STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES
Bachelor of Business Science in Financial Engineering
END OF SEMESTER EXAMINATION
BSM 4124: Computational Methods in Finance

DATE: 1/08/2022 TIME: 3 HOURS

INSTRUCTIONS

1. ***This examination consists of FIVE questions***
2. ***Answer Question 1 (COMPULSORY) and 2 optional questions out of Question 2 - 5***

Question 1 (30 Marks)

A. The main theoretical ingredients that allow to compute approximations of expectation from a sample of a given distribution are provided by the Law of Large Numbers and the Central Limit Theorem.
   i. What is the difference between Law of Large Numbers and the Central Limit Theorem? [2 marks]
   ii. Consider a European-style option $\varphi(S_T)$ with the payoff function φ depending on the terminal stock price. We can compute a fair price for $\varphi(S_T)$ taking the discounted expectation $E\left(e^{-rT}\varphi(S_T)\right)$. Write a pseudocode for an algorithm that will compute this expectation, its variance and the confidence interval. [6 marks]
   iii. The size of the confidence interval is determined by the variance and the size of the sample. Explain how the method of Antithetic variate works as a variance reduction technique. [4 marks]

B. Solve the following system of linear equations using the Doolittle's method

$$6x_1 - 15x_2 + 55x_3 = 76$$

$$15x_1 + 55x_2 + 225x_3 = 295$$

$$55x_1 + 225x_2 + 979x_3 = 1259$$

[5 marks]

C. Using the Taylor Series expansion, show that the four-point forward difference for the second derivative can be expressed as:

$$f'(x_i) = \frac{2f(x_i) - 5f(x_{i+1}) + 4f(x_{i+2}) - f(x_{i+3})}{h^2}$$

[4 marks]

D. Using the inverse transform method for generating random numbers, generate a random variable X with distribution:
$P(X = 1) = 0.4$, $P(X = 2) = 0.3$, $P(X = 6) = 0.1$, $P(X = 7.5) = 0.2$
[3 marks]

E. Show that the number of iterations of the Rejection method, for a random variable, X, with density function $f(x)$, follows a geometric distribution with mean $c = \max_x f(x)/g(x)$. Assume $g(x)$ has a heavier tail than $f(x)$. [4 marks]

F. Under the context of numeric schemes in SDEs, the Euler Maruyama and the Milstein Schemes can be applied to pricing options. Identify which can be used to price path-dependent and path-independent options and explain why each is suitable for the choice of option you have made. (In your answer, make sure to explain what it means for a numeric scheme to be strongly and weakly convergent with order γ).
[2 marks]

<u>Question 2 (20 Marks)</u>

Show that:

i. The Black-Scholes initial boundary value problem can be expressed as:

$$\left[\frac{\delta C}{\delta t} + rS\frac{\delta C}{\delta S} + {}^{1}\!/_{2}\frac{\delta^2 C}{\delta S^2}\sigma^2 S^2 - rC\right] = 0$$

Make sure to clearly highlight what the initial and boundary conditions are for the above BS-IBVP. [4 marks]

ii. Using the finite difference methods, the approximation for the BS-IBVP obtained in (i) can be expressed in the form:

$$C_j^{n+1} = \alpha_i C_{j-1}^n + \beta_i C_j^n + \gamma_i C_{j+1}^n$$

Clearly state what α_i, β_i and γ_i are and ~~also~~ make sure to express the finite difference approximation for your initial and boundary conditions.
(NOTE: use the central difference to approximate the first derivative wrt t, the backward difference for the first derivative wrt S and the 3-point central difference for the second derivative) [8 marks]

iii. Consider the most general linear second order differential equation that is of the form:

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x), a \leq x \leq b \text{ with } y(a) = A \text{ and } y(b) = B$$

Using the central difference approximation for both derivatives, show that the equation can be written in the form $A_h Y_h = R_h$ where:

$A \in \mathbb{R}^{(N-1)\times(N-1)}$ that you will specify, $R \in \mathbb{R}^{(N-1)\times 1}$ that you will also specify and $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$. [8 marks]

Question 3 (20 Marks)

A. Assume that a stock price $S(t)$ is driven by the SDE

$dS(t) = \mu(S(t), t)dt + \sigma(S(t), t)dW_t$.

i. Write the integral form of the above equation by integrating $dS(t)$ from t to t + dt. [2 marks]

ii. Obtain the Euler discretization of your equation from (i) above. [2 marks]

iii. Using Ito's lemma with $f = B_t^2$, verify that $\int B_s dB_s = \frac{B_t^2 - t}{2}$. [4 marks]

