the oprimal portfolio.

The equation (13) is not used to determine the optimal portfolio as β_{iP} and $\bar{R}_P$ are not known until the optimal portfolio is determined. It is instead used in interpreting the economic significance of the procedure.

Recall that securities are added into the portfolio if

$$\frac{\bar{R}_i - R_f}{\beta_i} > C_i$$

Rearranging and substituting in equation (13)

$$(\bar{R}_i - R_f) > \beta_{iP}(\bar{R}_P - R_f) \qquad (14)$$

The RHS is the expected excess return on a particular stock based on the expected performance of the optimum portfolio. The LHS is security analyst's estimate of the expected excess return on the individual stock. Therefore, if the analysis of a particular stock leads the portfolio manager to believe that it will perform better than would be expected, based on its relationship to the optimal portfolio, it should be added to the porfolio.

3.3 Single-Index model

The single-index model can be expressed as

$$R_i = \alpha_i + \beta_i R_m + e_i$$

which will be used to compute the historic beta. Where, R_m is the return on the NSE-20 index, a_i is security i's return independent of the market's performance and β_i is the sensitivity of stock return (R_i) with changes in NSE-20 index return.

The model used to compute the expected return and risk to construct the efficient frontier was based on the journal by Elton & Padberg (1976). There are two scenarios where first, short sales are allowed and second, short sales are not allowed but in this paper, we considered a case where short sales are not allowed as there are no short sales in Kenya.

The excess return can be expressed as:

$$\bar{R}_p = \sum_{i=1}^{N} X_i \bar{R}_i \tag{15}$$

Where the excess return is the return less the risk free rate

And the standard deviation can be expresssed as:

$$\sigma_p = \left[\sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{j=1_{j\neq 1}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{e_i}^2\right]^{1/2} \tag{16}$$

Where, X_i is the relative weights we place on each security, β_i is a measure of the responsiveness of security i to changes in the market index, σ_m^2 is the variance of the market index and σ_e^2 is the variance of the random error.

Solver is then used to construct the efficient frontier given the return and standard deviation.

3.4 Multifactor model

Recall that the multifactor model can be expressed as

$$R_i = a_i + b_1 I_1 + b_2 I_2 + \cdots + b_n I_n + e_i$$

$$R_i = a_i + \sum_{i=1}^{N} b_i I_i + e_i$$

Where, I_j^* is the actual level of index j, b_{ij}^* is the sensitivity of the return on the stock i to changes in the index j, a_i^* is the expected value of the unique return and c_i is the random component of the unique return and $c_i \sim N(0, \sigma_{ci}^2)$. This equation will be used to compute the historic beta

$\sum_{i=1}^{N} b_i I_i$ is the summation of the factors, including, consumption, treasury bills, inflation, exchange rate, M2 and GDP.

The characteristics of the factor model borrowed from Asl & Etula (2012) include: first, each factor reflects a distinct risk premium that is largely independent of the others, second, each risk premium has a clear economic rationale as shown by some market participant's willingness to pay a premium to offload the risk, third, the reward associated with each factor reflects compensation for systematic risks in the cross-section of expected return, which implies that assets that have higher exposures to our factors are expected to earn higher return. Fourthly, each risk premium is investable as factor positions can be achieved via long and short positions in liquid tradable assets.

In analyzing a multifactor model, scholars have used either factor analysis, for example, Ross (1976) or macroeconomic factor approach for example Burmeister & Wall (1986). Burmeister & McElroy (1992) argue for using macroeconomic factor approach rather than factor analysis approach. The advantages include, the factors can be given economic interpretations, while with factor analysis approach it is unknown which factors are being priced and rather than using only asset prices to explain asset prices, measured macroeconomic factors introduce additional macroeconomic information.

