2018

Exotic derivatives pricing using copula-based martingale approach

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EXOTIC DERIVATIVES PRICING USING COPULA-BASED MARTINGALE APPROACH.

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072559

Submitted in partial fulfillment of the requirements for the Degree of Master of Science in Mathematical Finance at Strathmore University

Institute of Mathematical Sciences (IMS)
Strathmore University
Nairobi, Kenya
June, 2018

DECLARATION

I declare that this work has not been previously submitted and approved for the award of a degree by this or any other University. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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Abstract

This study examines the pricing of bivariate exotic derivatives, namely: capped spread option and bivariate digital options, using martingale approach and pair copulae formulations. Pair copulae is used to capture the joint distribution of asset price process and varying dependence structure rather than the univariate marginal distribution used in pricing univariate options. Unique payoff conditions for these exotic options are developed and the prices of these exotic options are obtained under the best fitting pair copulae. We then assess the sensitivity of the exotic option prices to the copula parameter, by formulating a 'dependence delta' and 'dependence gamma' formula obtained by application of chain rule decomposition to the copula derivative to have h-function and density function representation. Data from 2012 to 2018 from the NYSE of Equity ETFs and Bond ETFs of Frontier Markets, Emerging Market and Developed Markets to construct 10 pair combination of Equity and Bond ETFs as underlyings for the bivariate exotic options. The findings reveal that the t-copula captures best the dependence between the 10 pair combinations of underlyings. The prices of the bivariate exotic options are affected by the strength of the dependence of underlyings. Emerging and Developed market equity ETFs combination are more sensitive to changes in copula parameter. However, Emerging market equity ETF and Developed market bond ETF exhibit lower downside dependence and have lower dependence delta. Dependence gamma is generally of similar strength and signage as the dependence delta.

Key words: pair copulae, exotic options, bivariate exotic options, dependence delta

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List of Abbreviations

OTC – Over-the-counter

CCP – Central Clearing Party

BIS – Bank of International Settlements

VIX – Implied Volatility Index on the S&P 500

NYSE – New York Stock Exchange

CDF – Cumulative Distribution Function

PDF- Probability Density Function

IFF – If and only if

IID – Independent and identically distributed

PDE – Partial differential equation

MSCI – Morgan Stanley Capital International (formerly)

EAFA – Europe, Asia and Far East

ETF – Exchange Traded Fund

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Chapter One: Introduction

1.1 Background of study

1.1.1 Derivatives

A derivative can be defined as a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables. Very often the variables underlying derivatives are the prices of traded assets (Hull, 2015). Many types of derivatives exist such as forwards, swaps, futures, options and other derivatives. Futures and options are actively traded on many exchanges throughout the world. These derivatives are used by financial institutions, banks, insurance companies, fund managers, hedge funds, and corporate treasurers in the over-the-counter market or in exchanges for hedging or speculation purposes. Derivatives also play a key role in risk management in transferring of risk, diversification of portfolio, stabilization of earnings and cash flows, reduction of price fluctuation exposure and also ultimately reduce bankruptcy or financial distress costs that may have otherwise ensued.

The works of Black and Scholes(1973) on option pricing considered pricing of options in frictionless, continuous and competitive markets. As Jarrow (1996) defines that a frictionless market is one in which there are no transaction costs and no restrictions on trade. A competitive market is one in which all traders act as price takers, believing that they can buy or sell as much of any security as desired without affecting the price. A continuous trading market is one in which a trader is able to revise his portfolio position continuously. These assumptions on frictionless, competitive and continuous trading are only approximations to reality. These assumptions yielded hedging and replication strategies for contingent contingent claims and the Black-Scholes pricing partial differential equation.

Following the works of Black and Scholes, a rigorous underpinning of the heuristic no-arbitrage argument was developed in a sequence of papers by Cox, Ross and

Rubinstein (1979), Cox and Ross (1976) and Harrison and Pliska (1981). The rigorous formulation of these no-arbitrage arguments is often called the *martingale* or *risk-neutral valuation* approach. This approach takes the intuitive economic arguments concerning the formulation of portfolios for an instant in time and translates them into mathematical theories (Jarrow, 1996). Bingham and Kiesel (2004) opined that the Black-Scholes theory had become the main theoretical tool for pricing options and given that the theory is well-established, the profit margins on the standard –'vanilla'-options are so slender that practitioners seek to develop new- nonstandard – 'exotic' option could be traded more profitably. These new nonstandard products (exotics) have to be priced as well.

Plain vanilla options are the European and American call and put options which are traded in both organized exchanges and over-the-counter. The difference between the two being the additional feature in American options that they may be exercised at any time prior to their expiry or at the expiry date. These American options are then termed to be path dependent and European options are said to be strictly path independent. Exotic options are any other which are not plain vanilla, as described.

Exotic options are classified in different ways. Just like the European and American options they may be path-dependent or path-independent. Examples of these path dependent exotic options are barrier options, double barrier options, lookback options, Russian and Asian options. Some exotic options are constructed as portfolios of plain vanilla options. These type of exotic options are known as packages such as range forward derivatives and double options. There exist also exotics option on a single underlying asset whose payoff depends on underlying price at different dates such as call-on-call options and calendar spreads. Other exotic derivatives that are traded are those whose payoff at a future date is dependent on two or more underlying assets whose prices may be correlated. Under this category of exotic derivatives we have basket options, two asset rainbow (correlation) options, best/worst option, exchange(spread) option and bivariate digital options.

Another type of exotic derivatives are based on the exercise style these include Bermudan, Canary, Verde, shout and evergreen options. Others are based on the payoff style a future date such as chooser and digital options. There also exist another category of exotic options known as moment derivatives whose value depends on the realized higher moments of the underlying. Their payoff is a function of powers of the (daily) log-returns and allows to cover different kinds of market shocks. Finally the last category of exotic derivatives are those that an exotics of other exotics know as compound exotics and those that are written on unconventional underlyings such as weather derivatives.

1.1.2 OTC Derivatives Market

Over-the-counter (OTC) derivatives markets have grown significantly over the past two decades, and constitute a systemically important component of financial services activity. The market for OTC derivatives as of December 2014 stood at approximately US$630 trillion in terms of outstanding notional value (Bank of England, 2015). The sum of the cash flow obligation of the outstanding OTC contracts between the two parties is known as "gross market value" . According to BIS OTC derivatives statistics, the gross market value of the global OTC derivatives market rose to fell from $21 trillion to $15 trillion over the same period end-June 2016 to end-December 2016 having been at $14.5 trillion at end of December 2015.

Academics and industry professionals have addressed the necessary ingredients vital to new product's advancement: The commercial demand for hedging, attractiveness of product to speculators and co-operative public policy that is receptive to the ideology on which product is based and one that ensures that the market will operate without interruptions have been put forth as the desired inputs for the longevity and viability of a new product in the market according to an article by the Business Standard (2009).

On matters of risk management of new and exotic OTC products, regulatory agencies have since the financial market turmoil of 2007 investigated the cause of the financial crisis and identified OTC transactions as one of the potential causes of the global financial crisis. There has since been recommendations of securing of this market. The common agreement has been moving the OTC trading system to a Central Clearing Party (CCP) platform. Here the idea is to introduce CCPs, which are trust worthy financial institutions in a bid to replace the bilateral relationships that prevailed between two counterparties. The CCPs will be able to cater for multilateral relationships. The seller would sell the contract to the CPP and the buyer buys the contract from the CCP. The CCP acts as buyer to every seller, and seller to every buyer, simplifying the network of exposures within the system. This will bring about effective monitoring since the CCP can stipulate the required collateral and monitor the positions of the two parties under new regulatory rules. This new infrastructure implemented for the OTC Derivative market will considerably reduce the global counterparty risk observed into this market.

Highlights from the combined semiannual and triennial surveys of positions in over-the-counter (OTC) derivatives markets at end-June 2016 by the Bank of International Settlements (BIS), show that central clearing has made very significant inroads into OTC interest rate derivatives markets but is less prevalent in other OTC derivatives segments. The interest rate segment accounted for the vast majority of outstanding OTC derivatives. From the OTC derivative market statistics provided by BIS surveys the notional amount of outstanding OTC interest rate derivatives contracts totalled $438 trillion, which represented 80% of the global OTC derivatives market. This is down from $581 trillion, or 83%, at end-June 2013. Trade compression to eliminate redundant contracts appears to have been a major driver of the decline in notional amounts. Compression was aided by a shift towards CCPs in recent years, which in effect multilateralised the compression process. FX derivatives make up the second largest segment of the global OTC derivatives market. In contrast to interest rate derivatives, the notional amount of outstanding FX contracts has continued to climb in recent years, rising to a record high of $86 trillion at end-June 2016. As a share of all OTC derivatives, FX instruments rose from 12% at end-June 2013 to 16% at end-June 2016 when measured in notional amounts – which determines contractual payments – and from 13% to 17% when measured at gross market value – which is the cost of replacing all outstanding contracts at market prices prevailing on the reporting date.

The credit derivatives market, in 2007 was briefly as large as the FX derivatives market in notional amounts, but it has declined steadily in size since then. The notional principal of outstanding credit derivatives fell to $12 trillion at end-June 2016, from $25 trillion at end-June 2013 and a peak of $51 trillion in 2007. As a share of all OTC derivatives, credit derivatives fell from 10% to 2% between end-June 2007 and end-June 2016 when measured in notional amounts, and from 8% to 2% when measured at gross market value. The smallest segments remained OTC derivatives linked to equities and commodities, which totalled $7 trillion and $2 trillion, respectively, at end-June 2016. Together, equity and commodity derivatives accounted for only 2% of notional amounts outstanding, but a larger proportion of market value. At their peak in 2007, equity and commodity derivatives had collectively accounted for over 15% of the gross market value of all OTC derivatives, but this proportion fell to 4% at end-June 2016.

The majority of OTC equity derivatives consists of equity options, equity swaps, portfolio swaps, contracts for difference, variance swaps, dividend swaps, and exotics. The private nature and the flexibility in terms of product design have helped the OTC market to thrive. OTC derivatives exist because there are a significant number of products that are not offered by exchanges: in the latter, products are limited in tenor, size and strike ranges. Equity derivatives differ from Interest Rate and Credit derivatives in the sense that there is considerable transparency in pricing due to the existence of listed equities and equity derivatives exchanges. One important example of this transparency is in the emerging area of variance swaps (Avellaneda & Cont, 2010)

1.2 Problem Statement

From the stylized facts of financial markets it is seen that correlations that are observed during normal market conditions differ considerably from those that are observed during periods of market stress. Assets have greater tendency to move together during periods of market stress. It is also a stylized fact that financial asset returns display other empirical properties such as heavy tails, skewness (gain and loss asymmetry) and leverage effects (negative correlation of return volatility with returns). The dependence and consequently the price of the exotic options written on these underlyings will be affected by these empirical observations.

The use of copula method in valuation of exotic option is able to capture the dependence structure of the underlying financial assets. A copula is a multivariate distribution function each of whose marginals is uniform on the unit interval. The advantage of the copula-based approach to modelling is that appropriate marginal distributions for the components of a multivariate system can be selected by any desired method, and then linked through a copula or family of copulas suitably chosen to represent the dependence prevailing between the components. With this feature we are then able to price a number of exotic options given assumptions about the dependence structure between the assets price processes or given the observed dependence structure of the asset price process. This will provide more accurate exotic option prices and allow us to examine sensitivities of these exotic options to changes in the underlying asset prices and the dependence parameters.

1.3 Research Objectives

Main Objective

1. To use copula based methods and martingale approach to formulate a pricing equation of two bivariate exotic derivatives, namely: capped spread option and bivariate digital option.