Now consider a homogeneous scalar SDE of the type

$$dS(t) = \mu\big(S(t)\big)dt + \sigma\big(S(t)\big)dB_t, \quad S(0) = s$$

Using (iii) above and the integral form of Ito's lemma

$$f(X_s) = f(X_s) + \int_{t_i}^{s}\left(f'(X_u)\mu(X_u) + {}^1/_2\, f''(X_u)\,\sigma(X_u)^2\right)du + \int_{t_i}^{s}\left(f'(X_u)\sigma(X_u)\right)dB_u,$$

derive an expression for the Milstein Scheme. [12 marks]

Question 4 (20 Marks)

A. Let $f(t, S_t)$ be a smooth function of two variables that is twice differentiable in S_t and once in t.

i. Use the Taylor series expansion to find an expression for $df(t, S_t)$ (ignoring terms with order higher than 2). [2 marks]

ii. Let $S(t)$ be governed by the following SDE where $\mu(t,\omega) > 0$ and $\sigma(t,\omega) > 0$:

$$dS(t) = \mu(t,\omega)dt + \sigma(t,\omega)dW(t,\omega)$$

Using your answer in (i) above, show that $df(t, S_t)$ can be written as:

$$df(t, S_t) = \left[\frac{\delta f}{\delta t} + \frac{\delta f}{\delta S}\mu(t,\omega) + {}^1/_2\frac{\delta^2 f}{\delta S^2}\sigma^2(t,\omega)\, dW(t,\omega)\right] dt + \frac{\delta f}{\delta S}\sigma(t,\omega)\, dW(t,\omega)$$

[4 marks]

iii. Let $\mu(t,\omega) = \mu S(t)$, $\sigma(t,\omega) = \sigma S(t)$ and $f(S) = \ln(S)$. Using the equation in (ii) above, derive the closed form solution of $S(t)$ [5 marks]

B. For Asian options, the average put option traded on the market has a payoff of the form:

$$\left(\frac{1}{n}\sum_{i=1}^{n} K - S_{t_i}\right)^{+}$$

The Black-Scholes model does not have an analytical expression for this price. However, the geometric average of the Asian option, which is of the form:

$$\left(K - \left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n}\right)^{+}$$

has an analytical formula.

Using the appropriate control variate (which you will need to express using the information above), for M simulated sample paths, write down the pseudocode of how you will implement the antithetic method of variance reduction for the price of an average put Asian option. (Note that the price is the discounted expected payoff).

[9 marks]

<u>Question 5 (20 Marks)</u>

Given that the payoff of a hedged portfolio has a lower standard deviation than the payoff of an unhedged one, using hedges can reduce the volatility of the value of the portfolio. A delta hedge consists of holding $\Delta = {}^{\delta C}/_{\delta S}$ shares in the underlying asset, which is rebalanced at the discrete time intervals. At time T, the hedge consists of the savings account and the asset, which closely replicates the payoff of the option.

i. Using the martingale property and the fact that the option price is the value of the hedging portfolio, obtain the hedged control variate. [5 marks]

ii. The price of an option is obtained by finding $E_{\mathbb{Q}}\varphi(S_T^x)$. Write an expression for the 3-point forward finite difference of $\Delta = {}^{\delta\varphi}/_{\delta S}$ where S_T^x denotes the value of S_T under the condition $S_0 = x$ [3 marks]

iii. Δ can be estimated using Monte Carlo methods. Under the BS Model

$$S_T^x = x \times exp\left(\left(r - {}^{\sigma^2}/_2\right)t + \sigma W_t\right)$$

Write an expression for the estimate of Δ using the principles of Monte Carlo estimation and substitute S_T^x with the appropriate expression given the above BS equation. **[4 marks]**

iv. Since for vanilla options the payoff function is not twice differentiable, Gamma cannot be computed by path wise differentiation of the payoff function. However, the likelihood ratio is a way that can be used to calculate gamma. Assume that the transition density is known to be

$$g(S_T^x) = g(x, S_T^x)$$

Further assume that the transitional density is smooth and all derivatives can be computed. Derive the expression for gamma $\Gamma = \frac{\delta^2}{\delta x^2} E[\varphi(S_T^x)]$ using the likelihood ratio method **[8 marks]**

END OF EXAMINATION PAPER