We will use the mean variance optimization to estimate the expected return, which is given by equation (8), the variation of each asset class will be given by (9) and since Markowitz MVO technique accounts for the covariances between stocks so as to reduce the risk of an entire portfolio (Markowitz, 1952), the variation of the portfolio with n stocks is given by equation (10) and the covariance matrix for a n stock portfolio case is given by,

$$\delta_{i\ldots n} = \begin{bmatrix} \delta_1^2 & \delta_{n-1,1} & \delta_{n,1} \\ \delta_{1,n-1} & \delta_{n-1}^2 & \delta_{n,n-1} \\ \delta_{1,n} & \delta_{n-1,n} & \delta_n^2 \end{bmatrix} \tag{17}$$

Where,

$$\delta_{i,j} = E\left[(R_i - \mu_i)(R_j - \mu_j)\right] \tag{18}$$

And δ_i^2 the variance of the stock as described in equation (9) and the main diagonal contains the variances.

The standard deviation of the portfolio is computed as,

$$X'CX \tag{19}$$

Every investor is tasked with the problem of creating portfolios where they would minimize the portfolio's risk given a certain return. The risk is constrained by the availability of stocks. This is represented mathematically as;

$$\text{Minimize } z = \frac{1}{2} X^T VX \tag{20}$$

$$\text{Subject to } X \epsilon\, \mathbb{R} \,/\, X^T,\ \mu = \mu_p, X^T e = 1 \tag{21}$$

Where, $X = [X_1, X_2 \ldots X_n]^T$ is a column vector of portfolio weights for each security, V is a covariance matrix for the returns and μ_p is the desired portfolio return.

To solve the equation (19) and (20), Taha (2007) suggests the use of the Lagrangian Function given by;

$$L(X,\lambda) = \frac{1}{2} X^T VX - \lambda_1 X^T e - 1 - \lambda_2 X^T \mu - \mu_p \tag{22}$$

Assuming that all of the first and second moments of the random variables (X) exist, the vectors are linearly independent and the covariance matrix is strictly positive definite, the solution to equation (21) gives the Optimal Portfolio (X*).

The Optimal Portfolio is;

$$X^* = V^{-1}(\lambda_{1e} + \lambda_{2\mu}) \tag{23}$$

Where the parameters λ_1 and λ_2 are given by;

$$\lambda_1 = \frac{(c - b\mu_p)}{(ac - b^2)} \tag{24}$$

$$\lambda_2 = \frac{(a\mu_p - b)}{(ac - b^2)} \tag{25}$$

And,

$$a = e^T V^{-1},\ b = e^T V^{-1}\mu,\ c = \mu^T V^{-1}\mu \tag{26}$$

In order to reduce the number of factors, the insignificant factors from the regression are not put into consideration when computing the return of a stock. Solver is then used to construct the efficient frontier given the return and standard deviation.

4 Findings and results

4.1 Portfolio stocks selected

A total of 31 stocks remained from which a portfolio could be formed. After stocks from each portfolio were chosen, 25 stocks remained which were above the cut-off rate, the stocks are shown under Appendix D. The figure below shows an example of the portfolio that will be used to construct the efficient frontier using the single-index model and the multifactor model.

Stocks	Excess Mean Return	Beta	Unsystematic Risk	Potential Cut-off Rate
	$Ri - R_f$	β	σ_{ej}^2	$\frac{\sigma_m^2 \sum_{j=1}^{i} \frac{(\bar{R}_j}{}}{1 + \sigma_m^2 \sum_{j=}^{i}}$
SCM	27.3258	0.5418	24.2131	7.9602
ICDC	56.9461	2.4556	20.5942	98.3685
ARM	42.8512	1.9906	89.2940	111.1285
KQ	22.8566	0.1379	126.7122	111.4480
BAT	27.1889	1.4427	115.2194	116.0240
CFC	32.2275	1.7125	33.1364	138.4130
SASN	29.8794	1.8496	17.6308	180.9011
KENO	34.0687	2.4207	68.8442	197.2781

Table 2: Calculating the Cut-off Rate

The market risk for the portfolio is 0.0697 and the cut off rate is 197.2781.

4.2 Single index model

The returns for the stocks were calculated and the results were

Assets	α	R_m	β_i	$R_i = \alpha_i + \beta_i R_m$
SCM	0.0287	19.2099	2.1232	40.7881
ICDC	0.0597	19.2099	3.2874	63.1517
ARM	8.5892*	19.2099	2.5917	58.3761
KQ	0.0258	19.2099	3.0105	57.8321
BAT	0.0418	19.2099	1.7805	34.2045
CFC	0.0441	19.2099	1.9470	37.4017
SASN	2.9446	19.2099	3.3040	63.4696
KENO	6.3997*	19.2099	2.6812	57.9067

Table 3: Returns for the Stocks in a Single Index Model

The (*) indicates that the alpha is significant.