Specific Objective

1. To use historical time series data to calibrate pair copulae and obtain the fair value of these 2 types of exotic derivatives
2. To develop a dependence delta and dependence gamma formula for the exotic derivatives and obtain the option price sensitivities to changes in pair copula parameters

1.4 Research questions

The research is guided by the following questions:

1. Can copula methods be applied to price exotic derivatives and would a closed-form pricing formula be achievable if pair copulae are used?

2. How sensitive would the option prices be then to the pair copula parameters?

1.5 Significance of research

Risk Managers: For the risk managers the departure for joint normality assumption may cause exposures that may lead to huge losses. Thus the choice of inputs into the pricing algorithm and the pricing model may lead to risk exposures in the exotic derivatives trade. The risk manager strives to choose the best method and the best set of parameters in order to report exposures. The choice of model and parameters can be confirmed during backtesting. He is then capable of advising the trader which best model to use. If differences in model risk are small then most risk managers using a standard multivariate normal assumptions would be justified in maintaining a simpler and cheaper method as a fast approximation to the theoretically superior copula approach. If the differences in option prices are substantial then the risk manager's model or inputs, used by the traders, may lead to quoting option prices that are distant from the competitors prices and be virtually excluded from the market.

Financial Engineers: Origination of new financial products such as different exotic derivative types, strategies and portfolio structuring may lead to profitable business and meeting of customers investment and risk management needs. In the valuation of these new products, financial engineers will desire an array or statistical techniques together with derivative valuation knowledge to come up with fair values of these new products. Or design structured payoff derivatives as desired by their clients. Financial engineering is intended to split risk and return components of financial products/instruments and offering the combination which is best suited to investor's risk-return profile. Quantitative analysis has brought innovation, efficiency and rigor to financial markets and to the investment process. And thus with reliance on these modelling techniques and understanding the dependence structure of the underlyings of these new products may be useful to ascertain profitable business and meet client portfolio needs for investment banks, insurance companies and hedge funds.

Academia: Diffusion of copulas for pricing exotic options may become possible especially if clear best practices about parameters estimation and leading copula functions emerge. There is at present a lack in empirical literature on use of martingale approach and copulae to value capped spread options without prior simulations and using of a closed-form formula. This research will thus address these concerns in modelling of dependence among returns and application to pricing exotic derivatives. Dependence risk if not reduced by a balanced portfolio with 'correlation-bullish' and 'correlation-bearish' options to have dependence hedged portfolios, may have serious implications for associated bank portfolios of huge positions of complex equity derivatives. This then furthers research into exotic pricing by copula based methods.

Traders: Traders in exotic options desk typical activity is to price and then sell the exotic option if it meets a defined counterparties payoff requirement, and then hedge this position directly or hedge a part off the risk involved. The trader is thus concerned with being wrong in the pricing of the exotic option when he sells it and being wrong in the hedging position consequently taken. Poor correlation and dependence estimates and consequently copula model selection may lead the trader to misprice and lose out on premiums or may lead to inaccurate hedging positions being adopted. Correlations in the underlyings may change through time and these changes in the level of correlation may produce substantial change in the options' prices and consequently problematic hedging if sensitivity is not assessed. The trader is capable of using more sophisticated methods of dependence (in the case of multivariate options) such a copula methods, at least in order to control the prices that he is quoting for new options which may be misvalued under simple multivariate normal pricing algorithm.

Chapter Two: Literature Review

2.1 Introduction

This chapter provides a summary of the works that have been done in the area of study. It contains the theoretical literature on option pricing theory and risk-neutral valuation and the empirical studies on copula based methods applied to valuation of exotic options. It aims to provide a suitable position to assess knowledge gap existing from literature pertaining to copula methods as applied in financial engineering.

2.2 Theoretical review

There are two main approaches to the pricing of derivative securities. The first, due to Black and Scholes (1973) and Merton (1973), in which is considered the partial differential equation (PDE) approach. In this approach the partial differential equation is obtained by hedging or replication argument. This technique constructs a PDE along with the appropriate boundary conditions for the price of a derivative security. The PDE can then be solved using various analytical or numerical methods. Feyman-Kac formula is applied to achieve the Black-Scholes model for derivative securities.

The second approach, by Harrison and Kreps (1979), is the "martingale method." This approach consists of writing the value of the security as the expected value of the discounted pay of the derivatives under a risk neutral measure/equivalent martingale measure, say Q, and calculating this expectation using probabilistic methods. Absence of arbitrage and market completeness are assumed in these approaches.

2.3 Empirical review

Cherubini and Luciano (2002) study the pricing of bivariate options with copula. They link risk-neutral marginal with various Archimedean copula families such as gumbel, clayton and frank in a bid to capture the non-normality of returns and dependence problem. They show that the pricing kernel is the copula function and that the super-replication strategy is represented by the Fréchet bounds and provide no-arbitrage pricing bounds, as well as the values consistent with independence of the underlying assets. From each market they use vertical spreads and the interpolation technique by Shimko to recover the implied marginal probability distributions from the market. They then provide prices for binary digital options, options on minimum and exchange options written on four indices: MIB30, S&P500, FTSE, DAX; using daily return time series data from 1999 to 2000 to obtain empirical risk-neutral densities. They as well present a sensitivity analysis of the bivariate option values with respect to the dependence structure of the underlying assets.

Saita et al (2003) analyse the pricing of multiasset equity options with copulas, assessing the effects of uncertain correlation parameters and assessing the choice between traditional standard methods that assume joint normality of asset returns and copula-based methods by use of a monte carlo simulation applied to different exotic contracts on a basket of five US stocks. They carry out empirical test of different short-term OTC exotic contracts such as 5 year Asian basket options, Asian best options, napoleon options that pay periodic coupons adjusted for worst performing asset, if underlying basket outperforms a threshold, digital barrier options that pay a period coupon if underlying basket underperforms a threshold (up and out digital barrier barrier with Bermudan exercise style), european style digital basket option paying an worst performing asset adjusted coupon at maturity and a bermudan digital barrier option that pays a fixed coupon annually if at each month the basket outperforms a threshold. They price these options using monte carlo simulations of 10,000 simulation runs. Pricing is done under multivariate monte carlo simulation under joint normality assumption and under student t copulae with varying degrees of freedom. In the simulation they extracted first a vector of uncorrelated standard nomarl variables and transformed them in correlated returns, same random number generator was used. The generation of correlated returns in the multivariate normal case was performed through a simple Cholesky decomposition method. When assessing the option price sensitivities they found that for Asian basket the sensitivity to increases in correlation coefficients is positive since this increases the volatility of the basket; for the same reason the sensitivity is negative in the Napoleon option. They finally prescribe further that due to the criticality of the assumption of the marginal distribution of the underlyings and their dependence structure the risk manager should strive the define the best model once or adopt the industry standard and periodically redefine inputs. Alternatively, they suggest that the risk manager could maintain a simple model for day close portfolio repricing and use the more sophisticated models as pricing control tools when option is issued and also so as to capture model risk.

Van den Goorbergh et al (2005) examine multivariate option pricing given dependence between the underlying assets, they use parametric families of copula to model this dependence which they explicitly assume that its parameter is varying overtime as a function of the volatilities of the assets. Kendall's measure of association is taken to be a function of the objective conditional variance estimate, premised on the evidence of correlation suggesting that increased dependence occurs in periods of market stress. They employ rolling window technique and regression to estimate the parameters of the Kendall's tau-conditional volatility specification. They apply this dynamic copula model to pricing the better-of-two-markets and worse-of-two-markets options on the S&P500 and Nasdaq indexes daily returns from January 1, 1993 to August 30, 2002 was used . Their finding reveal that option prices implied by dynamic copula models differ substantially from prices implied by models prescribe a fixed dependence between the underlyings, particularly in times of high volatilities. They find also that the normal copula produces option prices that differ significantly from non-gaussian copula prices, irrespective of initial volatility levels. Within the non-gaussian copula families that they considered they find that option prices were robust with respect to the choice of copula. their findings suggest that unless the dependence between the S&P500

and Nasdaq stock indexes is well described by a normal copula, then alternative copula families should be considered.

Zhang and Guegan (2008) extend the work of Van den Goorbergh and analyse the pricing of bivariate options under GARCH processes with time varying copula parameters. They obtain the returns innovation by fitting two GARCH-GH models on two underlying assets. They then observe the dependence structure for the two series of innovation changes over time through rolling windows technique. A series of copulas are then selected for different subsamples according to AIC criterion. This method allows the changes of the copula can be observed and the change trend appears more clearly. In their method the dynamic evolution of the copula's parameter is a time-varying function of predetermined variables. They price options on the better performer based on two important Chinese equity index returns (Shanghai Stock Composite Index and Shenzhen Stock Composite Index). The Student t copula is selected as the best fitting copula based on minimal AIC value. They then provide the option prices implied by GARCH-NIG model with time-varying copula and compare them with those obtained by GARCH-Gaussian model. They show that that the prices implied by the GARCH-Gaussian are generally underestimated.

Slavchev and Wilkens (2013) in their study of valuation of multivariate equity options in the Eurupean derivative market by use of copula in determining the risk-neutral joint distribution of the stock prices, they consider a multivariate case as opposed to use of only two underlyings. They study in particular a tri-variate and hexa-variate cases, analyzing derivatives on these multiple underlyings such as multivariate call option on-maximum, multivariate put option-on-miniumum and multivariate digital option. They construct two baskets with three stocks each for insurance sector and technology sector. They then calibrated empirically elliptical and Archimeadean copua families to stock returns using the canonical maximum likelihood (CML) method, in order to find the one that best captures the dependence structure observable. They use option data on single stock underlyings on the European Exchange (Eurex) to extract the marginal distributions. They then compared the valuations of the Black-Scholes type to multivariate underlyings to that of alternative copula models. They find remarkable differences in the two approaches and state that a high model risk is noticeable given the parameter sensitivities of the copula models, that aim at capturing dependence structures that differ from that of the Gaussian copula.

2.4 Research gap

With exotic derivatives, financial engineers are capable of developing new products that have payoff structures that match the risk and return profile of investors. Copula methods in pricing of exotic options allow for the separate modeling of the marginal distributions and the dependence structure of the variables. In exotic derivatives that have more than one underlying asset, copula methods allow for capturing of the dependence and by Sklar's theorem copula distribution as pricing kernel for that option.

Given stylized facts of asset returns such as heavy tails, volatility clustering, skewness, leverage effects and correlations that are observed during normal market conditions seen to differ considerably from those that are observed during periods of market stress; the assumption of joint normal distributions and linear dependence alone would then lead to mispricing of these exotic securities and consequently improper adoption of hedging strategies. A knowledge gap exists on how to apply copula techniques allowing for different dependence structures to value special bivariate exotic derivatives other than those studied by Dufresne, Kierstead and Ross (1996) who assumed a joint normal distribution. We therefore attempt to obtain a Black-scholes type formula capturing dependence for capped spread option and bivariate digital option. And consequently analyzing price sensitivity to pair copulae parameters. Correlations and dependence in the underlyings may change through time and these changes in the level of correlation may produce substantial change in the options' prices. Depending on the type of risk-neutral copula selected as best model for pricing the exotic derivatives, different hedge positions may be taken up to offset correlation and dependence risk in the exotic derivative portfolio. These copula methods to exotic derivatives will also inform evaluation of mispriced exotic derivatives by methods supplementary to the multivariate and joint normal pricing algorithm. It is from these identified gaps that research seeks to explore further copula methods in exotic derivatives pricing.

2.5 Conceptual Framework

Figure 1 below describes the copula approach applied to exotic derivatives. The copula approach captures the joint distribution and dependence in the underlying asset price processes. Investors are interested in these bivariate exotic options, which are traded majorly over-the-counter, for speculation and hedging purposes. The prices of these bivariate exotic derivatives are affected by the nature of the dependence between the underlying assets.

