Strathmore University

Strathmore Institute of Mathematical Sciences
Bachelor of Business Science (Financial Engineering)
End of Semester Examination
BSM 3114 - Differential Equations with Financial Applications

Date: 10th September 2021 | Time: 2 Hours

Instruction

1. Answer **QUESTION ONE** and any other **TWO QUESTIONS**

QUESTION ONE [30 Marks]

a) Determine which of the following differential equations can be expressed in the form $\frac{dy}{dx} = f(x)g(y)$, if possible, express each equation in this form. [3 Marks]

   i. $\frac{dy}{dx} = x^2y^{-2}$

   ii. $y^{'} = 4x^2 + 2y^2$

   iii. $yy^{'} + 3x = 7$

b) Use the method of separation of variables to solve

$$x\frac{dv}{dx} = \frac{1-4v^2}{3v}$$

[4 Marks]

c) Use the method of integrating factor to solve

$$(x + y + 1)dx - dy = 0$$

[3 Marks]

d) By making the substitution $v = ax + by$ solve the differential equation $y' = e^{9y-x}$ given that $y(0) = 0$. [6 Marks]

e) Express each differential equation given below below as a system of linear equations;

i. $\theta'' + \frac{m}{l} Sin(\theta) = 0$ given that $\theta(0) = \theta_0$ and $\theta'(0) = V_0$ [2 Marks]

ii. $y''' + 5ty'' + 3y = 0$ with $y(t_0) = a_0$, $y'(t_0) = a_1$ and $y''(t_0) = a_2$. [3 Marks]

f) Determine the general solution of $\mathbf{x}(t)' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t)$ [6 Marks]

g) Given the following differential equations, determine which of them is linear, quasi-linear or semi-linear. [3 Marks]

i. $a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$ with $a, b, c \in \mathbb{R}$

ii. $yu_{xx} + (2+y)u_{yy} + u^3 = 1$

iii. $yu_{xx} + 2xyuu_{yy} + u = 1$

QUESTION TWO [20 Marks]

a) Find all functions y_1, y_2 and y_3 such that [9 Marks]

$$\begin{aligned} y_1 - 3y_2 + 7y_3 &= y_1' \\ -y_1 - y_2 + y_3 &= y_2' \\ -y_1 + y_2 - 3y_3 &= y_3' \end{aligned}$$

b) Find all real valued functions y_1 and y_2 such that $y_1' = y_2$ and $y_2' = -y_1$. [8 Marks]

c) By letting $x_1 = y$ and $x_2 = y'$, transform the differential equation given below into its corresponding linear system; [3 Marks]

$$\begin{cases} ty'' - 2y' + t^2 y = ln(t) \quad given \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$

QUESTION THREE [20 Marks]

a) Solve the following; $\mathbf{y}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{y}$ given that

$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. [13 Marks]

b) Find the stability point(s) of the non-linear system: [3 Marks]

$$\begin{aligned} x' &= -(x-y)(1-x-y) \\ y' &= x(2+y) \end{aligned}$$

c) Compute the Jacobian matrix for the system: [4 Marks]

$$\begin{aligned} x' &= 10x - 5xy \\ y' &= 3y + xy - 3y^2 \end{aligned}$$

QUESTION FOUR [20 Marks]

a) Given the non-linear system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_2^2 - x_1x_2 - x_2 \\ \frac{dx_2}{dt} &= 2x_1^2 + x_1x_2 - 7x_1 \end{aligned}$$

find the stability point(s), phase plane diagram and an equation relating the two variables. [10 Marks]

b) Find all the critical **points** of the non-linear system:

$$\begin{aligned} \frac{dx}{dt} &= x - y - x^2 + xy \\ \frac{dy}{dt} &= -x^2 - y \end{aligned}$$

Identify the **types of critical points** obtained and sketch a possible phase plane diagram. [10 Marks]

QUESTION FIVE [20 Marks]

a) Given the system $\mathbf{x}^{'} = A\mathbf{x}$ and depending on the eigenvalues of $\mathbf{A}$, give the general solution when:

   i. We have two distinct real eigenvalues θ_1 and θ_2. [2 Marks]

   ii. We have two complex conjugate eigenvalues $\theta_1 \pm i\theta_2$ where $\theta_1 + i\theta_2$ has an eigenvector $\mathbf{v} = v_1 + iv_2$. [3 Marks]

   iii. We have a repeated eigenvalue r. [5 Marks]

b) Find the critical points and linearizations of the following systems:

   i. $x^{'} = Sin(\pi y) + (x-1)^2$, $y^{'} = y^2 - y$ [4 Marks]

   ii. $x^{'} = x + y + y^2$, $y^{'} = x$ [3 Marks]

   iii. $x^{'} = (x-1)^2 + y$, $y^{'} = x^2 + y$ [3 Marks]