Using solver, the portfolio returns and standard deviations used to construct the efficient frontier are,

Efficient Frontier		Sharpe Ratio	
Portfolio Returns	Standard Deviation	Rf	Sharpe Ratio
34.21	11.9833	8.9267	2.10987
36	10.3388		2.61862
38	9.9167		2.93177
40	9.2445		3.36129
42	8.8843		3.72269
44	8.6279		4.06512
46	8.4782		4.37278
48	8.4853		4.60483
50	8.6023		4.77467
52	8.8255		4.88053
54	9.1482		4.92700
56	9.5619		4.92301
58	10.0598		4.87815
60	10.6330		4.80330
62	11.2796		4.70524
63.46	11.8575		4.59906

Table 4: Efficient Frontier and Sharpe Ratio Data Points

The resultant efficient frontier and the Sharpe ratio graph are

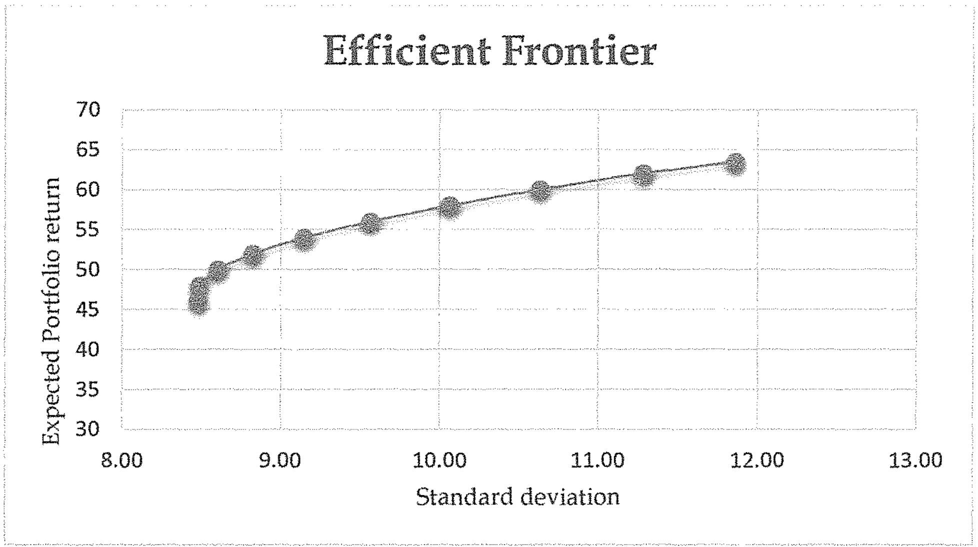

Figure 1: Single Index Model Efficient Frontier

4.3 Multifactor model

The returns for the stocks were calculated and the results were

Assets	SCM	ICDC	ARM	KQ	BAT	CFC	SASN	KENO
α	-76.353	-	6.8512	21.8872	-	16.8917	-	-
β_i	-	-	-	-	-	-	-	0.6925
I	5.7159	5.7159	5.7159	5.7159	5.7159	5.7159	5.7159	5.7159
β_{fx}	-318.27	-	-	-	-	-	-	-
FX	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
β_{nse}	172.111	328.74	276.653	296.512	179.168	178.291	3.3632	277.103
NSE	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481
β_{t-bill}	-	-	-	-	-	-1.5686	-	-
T-bill	8.9267	8.9267	8.9267	8.9267	8.9267	8.9267	8.9267	8.9267
β_{gdp}	-	-	2401.76	-	-	-	-	-
GDP	0.0038	0.0038	0.0038	0.0038	0.0038	0.0038	0.0038	0.0038
β_C	-	-	-2317.64	-	-	-	-	-152.30
C	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036
β_{M2}	0.6074	-	-	-0.0975	0.0234	-	-	-
M2	362.497	362.49	362.497	362.497	362.497	362.497	362.49	362.497
Return	151.969	15.811	20.9813	0.7971	17.0896	11.4648	0.1618	16.7363