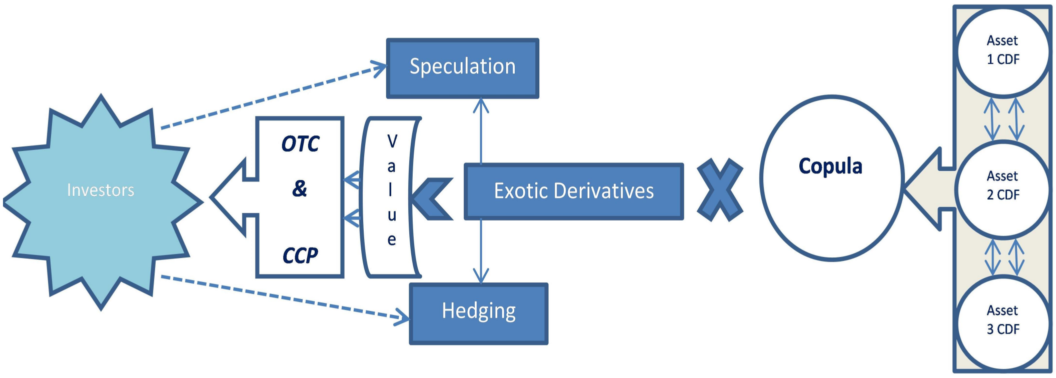

Figure 1: Conceptual Framework

Chapter Three: Methodology

3.1 Introduction

This chapter outlines the general methodology to be used to conduct the study. It specifies the research design, sampling design, data collection method and data analysis techniques of the study. The objective of this section is to provide insight into methodology of the study.

3.2 Research design

The study adopts a quantitative research design. A quantitative research design involves the collection of empirical data and applying modeling and analysis of data techniques. In the quantitative research design adopted methods to test relationships described by theory, examine causes, effects and relationships between the variables will be used. Through statistical analysis, it will be possible to make objective deductions of valuation of exotic derivatives and explore exotic derivatives sensitivity to dependence of the underlying variables.

3.3 Data Description

3.3.1 Sample and Data collection

The data to be used is secondary data. The data is obtained from *Yahoo Finance.* We use price data of Index Funds (ETFs) on the NYSE from October 24, 2012 to February 24, 2018 to calibrate the geometric Brownian motion process and obtain the pair copulae parameter estimates. The price (USD) data set is consists of 1341 observations of Frontier Market Equity Index Fund, Emerging Market Equity Index Fund and Developed Market Equity Index Fund which are the iShares MSCI Frontier 100 ETF(FM), iShares Core MSCI Emerging Market ETF (IEMG) and iShares MSCI EAFE ETF (EFA) respectively. We obtained also 1341 observations of Emerging Market Bond Index Fund and Developed Market Bond Index Fund which are the iShares JP Morgan Emerging Markets Local Currency Bond ETF (LEMB) and the iShares International Treasury Bond ETF (IGOV). The returns on these index funds correspond to the performance the respective benchmark index. We construct bivariate exotic option based on pair combinations of bond and equity ETFs.

3.4 Model Specification

3.4.1 Joint probability and Marginals

If X and Y are absolutely continuous and normally distributed, then X and Y are independent *iff* the joint density is equal to the product of the marginal ones, that is *iff*

$$\varphi_{X,Y} = \varphi_X(x)\varphi_Y(y)$$

Where $\varphi_X(x)$ and $\varphi_Y(y)$ are normally distributed marginal densities.

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

Via fubini's theorem and partial integration it holds that the bivariate cumulative distribution function in the case of independence becomes

$$\begin{aligned}
\phi_{X,Y}(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} \varphi_{X,Y}(w,s)dwds \\
&= \int_{-\infty}^{x}\int_{-\infty}^{y} \varphi_X(w)\varphi_Y(s)dwds \\
&= \int_{-\infty}^{x} \varphi_X(w)dw \int_{-\infty}^{y} \varphi_X(s)ds \\
&= \phi_X\phi_Y
\end{aligned}$$

In the case of dependence the bivariate joint cumulative distribution assuming normality becomes

$$\phi_{X,Y}\left(x,y\right) = \int_{-\infty}^{x}\int_{-\infty}^{y} \varphi_{X,Y,\rho}\left(w,s\right) dw\ ds \tag{2}$$

where the bivariate joint density is

$$\varphi_{X,Y,\rho} = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X} + \frac{(y-\mu_Y)^2}{\sigma_Y} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right)$$

3.4.2 Copula

Definition: A copula function is defined in the bivariate case as follows:
A two-dimensional copula is a function $C : [0,1] \times [0,1] \to [0,1]$ with the following properties:

1. For every $u, v \in [0,1]$: $C(u,0) = C(0,v) = 0$
2. For every $u, v \in [0,1]$: $C(u,1) = u$ and $C(1,v) = v$
3. For every $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \leq u_2$ *and* $v_1 \leq v_2$:

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$$

Given the copula C for uniform distributed variables u and v we have that

$$\begin{aligned} C(\phi_X(x), \phi_Y(y)) &= C(u,v) \\ &= \Pr(u \leq \phi_X(x), v \leq \phi_Y(y)) \\ &= \Pr(\phi_X^{-1}(u) \leq x, \phi_Y^{-1}(v) \leq y) \\ &= \Pr(X \leq x, Y \leq y) \\ &= \phi_{X,Y}(x,y) \end{aligned}$$

According to Sklar's theorem,we thus have

$$C(\phi_X(x), \phi_Y(y)) = \phi_{X,Y}(x,y) \tag{3}$$

one splits the joint probability into the marginals and a copula so that the latter only represents the "association" between X and Y. Copulas separate marginal behavior as represented by ϕ_i from the association: at the opposite the two cannot be disentangled in the usual representation of joint probabilities via distribution functions

3.4.3 Copula density and h-functions

If a copula is sufficiently differentiable the copula density c(u,v) can be defined as follows:

$$c\left(u,v\right)=\frac{\partial^{2}C(u,v)}{\partial u\partial v} \tag{4}$$

For continuous random vectors the copula density is related to the density of the distribution , denoted as φ. The copula density is equal to the ratio of the joint density $\varphi_{x,y}$ to the product of the marginal densities φ_{x} and φ_{y} , given as

$$c\left(u,v\right)=\frac{\varphi_{x,y}(x,y)}{\varphi_{x}\ (x)\varphi_{y}\ (y)}$$

By Sklar's theorem the following canonical representation holds

$$\varphi_{x,y}(x,y)=c(u,v)\varphi_{x}(x)\varphi_{y}(y) \tag{5}$$

The partial derivatives of a copula distribution have a probabilistic meaning. More precisely, if u,v has the copula C as joint distribution function, then the function

$$h_{v}\rightarrow\frac{\partial}{\partial v}C\left(u,v\right),\ v\in(0,1)$$

can typically be computed for almost every v ∈ (0,1). It equals the distribution function of u conditional on the event v. For some bivariate copulas, the function h_v can be computed, and even inverted in closed form. This function is known as the h-function and represents the cumulative distribution function X conditional on Y.

$$\Pr\left(X\leq x\mid Y=y\right)\ =C_{X|Y}\left(\phi_{X}\left(x\right),\phi_{Y}\left(y\right)\right)=C_{X|Y}\left(u,v\right)=\frac{\partial C(u,v)}{\partial v}=h_{v}$$

The h functions $\frac{\partial C(u,v)}{\partial v}$ and $\frac{\partial C(u,v)}{\partial u}$ exist for almost all u and v, as shown by Nelsen (2006) we have that $0\leq\frac{\partial C(u,v)}{\partial v}\leq1$ and $0\leq\frac{\partial C(u,v)}{\partial u}\leq1$. This then suggests an alternative way to writ the copula function

$$\phi_{X,Y}\left(x,y\right) = C\left(u,v\right) = \int_0^v \frac{\partial C(u,\omega)}{\partial \omega} d\omega \tag{6}$$

3.4.4 Parametric Estimation

We use the Maximum likelihood method to obtain pair copula parameter estmates. the maximum-likelihood method starts from the assumption of having a parametric model for the distribution in concern. It then estimates all parameters of the marginals by the maximum likelihood method (by maximizing the log-likelihood function). The joint density is the represented in the canonical form as a product of the copula density and the marginal densities. Log transform is then applied and splits the problem into a sum of terms that are alike in the univariate maximum-likelihood procedure (for each of the marginal laws) and a term that depends on both: parameters of the marginal laws and parameters of the dependence structure(log of copula density).

3.4.5 Martingale Option Pricing Approach

The value of a European call option is:

$$C\left(S,t\right) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(h(S_t)) \tag{7}$$

Where $h(S_t)$ is the payoff function , in the case of european call option it is $\max(S_T - K, 0)$. Dufresne, Keirstead and Ross (1996) show by using the martingale approach and auxiliary probability measure adopted by Girsanov Theorem(*see appendix theorem 1.1*) that the Black-Scholes equation is derived with computation simplicity as:

$$\begin{aligned} C\left(S,t\right) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(S_T - K)^{+} \\ &= \mathbb{E}_{\mathbb{Q}}(S_T e^{-r(T-t)} 1_{S_T>K}) - \mathbb{E}_{\mathbb{Q}}(K e^{-r(T-t)} 1_{S_T>K}) \end{aligned}$$

From the definition of the indicator function, $\mathbb{E}_{\mathbb{Q}}\left(1_{S_T>K}\right)=\mathbb{Q}(S_T > K)$

$$\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}}(S_T e^{-r(T-t)} 1_{S_T>K}) - Ke^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(1_{S_T>K}) \\
&= \mathbb{E}_{\mathbb{Q}}(S_T e^{-r(T-t)} 1_{S_T>K}) - Ke^{-r(T-t)}\mathbb{Q}(S_T > K) \\
&= A_1 - A_2
\end{aligned}$$

Assuming that S_T follows a geometric Brownian motion under equivalent martingale measure $\mathbb{Q}$,

$$dS = rSdt + \sigma S dW^{\mathbb{Q}} \tag{8}$$

$$S_T = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma dW^{\mathbb{Q}}} \tag{9}$$

With $\ln \frac{S_T}{S_t} \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$ and $dW^{\mathbb{Q}} \sim N(0, T-t)$. We have therefore that second term A_2 is

$$\begin{aligned}
A_2 &= \mathbb{E}_{\mathbb{Q}}(Ke^{-r(T-t)} 1_{S_T>K}) \\
A_2 &= Ke^{-r(T-t)}\mathbb{Q}(S_T{>}K) \\
A_2 &= Ke^{-r(T-t)}\mathbb{Q}(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma dW^{\mathbb{Q}}}{>}K) \\
A_2 &= Ke^{-r(T-t)}\mathbb{Q}(-\sigma dW^{\mathbb{Q}} {\leq} \ln\frac{S_t}{K} {+} \left(r{-}\frac{1}{2}\sigma^2\right)(T{-}t)\,) \\
A_2 &= Ke^{-r(T-t)}\mathbb{Q}\left(\frac{-\sigma dW^{\mathbb{Q}}}{\sigma\sqrt{T-t}} {\leq} \frac{\ln\frac{S_t}{K} + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
A_2 &= Ke^{-r(T-t)}\mathbb{Q}\left(\xi {\leq} \frac{\ln\frac{S_t}{K} + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

$$A_2 = K\mathrm{e}^{-\mathrm{r(T-t)}}\phi(\mathrm{d}_2) \tag{10}$$

Where $\mathrm{d}_2 = \frac{\ln\frac{\mathrm{S_t}}{\mathrm{K}} + \left(\mathrm{r}-\frac{1}{2}\sigma^2\right)(\mathrm{T-t})}{\sigma\sqrt{\mathrm{T-t}}}$. The first integral $A_1 = \mathbb{E}_{\mathbb{Q}}(S_T e^{-r(T-t)} 1_{S_T>K})$ by application of girsanov theorem becomes $A_1 = S_t\phi(d_1)$ as shown in the derivations below.

