



Strathmore  
UNIVERSITY

STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCE (SIMS)  
MASTERS OF SCIENCE IN BIOMATHEMATICS  
END OF SEMESTER EXAMINATION  
BMA 8201: PARTIAL DIFFERENTIAL EQUATIONS MODELLING IN BIOLOGY

DATE: 5th September, 2022

TIME: 3 Hours

INSTRUCTIONS

1. This examination consists of FOUR questions.
2. Answer Question ONE (COMPULSORY) and any other TWO questions.

QUESTION ONE (30 MARKS)

- a) Determine the general solution of the quasi-linear PDE  
 $x(y - u)u_x + y(x + u)u_y = (x + y)u.$  (6 marks)
- b) Given that  $u = u(x, y)$  and the change of variables  $\theta = \theta(x, y), \eta = \eta(x, y)$  such that  $u_y = u_\theta \theta_y + u_\eta \eta_y$ , show that;  
 $u_{yy} = u_{\theta\theta} \theta_y^2 + 2u_{\theta\eta} \theta_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\theta \theta_{yy} + u_\eta \eta_{yy}$  (5 marks)
- c) Suppose that the steady state distribution of a certain chemical in a slender rectangular body tissue is described by  $u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b$ . Show that the chemical concentration is generally given by;  
 $u(x, y) = (A \cos ax + B \sin ax)(C \cosh ay + D \sinh ay)$  (6 marks)
- d) Consider the heat equation  $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < L, 0 < t < T$  together with the boundary conditions  $u(0, t) = u(L, t) = 0, 0 < t < T$ , and initial condition  $u(x, 0) = f(x), 0 \leq x \leq L$ . Obtain the Crank-Nicolson approximation for the heat equation and present it in the matrix form  $Aw^{(j+1)} = Bw^{(j)}$  for each  $j = 0, 1, 2, \dots$  where A and B are tridiagonal matrices. (7 marks)
- e) Applying the method of separation of variables to an initial-boundary value problem for a vibrating circular membrane gives rise to the radial equation;  
$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (\lambda r^2 - n^2)R = 0, \quad 0 < r < a$$
whereby  $|R(r \rightarrow 0^+)| < \infty, R(a) = 0$ 
  - i) Show that the radial equation is a singular Sturm-Liouville problem. (3 marks)
  - ii) Transform the radial equation into a Bessel's equation of order n, by letting  $x = \sqrt{\lambda} r$ . (3 marks)

### QUESTION TWO (20 MARKS)

- a) Given the second order PDE  $u_{xx} + x^2 u_{yy} = 0$  where  $x \neq 0$ ;
- Classify the given differential equation. (2 marks)
  - Show that the change of variables can be expressed as  $\alpha = 2y$ ,  $\beta = -x^2$  (5 marks)
  - Obtain its canonical form. (5 marks)
- b) Given a domain  $R = \{(x, y) | 0 < x < 2, 0 < y < 2\}$ ;
- Sketch the mesh grid of the domain discretized using step sizes  $\Delta x = \Delta y = 0.5$  (3 marks)
  - Write a MATLAB code, mesh\_BC, that appropriately discretizes R and prescribes the following boundary conditions on the four sides of R for some function  $u(x, y)$ ;  
 $u(x, 0) = x^2$ ,  $u(x, 2) = e^{-x}$ ,  $0 \leq x \leq 2$   
 $u(0, y) = y^2$ ,  $u(2, y) = e^{-y}$ ,  $0 \leq y \leq 2$  (5 marks)

### QUESTION THREE (20 MARKS)

- a) Classify the first order PDE  $(y - u)u_x + (u - x)u_y = x - y$  and hence show that it has the general solution  $F(x + y + u, x^2 + y^2 + z^2) = 0$ . (7 marks)
- b) State the 3 types of boundary conditions that can be prescribed on the boundary  $\delta\Omega$  of a domain  $\Omega$  in which an initial-boundary value problem is defined. (3 marks)
- c) The following MATLAB code implements the finite difference method to solve a given Poisson equation subject to the specified boundary conditions;

```
function FDPoissonEqn()  
clear all;clc;  
x0=0; xn=2; n=16; dx=(xn-x0)/n; x=x0:dx:xn;  
y0=0; ym=1; m=15; dy = (ym-y0)/m; y=y0:dy:ym;  
%%=====
```

$$A = \frac{dy \cdot dy}{2 \cdot (dx^2 + dy^2)}; B = \frac{dx \cdot dx}{2 \cdot (dx^2 + dy^2)};$$

```
C = ((dx*dy)^2) / (2*(dx^2+dy^2));  
%%=====
```

$$u = \text{zeros}(n, m);$$

```
%%boundary conditions specification  
u(1,:) = 0; u(n,1:m) = 2*exp(y(1:m));  
u(1:n,1) = x(1:n); u(1:n,m) = exp(1)*x(1:n);  
%% approximate solution by finite difference method  
for i=2:n-1  
    for j=2:m-1  
        u(i,j) = A*(u(i+1,j)+u(i-1,j)) + B*(u(i,j+1)+u(i,j-1)) -  
        C*x(i)*exp(y(j));  
    end  
end  
mesh(u(:,:,K))% this is for approximation solution  
end
```

- i) Modify the code to implement the Gauss-Siedel iteration method in order to improve the approximation of the solution. (5 marks)
- ii) Given the exact solution to the problem as  $u(x, y) = xe^y$ , write the code segments needed to compute the exact values within the `for` loops and plot the curves of both the exact and approximate solution at  $y = 0.5$ , on the same graph. (5 marks)

**QUESTION FOUR (20 MARKS)**

- a) Magnetic resonance elastography (MRE) is a phase-contrast-based MRI imaging technique with possible applications in tumor detection (particularly in breast, liver, kidney and prostate), characterization of disease, and assessment of rehabilitation (particularly in muscle). MRE directly visualizes and quantitatively measures propagating acoustic waves in tissues subjected to harmonic mechanical excitation. Given that the wave motion in the medium is basically modelled by the wave equation;

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0;$$

together with the conditions,

$$\begin{aligned} u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L \\ u(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0 \end{aligned}$$

- i) Use the separation of variables method to obtain the eigenvalue problem  $X'' + \omega^2 X = 0, X(0) = 0, X(L) = 0$  (5 marks)
- ii) Solve the eigenvalue problem in i) above to obtain the eigenvalue  $\omega_n = \frac{n\pi}{L}, n = 1, 2, 3, \dots$  (5 marks)
- iii) Hence, obtain the eigenfunctions  $u_n(x, t), n = 1, 2, 3, \dots$  (6 marks)
- b) Determine the appropriate Fourier series expansion of  $f(x) = x$  on  $-\pi \leq x \leq \pi$  (4 marks)