Table 5: Stock Returns in a Multifactor Model

Using solver, the portfolio returns and standard deviations used to construct the efficient frontier are,

Efficient Frontier		Sharpe Ratio	
Portfolio Returns	Standard Deviation	Rf	Sharpe Ratio
0.16	0.19	8.9267	-45.13225022
10	3.23		0.331879327
20	12.82		0.86383973
30	16.21		1.300337984
40	18.87		1.646358185
50	18.93		2.169567836
60	21.11		2.41975302
70	23.61		2.586458904
80	26.36		2.696604213
90	29.27		2.769542866
100	32.32		2.818149883
110	35.46		2.850709179
120	38.67		2.872550868
130	41.94		2.88713238
140	45.25		2.896729922
150	48.60		2.90287492

Table 6: Efficient Frontier and Sharpe Ratio Data Points

The resultant efficient frontier and the Sharpe ratio graph are

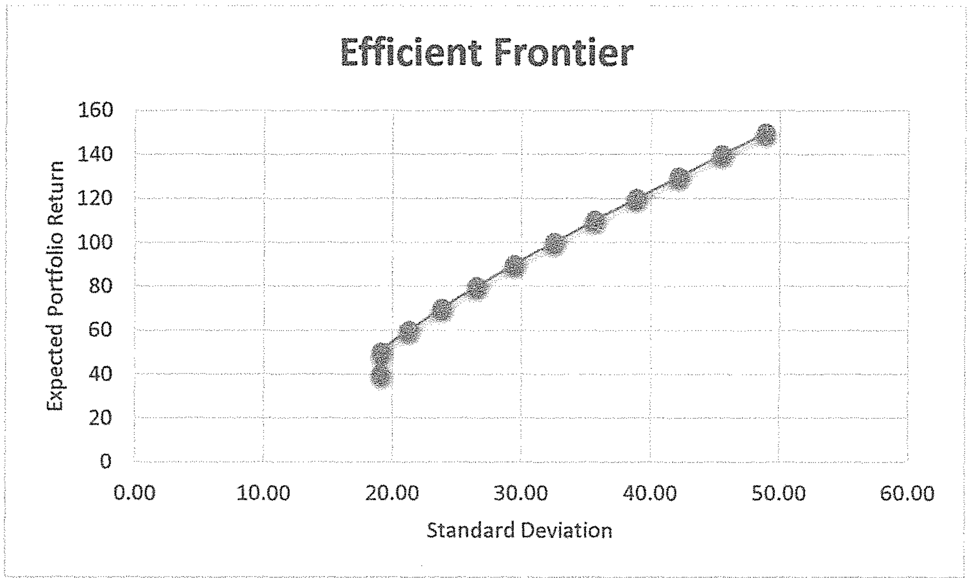

Figure 2: Multifactor Model Efficient Frontier

5 Conclusion

This study focused on comparing the relative performance of the single index model and the multifactor model in determining the efficient frontier. We find that the single index model has higher Sharpe ratios, which is attractive to investors. Therefore, we can conclude that the single index model is superior to the multifactor model. This can also be explained by the fact that the market model in the single index model contains the characteristics of the macroeconomic variables.

There is still more room for further research in the study due to the following assumptions, 1) only one portfolio was considered, 2) the stocks in the portfolio were equity only, other asset classes were not considered, 3) only one period, 2013, was considered and 4) we did not consider the observed returns for 2013

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Appendices

Appendix A: Derivation of the risk of the portfolio with equal weights

The expected return of the security is,

$$E(R_i) = \alpha_i + \beta_i \bar{R}_m \tag{27}$$

The variance of the security's return is,

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 \tag{28}$$

The covariance of returns between securities i and j is

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 \tag{29}$$

When we look at a portfolio, β_p is a weighted average of the individual β_is on each security in the portfolio and $\alpha_p$is a weighted average of the individual α_is. Thus,

$$\beta_p = \sum_{i=1}^{N} X_i \beta_i \tag{30}$$

$$\alpha_p = \sum_{i=1}^{N} X_i \alpha_i \tag{31}$$

Where X_i denote the weights of the securities in the portfolio.