We have that $\mathbb{E}_{\mathbb{F}}\left(1_{S_T>K}\right) = \mathbb{E}_{\mathbb{Q}}\left(\eta_T 1_{S_T>K}\right)$. The solution of the first term A_1 yields,

$$
\begin{aligned}
A_1 &= \mathbb{E}_{\mathbb{Q}}(S_T e^{-r(T-t)} 1_{S_T>K}) \\
A_1 &= S_t \mathbb{E}_{\mathbb{Q}}(\eta_T 1_{S_T>K}) \\
A_1 &= S_t \mathbb{E}_{\mathbb{F}}(1_{S_T>K}) \\
A_1 &= S_t \mathbb{F}(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma dW^{\mathbb{F}}+\sigma^2(T-t)} > K) \\
A_1 &= S_t \mathbb{F}\left(-\sigma dw^{\mathbb{F}} \leq \ln\frac{S_t}{K} + \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right) \\
A_1 &= S_t \mathbb{F}\left(\frac{-\sigma dW^{\mathbb{F}}}{\sigma\sqrt{T-t}} \leq \frac{\ln\frac{S_t}{K} + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
A_1 &= S_t \mathbb{F}\left(\xi \leq \frac{\ln\frac{S_t}{K} + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

$$A_1 = S_t \phi(d_1) \tag{11}$$

Where $d_1 = \frac{\ln\frac{S_t}{K} + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$. We therefore have that the price of a european call option is

$$C(S,t) = S_t \phi(d_1) - K e^{-r(T-t)} \phi(d_2) \tag{12}$$

The $\mathbb{F}$ measure is the equivalent martingale measure when the stock mutual fund $X_t = S_t$ is used as numeraire, as opposed to the $\mathbb{P}$ measure which is the equivalent martingale measure when the money market fund e^{rt} is used as numeraire.

3.4.6 Exotic Derivatives Valuation

We are now able to apply the martingale approach above as outlined by Dufresne, Keirstead and Ross (1996) to the valuation problem of exotic derivatives without much computational burden to have closed-form pricing formulas. We price the following derivatives: capped spread option and bivariate digital option.

3.4.6.1 Capped Spread Option

(a) ***Description***

The capped spread option is a European type exotic derivative which pays off the difference between two assets at maturity given that one of the assets is below a certain cap value at maturity or is above a certain floor value at maturity but the value of one asset is great than the other.

(b) ***Payoff function***

Table 1 below shows the payoffs for asset price capped and asset price floored spread options.

Table 1: Capped spread option payoff

Payoff function, $h(S_T)$	Type	Payoff condition (1_D)
$(S_1 - S_2)^+$	Asset 1 Cap Asset 1 Floor	$S_1 > S_2$ *and* $S_1 \leq K$ $S_1 > S_2$ *and* $S_1 > K$
$(S_1 - S_2)^+$	Asset 2 Cap Asset 2 Floor	$S_1 > S_2$ *and* $S_2 \leq K$ $S_1 > S_2$ *and* $S_2 > K$

(c) ***Valuation of capped spread option***

We apply martingale approach and copulae theorem to price the capped/floored spread option.

$$C(S,t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}((S_{1,T} - S_{2,T})1_D) \tag{13}$$

This gives the price of the capped/floor spread option given the respective payoff condition D. We have that

$$\begin{aligned}
C(S,t) &= \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}S_{2,T}(Y_T - 1)\,1_D) \\
&= \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}S_{2,T}Y_T 1_D) - \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}S_{2,T}1_D) \\
&= S_{2,t}\mathbb{E}_{\mathbb{Q}}(Y_T 1_D) - S_{2,t}\mathbb{E}_{\mathbb{Q}}(1_D) \\
&= S_{2,t}[\mathbb{E}_{\mathbb{Q}}(Y_T 1_D) - \mathbb{E}_{\mathbb{Q}}(1_D)] \\
&= S_{2,t}V(Y,t)
\end{aligned}$$

where the constant $S_{2,t}$ is the current price of stock 2 and $V(Y,t) = \mathbb{E}_{\mathbb{Q}}\left(Y_T 1_D\right) - \mathbb{E}_{\mathbb{Q}}\left(1_D\right)$ is the price of a European call option written on Y_T with strike of 1 and risk-free rate of zero. Using the same approach as for the call option of equation (12) for $V(Y,t)$, we have

$$V(Y,t) = B_1 - B_2 \tag{14}$$

where $Y_T = \frac{S_{1,T}}{S_{2,T}}$. Using the change of measure by multidimensional girsanov theorem (see appendix II and III), we have that under the risk neutral measure $\mathbb{Q}$ the stochastic differential equations are

$$dS_1 = rS_1dt + \sigma_1 S_1 dw^{\mathbb{Q}} \tag{15}$$

$$dS_2 = rS_2dt + \sigma_2 S_2 dw^{\mathbb{Q}} \tag{16}$$

$$dY = \Gamma Y d\hat{w}^{\mathbb{Q}} \tag{17}$$

Where $d\hat{w}^{\mathbb{Q}}$ is a standard brownian motion under $\mathbb{Q}$ with $d\hat{w}^{\mathbb{Q}} \sim N(0,dt)$ and Γ is given as $\Gamma = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 - 2\rho_{1,2}\sigma_1\sigma_2}$ and $\mathbb{E}(dw_1^{\mathbb{Q}} dw_2^{\mathbb{Q}}) = \rho_{1,2}dt$.
We can now therefore proceed to obtain a closed-form formula for the capped/floored spread option. Continuing from equation (14), we have that the second term B_2 yields

$$B_2 = \mathbb{E}_{\mathbb{Q}}(1_D)$$
$$B_2 = \mathbb{Q}(D)$$

Taking the payoff condition D of Asset 1 Floor Spread Option to be $S_{1,T} > S_{2,T}$

and $S_{1,T} > K$

$$B_2 = \mathbb{Q}(S_{1,T} > S_{2,T}, S_{1,T} > K)$$
$$B_2 = \mathbb{Q}(Y_T > 1, S_{1,T} > K)$$
$$B_2 = \mathbb{Q}(Y_t e^{-\frac{1}{2}\Gamma^2(T-t)+\Gamma dW^{\mathbb{Q}}}>1, S_{1,t}e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma dW^{\mathbb{Q}}}>K)$$
$$B_2 = \mathbb{Q}(-\Gamma dW^{\mathbb{Q}}\leq \ln\frac{Y_t}{1}-\frac{1}{2}\Gamma^2\left(T-t\right), -\sigma dW^{\mathbb{Q}}\leq \ln\frac{S_{1,t}}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t))$$
$$B_2 = \mathbb{Q}\left(\frac{-\Gamma dW^{\mathbb{Q}}}{\Gamma\sqrt{T-t}}\leq\frac{\ln\frac{S_{1,t}}{S_{2,t}}-\frac{1}{2}\Gamma^2\left(T-t\right)}{\Gamma\sqrt{T-t}}, \frac{-\sigma dW^{\mathbb{Q}}}{\sigma\sqrt{T-t}}\leq\frac{\ln\frac{S_{1,t}}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$B_2 = \mathbb{Q}\left(\xi_1\leq\frac{\ln\frac{S_{1,t}}{S_{2,t}}-\frac{1}{2}\Gamma^2\left(T-t\right)}{\Gamma\sqrt{T-t}}, \xi_2\leq\frac{\ln\frac{S_{1,t}}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$B_2 = C_\theta(\phi(e_2), \phi(d_2))$$

Where values of e_2 and d_2 are obtained by $e_2 = \frac{\ln\frac{S_{1,t}}{S_{2,t}}-\frac{1}{2}\Gamma^2(T-t)}{\Gamma\sqrt{T-t}}$ and $d_2 = \frac{\ln\frac{S_{1,t}}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$. The first term of equation (14) upon using *Theorem 1.1* yields

$$B_1 = \mathbb{E}_{\mathbb{Q}}(Y_T 1_D)$$
$$B_1 = Y_t\mathbb{E}_{\mathbb{Q}}(\eta_T 1_D)$$
$$B_1 = Y_t\mathbb{E}_{\mathbb{F}}(1_D)$$
$$B_1 = Y_t\mathbb{F}(Y_t e^{-\frac{1}{2}\Gamma^2(T-t)+\Gamma dW^{\mathbb{F}}+\Gamma^2(T-t)} > 1, S_{1,t}e^{\left(r+\frac{1}{2}\sigma^2\right)(T-t)+\sigma dW^{\mathbb{F}}+\sigma^2(T-t)} > K)$$
$$B_1 = Y_t\mathbb{F}\ (-\Gamma dw^{\mathbb{F}}\leq \ln\frac{Y_t}{1}+\frac{1}{2}\Gamma^2\left(T-t\right), -\sigma dw^{\mathbb{F}}\leq \ln\frac{S_{1,t}}{K}+\left(r+\frac{1}{2}\sigma^2\right)(T-t))$$
$$B_1 = Y_t\mathbb{F}\left(\frac{-\Gamma dW^{\ \mathbb{F}}}{\Gamma\sqrt{T-t}}\leq\frac{\ln\frac{S_{1,t}}{S_{2,t}}+\frac{1}{2}\Gamma^2\left(T-t\right)}{\Gamma\sqrt{T-t}}, \frac{-\sigma dW^{\ \mathbb{F}}}{\sigma\sqrt{T-t}}\leq\frac{\ln\frac{S_{1,t}}{K}+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$B_1 = Y_t\mathbb{F}\left(\xi_1\leq\frac{\ln\frac{S_{1,t}}{S_{2,t}}+\frac{1}{2}\Gamma^2\left(T-t\right)}{\Gamma\sqrt{T-t}}, \xi_2\leq\frac{\ln\frac{S_{1,t}}{K}+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$B_1 = Y_t C_\theta(\phi(e_1), \phi(d_1))$$

Where values of e_1 and d_1 are obtained by $e_1 = \frac{\ln\frac{S_{1,t}}{S_{2,t}}+\frac{1}{2}\Gamma^2(T-t)}{\Gamma\sqrt{T-t}}$ and $d_1 = \frac{\ln\frac{S_{1,t}}{K}+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$.

We therefore have that the the price of the capped/floored spread options is given as

$$C\,(S,t) = S_{2,t}V(Y,t)$$
$$C\,(S,t) = S_{2,t}[Y_t C_\theta(\phi(e_1),\phi(d_1)) - C_\theta(\phi(e_2),\phi(d_2))]$$

$$C\,(S,t) = S_{1,t}C_\theta(\phi(e_1),\phi(d_1)) - S_{2,t}C_\theta(\phi(e_2),\phi(d_2)) \tag{18}$$

3.4.6.2 Bivariate Digital Option

(a) ***Description***

A bivariate digital call (or put) option give the holder a fixed payment at maturity provided that the price of two underlyings are above (or below) a certain strike price at maturity.

(b) **Payoff function**

Table 2 below shows the payoff and the payoff conditions for the bivariate digital call option and the bivariate digital put option.

