STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES
BBS Actuarial Science, BBS Finance & BBS Financial Economics
END OF SEMESTER EXAMINATION

BSA 3214: Financial Calculus

DATE: 5th December, 2017 TIME: 2 HOURS

INSTRUCTIONS:

(1) This Exam Contains five questions
(2) Attempt question one and any other two questions
(3) Question one is compulsory and carries 30 marks
(4) All the other questions carry 20 marks each

QUESTION 1 - COMPULSORY

(a) State and briefly explain the Fundamental Theorem of Asset Pricing [4 marks]

(b) Consider a European call option on a stock. The stock pays no dividends and the stock price follows an Ito process. Is it possible that, while the stock price declines between times t_1 and $t_2 > t_1$, the price of the Call increases? *Justify* your response. [3 marks]

(c) Let $(X_{t;t\geq 0})$ be a stochastic process satisfying $dX_t = \mu_t dt + \sigma_t dW_t$ where W_t is a standard Brownian motion. Further Let $f(t, X)$ be a function, twice partially differentiable with respect to X, once with respect to t. State the stochastic differential equation for $f(t, X_t)$. [4 marks]

(d) The prices of two non-dividend paying stocks are governed by the following stochastic differential equations:

$$\frac{dS_1(t)}{S_1(t)} = 0.06dt + 0.02dZ_t$$
$$\frac{dS_2(t)}{S_2(t)} = 0.05dt + kdZ_t$$

where Z(t) is a standard Brownian motion and k is a constant. The current stock prices are $S_1(0) = 100$ and $S_2(0) = 50$ and the continuously compounded risk-free interest rate is 4%. You now want to construct a zero-investment, risk-free portfolio with the two stocks and risk-free bonds.

If there is exactly one share of Stock 1 in the portfolio, determine the number of shares and the position you should take in Stock 2. [5 marks]

(e) Assume the Binomial Option Pricing Framework (BOPM). If we set $X_0 = C_0$ and define recursively forward in time

$$X_n = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

and $X_n(\omega_1, \cdots, \omega_n) = C_n(\omega_1, \cdots, \omega_n)$ such that $C_n(\omega_1, \cdots, \omega_n)$ is referred to as the option process at time n provided that the sequence $\omega_1, \cdots, \omega_n$ happened.

i. Show that under $\tilde{p}$ the discounted self financing process value

$$\left\{(1+r)^{-n} X_n | \mathcal{F}_n\right\}_{n=0}^{N}$$

is a martingale [4 marks]

ii. Show that, given this definition, the BOPM is complete. Specifically, prove that the expression below holds.

$$\begin{aligned} X_{n+1}(\omega_1, \cdots, \omega_n; T) &= \Delta_n(\omega_1, \cdots, \omega_n; T) S_{n+1}(\omega_1, \cdots, \omega_n; T) \\ &\quad + (1+r)(X_n(\omega_1, \cdots, \omega_n; T)) \\ &\quad - \Delta_n(\omega_1, \cdots, \omega_n) S_n(\omega_1, \cdots, \omega_n) \end{aligned}$$

[6 marks]

(f) Suppose that an investor is short a derivative with price process given by $F_t(S_t)$ and can hedge the position by taking positions in both the underlying S_t and a call option $C_t(S_t)$ such that the value of the position is given by:

$$V_t = -F_t + \varphi S_t + \omega C_t$$

Where φ and ω are the units for the underlying and the call option respectively. Calculate the number of units for the stock that will achieve perfect gamma hedging. [4 marks]

Total for Question 1: 30 marks

QUESTION 2 - OPTIONAL

(a) The payoff at maturity of an Average Price Asian option can be computed using either the geometric average of underlying asset price or the arithmetic average of the underlying asset price. Show that the price of an Arithmetic Average Price

Asian Put is less than or equal to a geometric average price Asian put. [3 marks]

(b) State the martingale representation theorem, including conditions for its application, defining all terms used. [4 marks]

(c) Let S_t denote the price of an underlying security at time t; r denotes the risk free rate of return expressed in continuously compounded form, B_t represents an accumulated bank account at time t that earns the risk free rate of return.