Therefore, it can be deduced that the expected return of a portfolio would be,

$$E(R_p) = \alpha_p + \beta_p \bar{R}_m \tag{32}$$

Where the weights are equal, the risk of the portfolio can be shown to be

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2 \tag{33}$$

Where the weights are equal, the risk of the portfolio can be represented as

$$\sigma_p = \beta_p \sigma_m = \sigma_m \left[\sum_{i=1}^{N} X_i \beta_i\right] \tag{34}$$

Appendix B: Procedure for reducing any multi-index model to a multi-index model with orthogonal indexes

Let,

$$R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + \cdots + b_{iL}^* I_L^* + c_i \tag{35}$$

Define I_1 as equal to I_1^*. To remove the impact of I_1^* fromI_2^*, we can establish the parameters of the following equation via regression analysis:

$$I_2^* = \gamma_0 + \gamma_1 I_1 + d_t \tag{36}$$

Where, γ_0 and γ_1 are regression coefficients and d_t is the random error term

By the technique of estimation used in regression analysis, $Cov(d_t, I_1) = 0$. Thus

$$d_t = I_2^* - (\gamma_0 + \gamma_1 I_1) \tag{37}$$

is an index of the performance of I_2^* with the effect of I_1 removed.

If we define

$$I_2 = d_1 = I_2^* - \gamma_0 - \gamma_1 I_1 \tag{38}$$

we have defined and index of I_2^* performance that is uncorrelated with the market. Solving for I_2^* and substituting into the return equation yields

$$R_i = a_i^* + b_{i1}^* I_1 + b_{i2}^* I_2 + b_{i2}^* \gamma_0 + b_{i2}^* \gamma_1 I_1 + c_i \tag{39}$$

Rearranging the terms gives

$$R_i = (a_i^* + b_{i2}^* \gamma_0) + (b_{i1}^* + b_{i2}^* \gamma_1) I_1 + b_{i2}^* I_2 + c_i \tag{40}$$

The first term is a constant we define as a_i.The coefficient of the second is b_{i1} and let $b_{i2}^* = b_i$. The equation becomes

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + c_i \tag{41}$$

Appendix C: The cut off rate

The C^* is computed from the characteristics of all of the securities that belong in the optimum portfolio. It is necessary to calculate C^* as if there were different numbers of securities in the optimum portfolio. C_i will be taken as a candidate for C^*.

Since securities are ranked from highest excess return to beta to lowest, then if a particular security beongs in the optimal portfolio, all higher ranked securities also belong in the optimal portfolio. The C_i is calculated as if the first ranked security was in the optimal portfolio, then the first and second were in the optimal portfolio, then the first, second and third were in the optimal portfolio and so forth. We know that we have found the optimum C^* when all securities used in the calculation of C_i have excess returns to beta above C_i and all securities not used to calculate C_i have excess returns to beta below C_i. There can only be one C^*.

Appendix D: The stocks that can be included in a portfolio

Kakuzi (KAKZ), Sasini (SASN), Barclays Bank Ltd (BBK), CFC Stanbic Holdings Ltd (CFC), Housing Finance Company Ltd. Kenya (HFCK), Kenya Commercial Bank (KCB), NIC Bank Ltd (NIC), Standard Chartered Bank Ltd (SCBK), Equity Bank Ltd (EQTY), The Cooperative Bank of Kenya Ltd (COOP), Kenya Airways Ltd (KQ), Nation Media Group (NMG), WPP Scangroup Ltd (SCAN), Athi River Mining (ARM), Bamburi Cement Ltd (BAMB), East African Cables Ltd (CABL), KenolKobil Ltd (KENO), British-American Investments Company (Kenya) Ltd (BRIT), Liberty Kenya Holdings (CFCI), CIC Insurance Group (CIC), Centum Investments Co. Ltd (ICDC), British American Tobacco Kenya Ltd (BAT), East African Breweries Ltd (EABL) and Mumias Sugar Co. Ltd, Safaricom Ltd (SCOM).