Table 2: Bivariate digital call and put payoff

Bivariate digital type	Payoff function, $h\,(S_T)$	Payoff condition (1_D)
Bivariate digital call	D	$S_1 > k_1$ *and* $S_2 > k_2$
Bivariate digital put	D	$S_1 < k_1$ *and* $S_2 < k_2$

(c) ***Valuation of bivariate digital call***

We similarly apply the martingale approach and copula theorem to the bivariate digital call and bivariate digital put options to obtain a closed-form price formula. The price of a bivariate digital option is given as

$$C(S,t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[D1_D] \tag{19}$$

Assuming that the stock 1 prices and stock 2 prices follow a geometric brownian motion under $\mathbb{Q}$ as that of equation (15) and (16).We therefore have that for the bivariate call option

$$
\begin{aligned}
C(S,t) &= \mathbb{E}_{\mathbb{Q}}[De^{-r(T-t)}1_D] \\
&= De^{-r(T-t)}\mathbb{Q}(S_{1,T} > k_1, S_{2,T} > k_2) \\
&= De^{-r(T-t)}\mathbb{Q}(S_{1,T} > k_1, S_{2,T} > k_2) \\
&= De^{-r(T-t)}\mathbb{Q}(S_{1,t}e^{\left(r-\frac{1}{2}\sigma_1{}^2\right)(T-t)+\sigma_1 dW^{\mathbb{Q}}}>k_1, S_{2,t}e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)(T-t)+\sigma_2 dW^{\mathbb{Q}}}>k_2) \\
&= De^{-r(T-t)}\mathbb{Q}\left(\xi_1\leq\frac{\ln\frac{S_{1,t}}{k_1}+\left(r-\frac{1}{2}\sigma_1{}^2\right)(T-t)}{\sigma_1\sqrt{T-t}}, \xi_2\leq\frac{\ln\frac{S_{2,t}}{k_2}+\left(r-\frac{1}{2}\sigma_2{}^2\right)(T-t)}{\sigma_2\sqrt{T-t}}\right) \\
&= D\mathrm{e}^{-\mathrm{r(T-t)}}C_\theta(\phi(d_2^1),\phi(d_2^2))
\end{aligned}
$$

The price of the bivariate digital call becomes

$$
C(S,t) = D\mathrm{e}^{-\mathrm{r(T-t)}}C_\theta(\phi(d_2^1),\phi(d_2^2)) \tag{20}
$$

Where the values of d_2^1 and d_2^2 are obtained as $\mathrm{d}_2^1=\frac{\ln\frac{\mathrm{S_{1,t}}}{\mathrm{k_1}}+\left(\mathrm{r}-\frac{1}{2}\sigma_1{}^2\right)(\mathrm{T-t})}{\sigma_1\sqrt{\mathrm{T-t}}}$ and $\mathrm{d}_2^2=\frac{\ln\frac{\mathrm{S_{2,t}}}{\mathrm{k_2}}+\left(\mathrm{r}-\frac{1}{2}\sigma_2{}^2\right)(\mathrm{T-t})}{\sigma_2\sqrt{\mathrm{T-t}}}$.

(d) ***Valuation of bivariate digital put***

It follows using the same approach that the price of the bivariate digital put option with the payoff condition being $S_1 < k_1$ and $S_2 < k_2$ becomes

$$
\begin{aligned}
P(S,t) &= \mathbb{E}_{\mathbb{Q}}[De^{-r(T-t)}1_D] \\
&= De^{-r(T-t)}\mathbb{Q}(S_{1,T} < k_1, S_{2,T} < k_2) \\
&= De^{-r(T-t)}\mathbb{Q}(S_{1,T} < k_1, S_{2,T} < k_2) \\
&= De^{-r(T-t)}\mathbb{Q}(S_{1,t}e^{\left(r-\frac{1}{2}\sigma_1{}^2\right)(T-t)+\sigma_1 dW^{\mathbb{Q}}}<k_1, S_{2,t}e^{\left(r-\frac{1}{2}\sigma_2{}^2\right)(T-t)+\sigma_2 dW^{\mathbb{Q}}}<k_2) \\
&= De^{-r(T-t)}\mathbb{Q}\left(\xi_1\leq\frac{\ln\frac{k_1}{S_{1,t}}-\left(r-\frac{1}{2}\sigma_1{}^2\right)(T-t)}{\sigma_1\sqrt{T-t}}, \xi_2\leq\frac{\ln\frac{k_2}{S_{2,t}}-\left(r-\frac{1}{2}\sigma_2{}^2\right)(T-t)}{\sigma_2\sqrt{T-t}}\right) \\
&= De^{-r(T-t)}\mathbb{Q}(\xi_1 \leq -d_2^1, \xi_2 \leq -d_2^2) \\
&= D\mathrm{e}^{-\mathrm{r(T-t)}}C_\theta(\phi(-d_2^1),\phi(-d_2^2))
\end{aligned}
$$

We thus have the price of the bivariate digital put as

$$P\left(S,t\right)=De^{-r(T-t)}C_{\theta}\left(\phi(-d_{2}^{1}),\phi(-d_{2}^{2})\right) \tag{21}$$

Where the values of d_2^1 and d_2^2 are obtained as $\mathrm{d}_2^1=\frac{\ln\frac{\mathrm{S}_{1,\mathrm{t}}}{\mathrm{k}_1}+\left(\mathrm{r}-\frac{1}{2}\sigma_1{}^2\right)(\mathrm{T}-\mathrm{t})}{\sigma_1\sqrt{\mathrm{T}-\mathrm{t}}}$ and $\mathrm{d}_2^2=\frac{\ln\frac{\mathrm{S}_{2,\mathrm{t}}}{\mathrm{k}_2}+\left(\mathrm{r}-\frac{1}{2}\sigma_2{}^2\right)(\mathrm{T}-\mathrm{t})}{\sigma_2\sqrt{\mathrm{T}-\mathrm{t}}}$. We will therefore use the resulting pricing equation(18), equation (20) and equation (21).

3.4.7 Exotic Derivatives Dependence Greek

Since the one parameter copula families have closed form expressions for the copula distribution function we are therefore capable of assessing the exotic derivatives sensitivities to changes in the dependence parameter by directly differentiating the copula distribution function and evaluating it at the estimated parameter values.

Proposition 1.1: Dependence Delta

The copula first derivative with respect to the dependence parameter is decomposed into the h-function h_u, the copula density and the h-function derivative with respect to the dependence parameter $h_{v,\theta}$ as shown in equation (22) below.

$$\frac{\partial C_{\theta}\left(u,v\right)}{\partial\theta}=\frac{h_{u}h_{v,\theta}}{c(u,v)} \tag{22}$$

Proposition 1.2: Dependence Gamma

The copula second derivative with respect to the dependence parameter is decomposed as follows

$$\frac{\partial^{2}C_{\theta}\left(u,v\right)}{\partial\theta^{2}}=\frac{h_{u}h_{v,\theta,\theta}}{c(u,v)}+\left(h_{v,\theta}\right)^{2}\left(\frac{h_{u,u}-\frac{h_{u}}{c(u,v)}\frac{\partial c(u,v)}{\partial u}}{c(u,v)^{2}}\right) \tag{23}$$

We use results obtained in equation(22) and equation(23) above in the analysis of the exotic derivatives price sensitivity to changes in copula parameter.

3.4.7.1 Dependence Delta

To obtain the dependence delta of the exotic derivatives we directly plug the results of equation (22) in the equations below and analyse the sensitivity at the respective values of the parameters.

(a) Spread option

$$\frac{\partial C\,(S,t)}{\partial\theta} = S_{1,t}\frac{\partial C_\theta(\phi(e_1),\phi(d_1))}{\partial\theta} - S_{2,t}\frac{\partial C_\theta(\phi(e_2),\phi(d_2))}{\partial\theta} \tag{24}$$

(b) Bivariate digital call

$$\frac{\partial C\,(S,t)}{\partial\theta} = De^{-r(T-t)}\frac{\partial C_\theta\left(\phi(d_2^1),\phi(d_2^2)\right)}{\partial\theta} \tag{25}$$

(c) Bivariate digital put

$$\frac{\partial P\,(S,t)}{\partial\theta} = De^{-r(T-t)}\frac{\partial C_\theta\left(\phi(-d_2^1),\phi(-d_2^2)\right)}{\partial\theta} \tag{26}$$

3.4.7.2 Dependence Gamma

To obtain the dependence gamma of the exotic derivatives we directly plug the results of equation (23) in the equations below and analyse the sensitivity at the respective values of the parameters.

(a) Spread option

$$\frac{\partial^2 C\,(S,t)}{\partial\theta^2} = S_{1,t}\frac{\partial^2 C_\theta(\phi(e_1),\phi(d_1))}{\partial\theta} - S_{2,t}\frac{\partial^2 C_\theta(\phi(e_2),\phi(d_2))}{\partial\theta} \tag{27}$$

(b) Bivariate digital call

$$\frac{\partial^2 C\,(S,t)}{\partial\theta^2} = De^{-r(T-t)}\frac{\partial^2 C_\theta\left(\phi(d_2^1),\phi(d_2^2)\right)}{\partial\theta} \tag{28}$$

(c) Bivariate digital put

$$\frac{\partial^2 P\,(S,t)}{\partial\theta^2} = De^{-r(T-t)}\frac{\partial^2 C_\theta\left(\phi(-d_2^1),\phi(-d_2^2)\right)}{\partial\theta} \tag{29}$$

Chapter Four: Results and Discussion

4.1 Introduction

This chapter gives descriptive statistics of the collected data, its analysis results and discussion of these results. We begin with summary statistics of the data and then proceed to copula fitting and exotic derivatives valuation.

4.2 Descriptive Statistics

In this section we present the summary statistics and time series plots of the Stock prices, log returns and percentage returns.

4.2.1 Summary Statistics

Table 3: Summary statistics of ETF price data

ETF	Min	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Max.	Std. dev
FM	21.88	26.48	29.65	30.03	32.47	39.57	4.200758
IEMG	34.69	44.84	48.57	48.00	51.28	62.69	4.959906
EFA	51.38	58.77	62.24	62.54	66.27	75.25	4.655523
LEMB	38.62	44.13	46.96	46.94	49.86	54.72	3.868289
IGOV	44.05	46.40	49.22	48.63	50.36	52.63	2.332305

Table 4: Summary statistics of ETF log returns

ETF	Min	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Max.	Std.dev
FM	-0.0957	-0.003892	0.0004046	0.0002093	0.004986	0.04545	0.00890
IEMG	-0.06031	-0.006444	0.0004725	0.0001619	0.007141	0.04249	0.01105
EFA	-0.08977	-0.004032	0.0005013	0.0002163	0.005301	0.03272	0.00913
LEMB	-0.05599	-0.003307	0.00000	-0.0000316	0.003402	0.02271	0.00589
IGOV	-0.01648	-0.002978	0.0001037	-0.000005	0.002994	0.02197	0.00491

Table 5: Summary statistics of ETF percentage returns

ETF	Min	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Max.	Std. dev
FM	-9.12600	-0.38840	0.04047	0.02488	0.49980	4.65000	0.885016
IEMG	-5.85300	-0.64230	0.04726	0.02228	0.71660	4.34100	1.103147
EFA	-8.58600	-0.40240	0.05014	0.02578	0.53150	3.32600	0.908807
LEMB	-5.44600	-0.33020	0.00000	-0.00143	0.34080	2.29700	0.587485
IGOV	-1.63500	-0.29740	0.01037	0.000711	0.29990	2.22200	0.491195