Let X be any derivative payment contingent on F_T, payable at some fixed future time T, where F_T is the sigma algebra generated by S_u for $0 \leq u \leq T$. You may assume that, under the equivalent measure Q, the process

$$D_t = e^{-rt} S_t$$

is a martingale and that

$$dS_t = B_t(rD_t dt + dD_t)$$

Show that the value of this derivative at time $t \leq T$ is

$$V_t = e^{-r(T-t)} E_Q[X|F_t]$$

[13 marks]

Total for Question 2: 20 marks

QUESTION 3 - OPTIONAL

(a) In an ideal world, a trader would like to rebalance his portfolio to get zero delta, zero gamma and zero vega. This however is not possible. We also know that using Taylor series expansion small changes in a call option can be approximated as:

$$dC(S_t, t) \approx \Delta_t dS_t + \frac{1}{2}\Gamma_t (dS_t)^2 + \theta_t dt$$

where Δ_t, Γ_t and θ_t are Option Greeks Delta, Gamma and Theta respectively.

i. Suppose that a portfolio is delta-neutral. Explain, using proper workings, how a hedger would go about making the portfolio gamma-neutral. [2 marks]

ii. Now suppose that the hedger requires a portfolio to be both gamma neutral and vega neutral. Explain, using proper workings, how the hedger would go about achieving this objective. [2 marks]

(b) The theta of a derivative security is given as $\theta = re^{rt}S^{\left(1-\frac{2r}{\sigma^2}\right)}$ whereas the delta is given as $\Delta = \left(1 - \frac{2r}{\sigma^2}\right)\left(e^{rt}S^{\frac{2r}{\sigma^2}}\right)$. The underlying does not pay any dividends. Using

Ito calculus, show that the derivative contract is backed by a tradable underlying asset. [7 marks]

(c) The interest rates in an economy are driven by a Cox Ingersol Ross process of the form:

$$dr_t = k(\theta - r)\,dt + \sigma\sqrt{r}dw_t$$

Derive the expression for the variance of $r(t)$ [9 marks]

Total for Question 3: 20 marks

QUESTION 4 - OPTIONAL

Let S_t be the price, in Shillings of a foreign currency. The shilling interest rate is r, and the foreign interest rate is f, both assumed to be constant and expressed on a continuously compounded basis.

(a) Explain what is meant by the forward price of a foreign currency. [2 marks]

(b) At time $t = 0$ an investor purchases a forward contract on the foregin currency which pays off $S - K$ at time $t = T$, where K is a fixed amount in Shillings. The investor pays C_0 for the contract. Assume that the dealer makes no profit or loss on the transaction, use the no arbitrage argument to show that $C_0 = 0$ if and only iff $K = \tilde{S}_T$, where $\tilde{S}_T = S_0 e^{(r-f)T}$. [8 marks]

(c) Now assume that the price process of the foreign currency is driven by

$$S_t = S_0 e^{\mu t + \sigma w_t - \frac{1}{2}\sigma^2 t}$$

where the drift μ and the volatility σ are both assumed to be constant and w_t is the Wiener process. Calculate the mean of S_T.

(d) Show that if the drift μ and the volatility σ are bot assumed to be constant, and w_t is the Wiener process then

$$S_t = S_0 e^{\mu t + \sigma w_t - \frac{1}{2}\sigma^2 t}$$

satisfies

$$\frac{dS_t}{S_t} = \mu dt + \sigma dw_t$$

and the initial condition, $S = S_0$ at time $t = 0$

Total for Question 4: 20 marks

QUESTION 5 - OPTIONAL

Consider the Black Scholes framework. A dealer, who delta-hedges, sells a three-month at-the-money European call option on a non-dividend-paying stock. The continuously compounded risk free interest rate is 10%, the current stock price is 50, the current call option delta is 0.61791 and there are 365 days in the year.

(a) If after one day, the dealer has zero profit, determine the price move over the day.

[4 marks]

(b) Show using Taylor series expansion that small changes in a call option can be approximated as:

$$dC(S_t, t) \approx \Delta_t dS_t + \frac{1}{2}\Gamma_t (dS_t)^2 + \theta_t dt$$

[5 marks]

(c) Pricing by arbitrage means in a complete market, all derivatives can be expressed in terms of a self-financing replicating strategy, and that this replicating strategy is unique. With this replicating strategy we can set up a replicating portfolio and use a risk neutral measure to calculate the value of the derivative. We can replicate a derivative V by forming a self-financing portfolio with the stock S and the bond B in the right proportion.

Show that by using the replicating strategy $(a(t); b(t))$ to form the replicating portfolio $V(t) = a(t)S(t) + b(t)B(t)$ and determine the value of $(a(t); b(t))$. The self-financing assumption means that:

$$dV = adS + bdB$$

where $a = a(t)$ and $b = b(t)$ one can obtain the Black Scholes PDE given as:

$$rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - rV_t(S_t, t) = 0$$

[8 marks]

(d) The time-t Black-Scholes price of a call with time to maturity $\tau = T - t$ and strike K when the spot price is S is:

$$V(S_t, K, T) = S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

where:

$$d_1 = \frac{log\frac{S}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

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What is the meaning of the above representation? [3 marks]

Total for Question 5: 20 marks