4.2.2 Time series plots

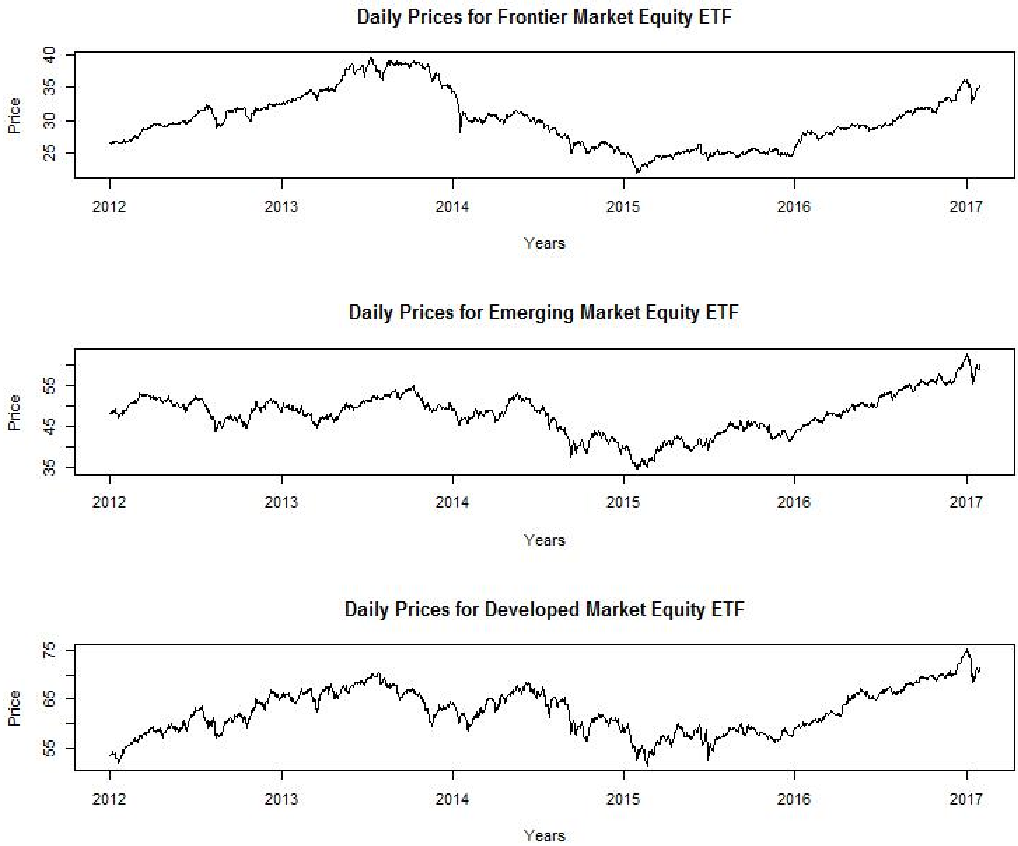

Figure 2: Daily prices of frontier, emerging and developed markets equity ETFs

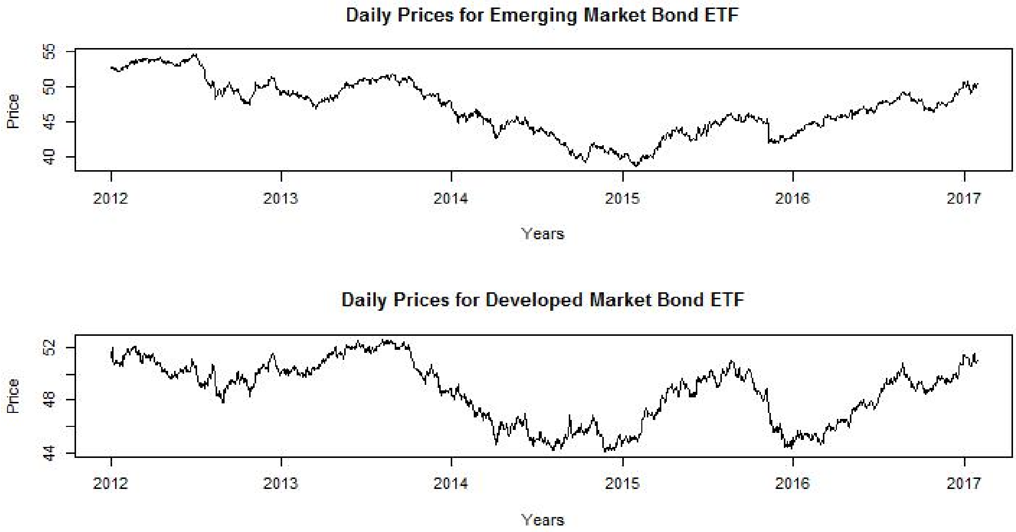

Figure 3: Daily prices for emerging and developed markets bond ETFs

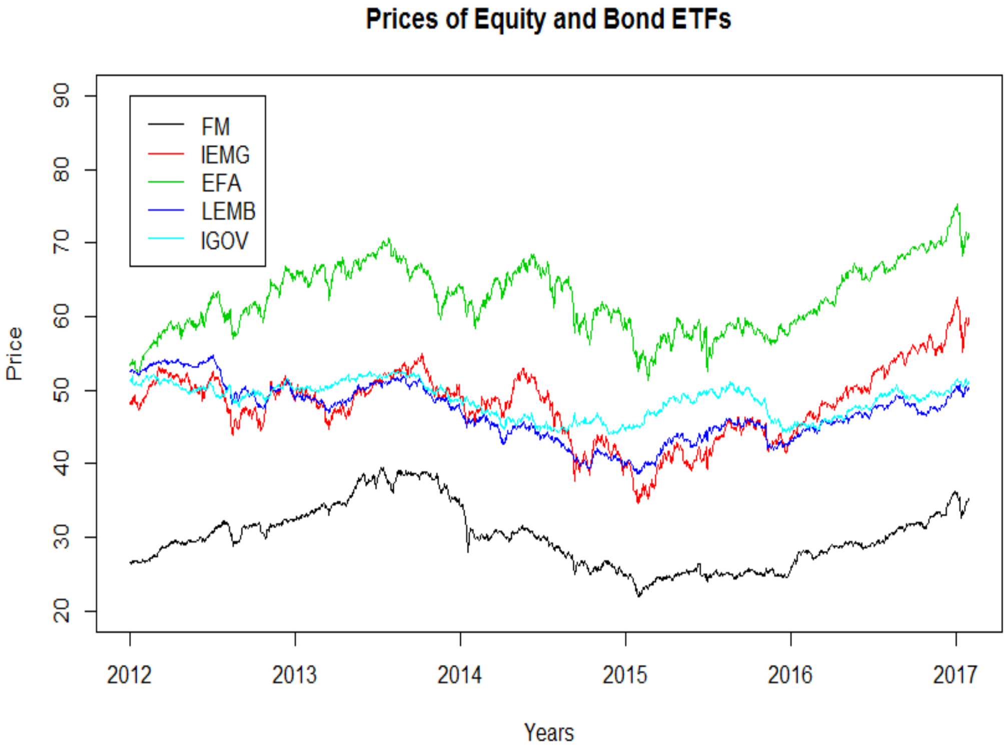

Figure 4: Daily price series plot of Equity and Bond ETFs

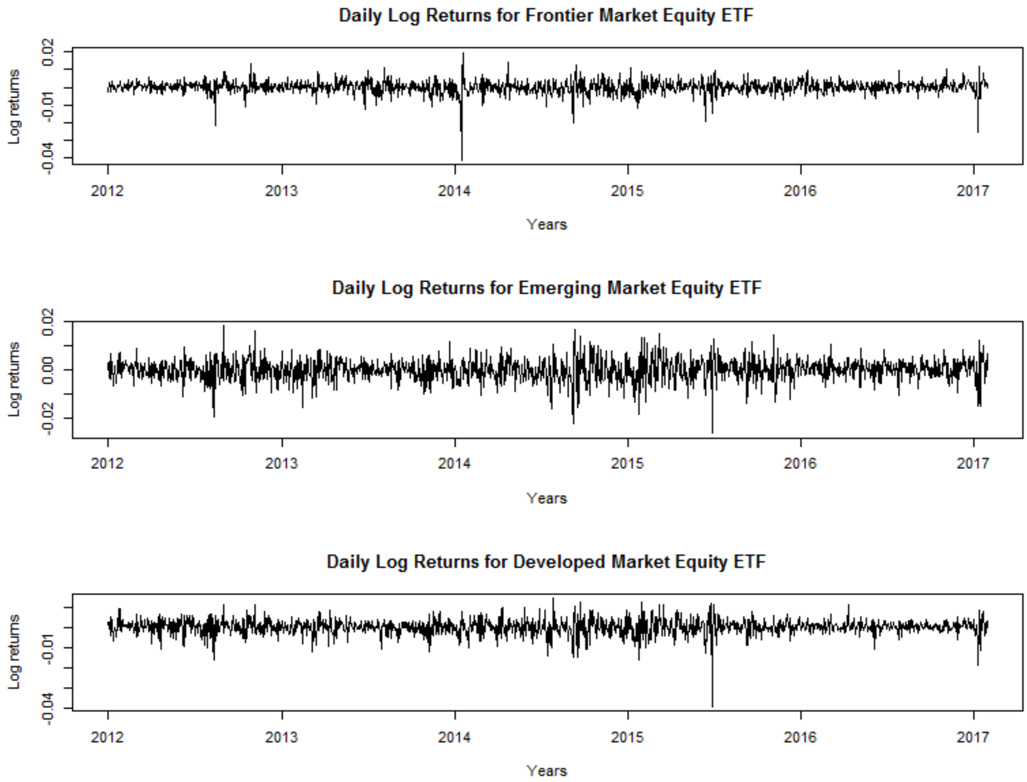

Figure 5: Daily log returns of frontier, emerging and developed markets equity ETFs

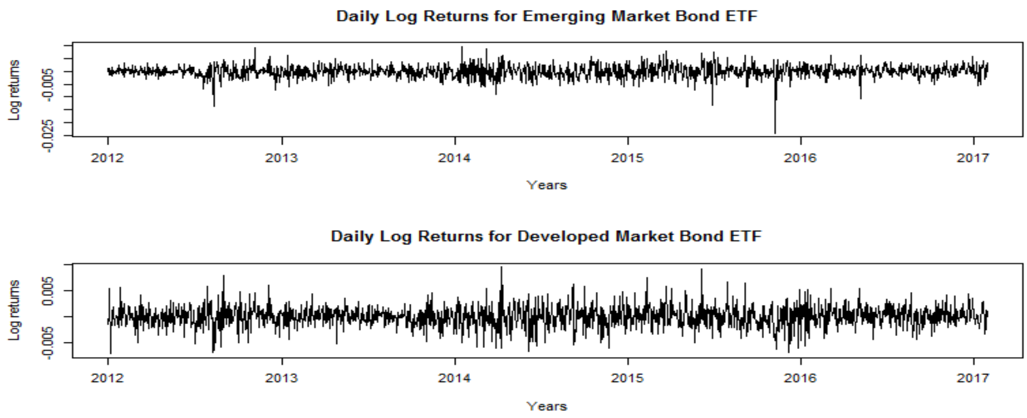

Figure 6: Daily log returns for emerging and developed markets bond ETFs

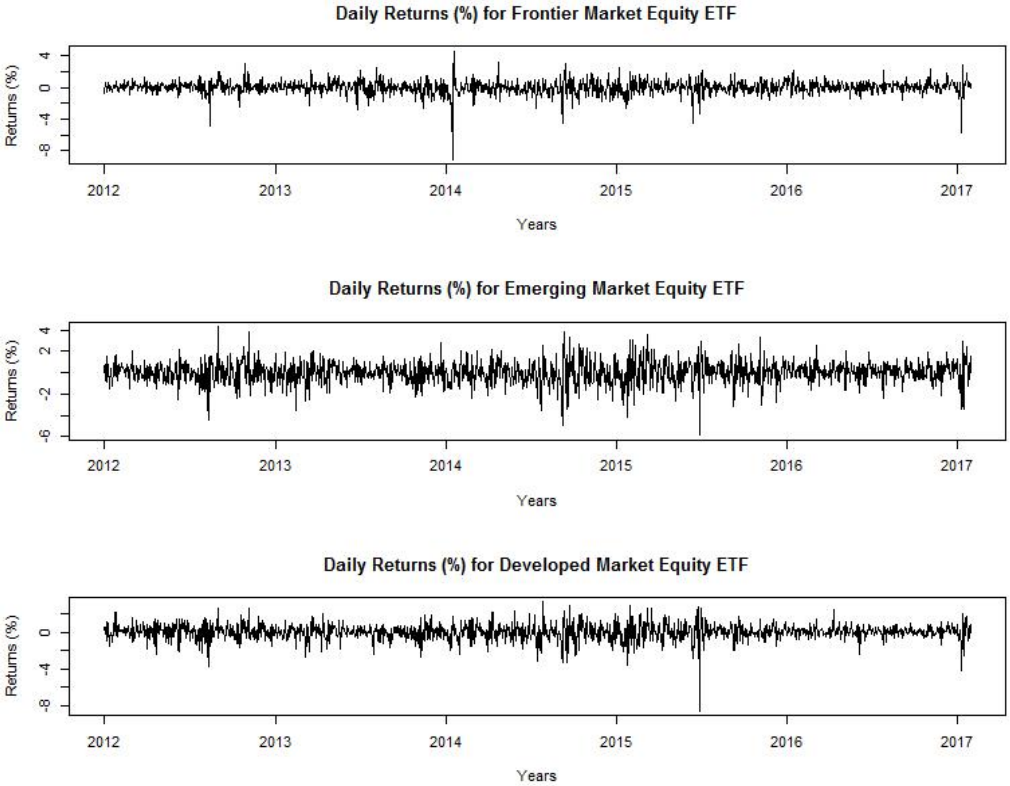

Figure 7: Daily returns (%) of frontier, emerging and developed markets equity ETFs

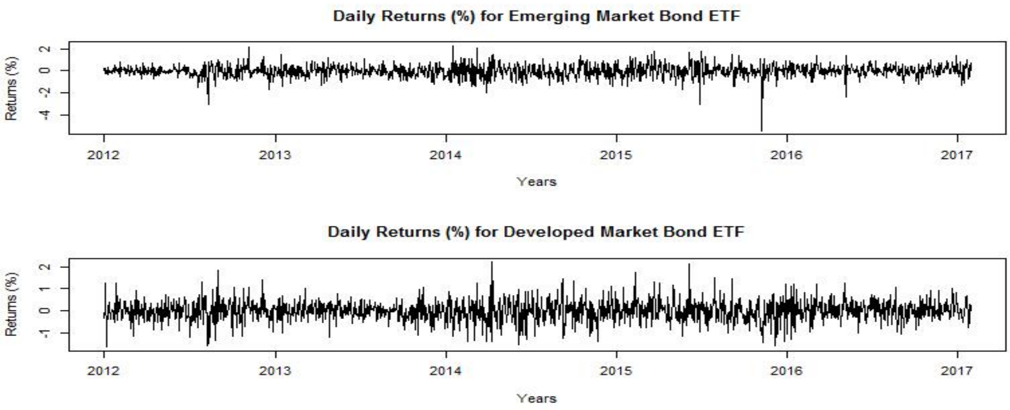

Figure 8: Daily returns (%) of emerging and developed markets bond ETFs

4.2.3 Correlation, dependence and distribution

The correlations matrix of equity and bond ETFs prices and the correlation matrix of equity and bond ETFs logreturns are shown in Table 6 and Table 7 below.

Table 6: Correlation matrix of Equity and Bond ETFs prices

	FM	IEMG	EFA	LEMB	IGOV
FM	1.0000000	0.6996012	0.7534489	0.6098728	0.5690132
IEMG	0.6996012	1.0000000	0.7577561	0.7260409	0.5072062
EFA	0.7534489	0.7577561	1.0000000	0.2978325	0.2438378
LEMB	0.6098728	0.7260409	0.2978325	1.0000000	0.8067641
IGOV	0.5690132	0.5072062	0.2438378	0.8067641	1.0000000

Table 7: Correlation matrix of Equity and Bond ETFs log returns

	FM	IEMG	EFA	LEMB	IGOV
FM	1.00000000	0.50877585	0.51228018	0.3298255	0.05472232
IEMG	0.50877585	1.00000000	0.81597554	0.6637604	0.08295951
EFA	0.51228018	0.81597554	1.00000000	0.5543888	0.08506275
LEMB	0.32982552	0.66376035	0.55438878	1.0000000	0.39277012
IGOV	0.05472232	0.08295951	0.08506275	0.3927701	1.00000000

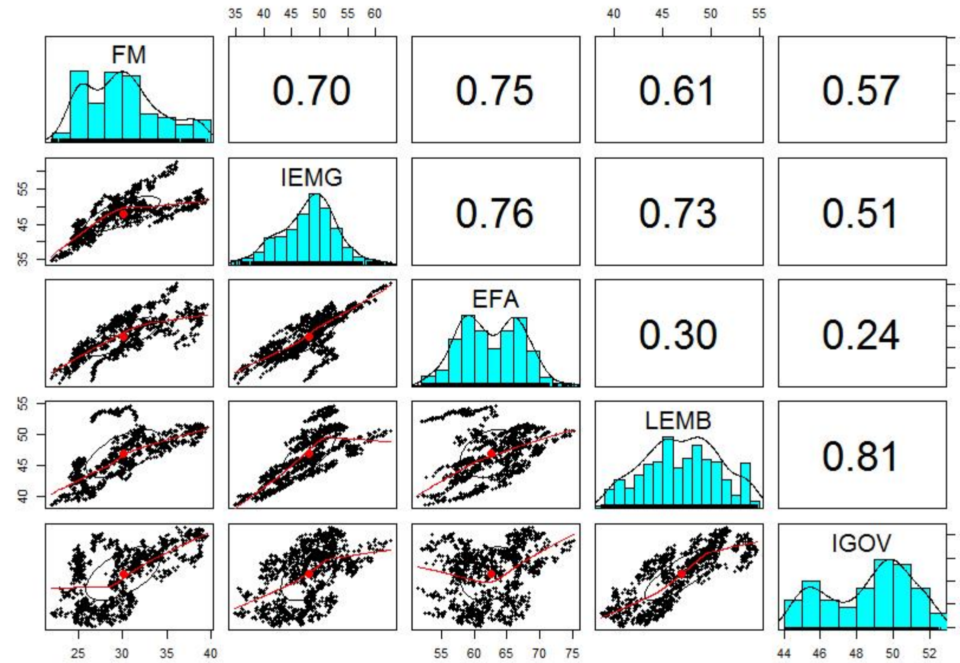

Figure 9: Pair plot showing empirical correlations plots, empirical histograms of Equity and Bond ETFs prices and correlation matrix

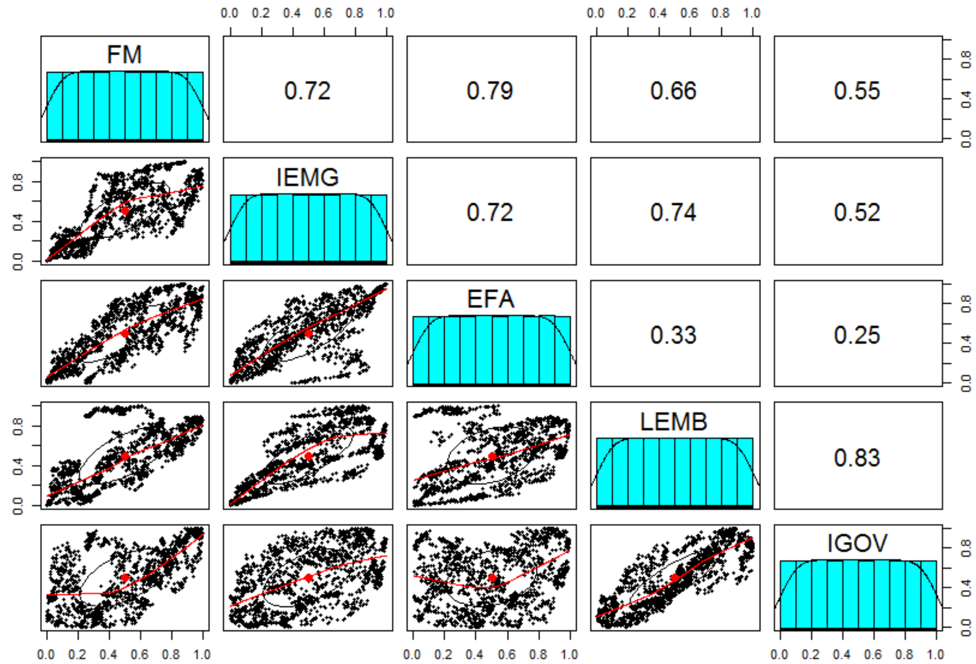

Figure 10: Pair plot showing empirical copula dependence plots, empirical histograms of Equity and Bond ETFs prices and correlation matrix

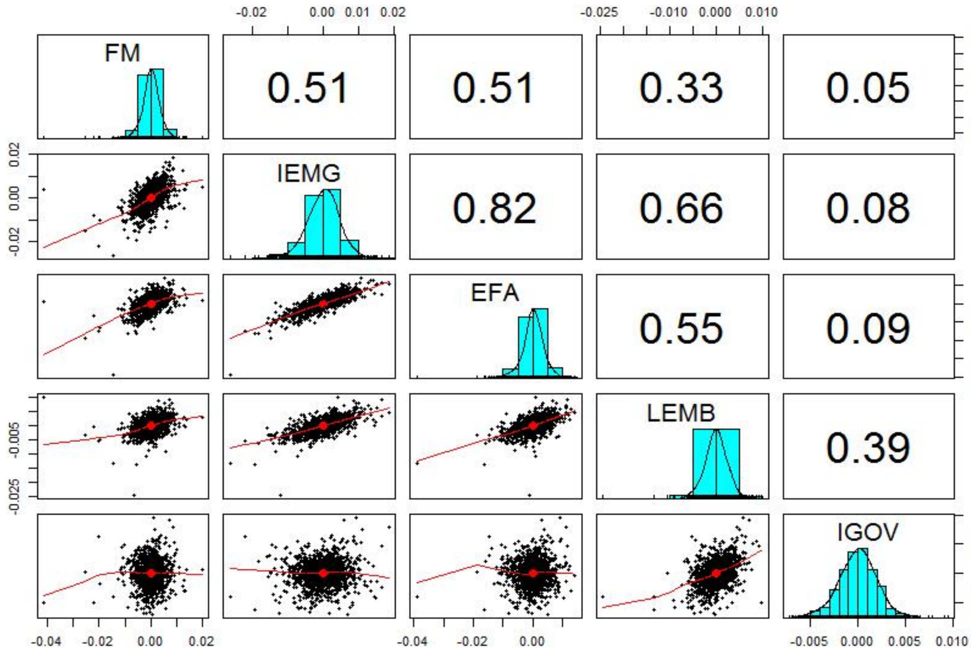

Figure 11: Pair plot showing empirical correlation plots, empirical histograms of log Equity returns and log bond returns of ETFs and the correlation matrix

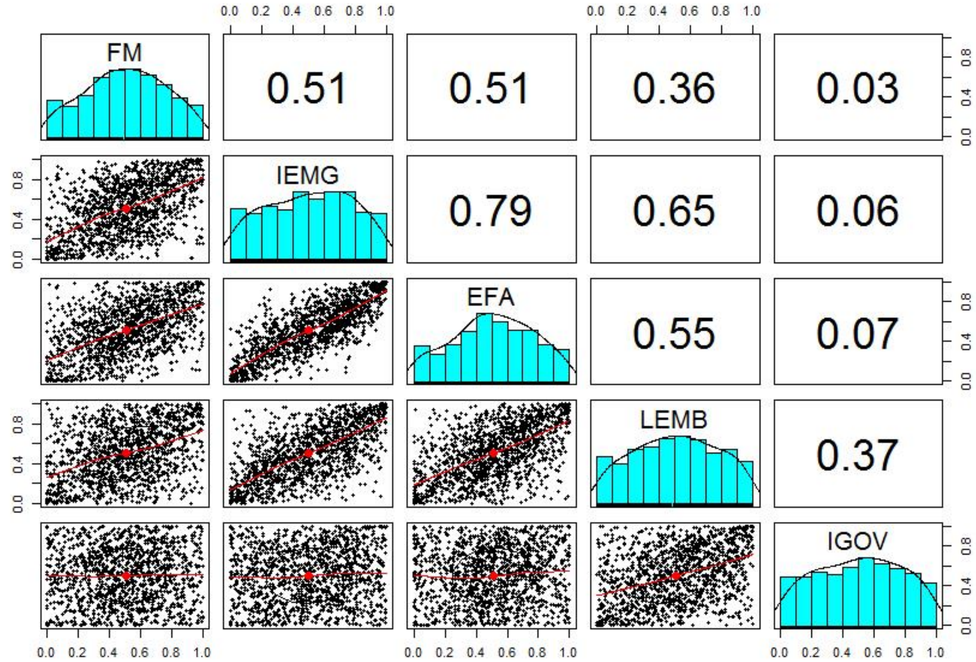

Figure 12: Pair plot showing empirical copula dependence plots, empirical histograms of probability transform of log Equity returns and log bond returns of ETFs and the correlation matrix

4.3 Tests for normality

4.3.1 Shapiro-Wilk test

The null-hypothesis of this test is that the population is normally distributed. Thus, if the p-value is less than the chosen 0.05 significance level, then the null hypothesis is rejected meaning there is evidence that the data tested are not from a normally distributed population. If the p-value is greater than the 0.05 significance level, then the null hypothesis that the data came from a normally distributed population cannot be rejected. The results of the Shapiro-Wilk test for the price series and return series are summarized in Table 8 below.

Table 8: Shapiro-Wilk normality test for prices and log returns of equity and bond ETFs

	Prices		Log returns	
	Test Stat. (W)	p-value	Test Stat.	p-value
FM	0.9576	2.2e-16	0.91148	2.2e-16
IEMG	0.99093	2.345e-07	0.98293	1.748e-11
EFA	0.98149	4.318e-12	0.94142	2.2e-16
LEMB	0.9813	3.609e-12	0.95572	2.2e-16
IGOV	0.93927	2.2e-16	0.98992	5.644e-08

For the prices of the Equity and Bond ETFs, we find that the p-values are below the significance level of 0.05. we thereby reject the null hypothesis that the price data is normally distributed. For the log returns of the Equity and Bond ETFs, we find that the p-values are below the significance level of 0.05. we thereby reject the null hypothesis that the log returns data is normally distributed. We confirm these results by using the QQ plots also.

4.3.2 Q-Q plots

Q–Q (quantile-quantile) plot is a probability plot, which is a graphical method for comparing two probability distributions by plotting their quantiles against each other. First, the set of intervals for the quantiles is chosen. A point $(x,\ y)$ on the plot corresponds to one of the quantiles of the second distribution (y-coordinate) plotted against the same quantile of the first distribution (x-coordinate). Thus the line is a parametric curve with the parameter which is the number of the interval for the quantile. If the two distributions being compared are similar, the points in the Q–Q plot will approximately lie on the line $y = x$. If the distributions are linearly related, the points in the Q–Q plot approximately lie on a line.

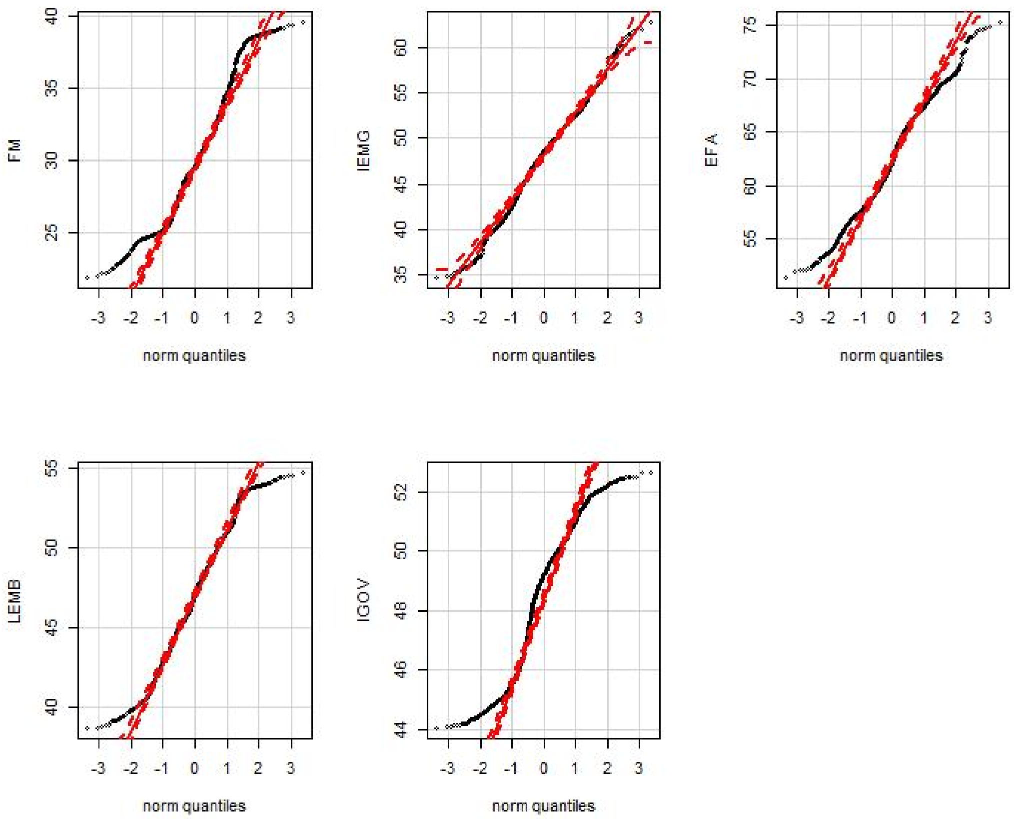

Figure 13: Q-Q plots of Equity and Bond ETF prices

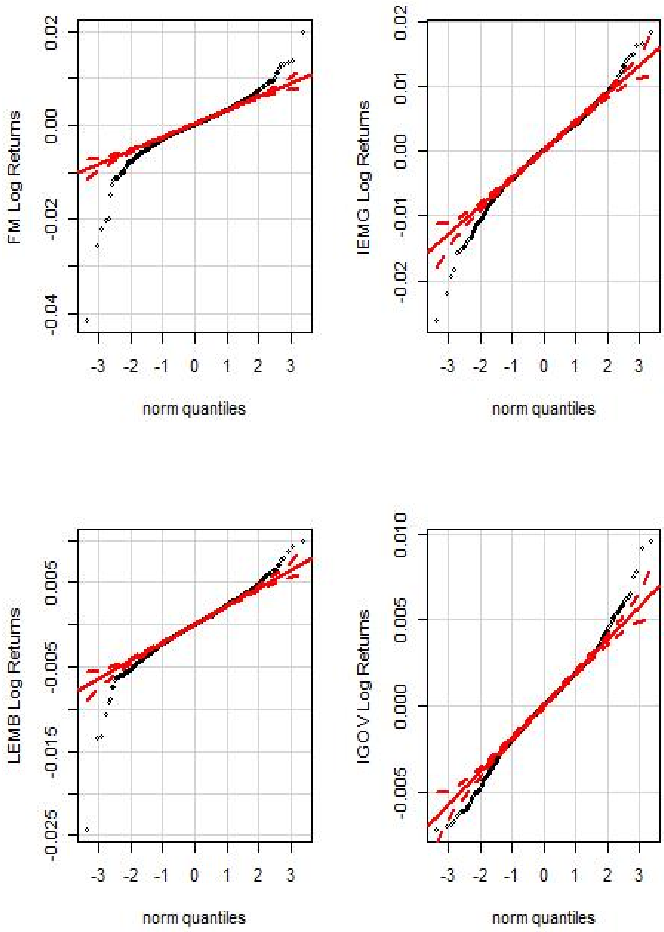

Figure 14: Q-Q plots of log returns of Equity and Bond ETFs

We find that the equity ETFs and bond ETFs prices are not normally distributed. For the log returns, the linearity of the points suggests that the log returns of equity and bond ETF data are much closer to being normally distributed that that of the equity and bond ETFs prices. This is however *strictly* rejected by the Shapiro-Wilk normality test results. Given the large size of 1341 observations, the test results of being statistically significant from a normal distribution in the large samples is explainable. We therefore expect that the log returns are normally distributed if we reduce the sample size further. We work with the assumption that the log returns are normally distributed when pricing the bivariate exotic derivatives. Proceeding to work with this assumption will not affect the empirical dependence captured by the copula function.

4.4 Copula fitting by MLE

We apply rank transformation to obtain the copula dependence plot rather than probabilty integral transformation since we find that the price series and log return series are not normally distributed, thus rank transformation assuming empirical distribution will still maintain the dependence structure. The summary of the three best fitting copula for each assset pair is show below in Table 9.

Table 9: Summary of top three copula fits by AIC selection for each of the underlying pairs

Asset Pairs	1	2	3
FM-IEMG	t-Copula	Frank	Normal
FM-EFA	t-Copula	Frank	Normal
FM-LEMB	t-Copula	Frank	Normal
FM-IGOV	t-Copula	Clayton	Gumbel
IEMG-EFA	t-Copula	Normal	Frank
IEMG-LEMB	t-Copula	Normal	Frank
IEMG-IGOV	t-Copula	Gumbel	Joe
EFA-LEMB	t-Copula	Frank	Normal
EFA-IGOV	t-Copula	Gumbel	Joe
LEMB-IGOV	t-Copula	Normal	Gumbel

In Table 10 and Table 11 below we show the empirical copula fitting results displaying the estimated parameters, it standard error (in brackets) and the information criteria of the fit. The maximum likelihood method is used to estimate the copula parameter in *R*, using the function *BiCopEst* of the *VineCopula* library. The pair copula fit results with the lowest AIC was selected as the best fitting copula. The findings summarized in Table 9 from Table 10 and Table 11 are as expected since the empirical pair copula plots of Figure 12 indicate that the empirical copula plots data points are more spread out typical of t-copula as opposed to the normal that are concentrated at both tails.

Table 10: Copula fitting by MLE results summary

Asset Pairs	Normal (1)		t-Copula (2)			Clayton (3)		Fit
	Par	Fit Stat.	Par	Par2	Fit Stat.	Par	Fit Stat.	
FM-IEMG	0.54 (0)	Loglik=217.82 AIC= -433.64 BIC=-428.44	0.55 (0.02)	15.96 (4.4)	Loglik=230.52 AIC= -457.04 BIC=-446.64	0.54 (0)	Loglik=131.31 AIC= -260.62 BIC=-255.42	**2**
FM-EFA	0.54 (0)	Loglik=220.38 AIC=-438.77 BIC=-433.57	0.56 (0.02)	18.23 (5.37)	Loglik=230.48 AIC=-456.96 BIC=-446.56	0.52 (0)	Loglik=140.25 AIC=-278.49 BIC=-273.29	2
FM-LEMB	0.37 (0)	LogLik: 90.73 AIC: -179.46 BIC: -174.26	0.41 (0.03)	13.79 (3.46)	LogLik: 110.98 AIC: -217.96 BIC: -207.55	0.28 (0)	LogLik: 50.76 AIC: -99.52 BIC: -94.31	2
FM-IGOV	0.05 (0)	LogLik: 1.58 AIC: -1.15 BIC: 4.05	0.04 (0.03)	20.43 (7.09)	LogLik: 7.77 AIC: -11.55 BIC: -1.14	0.04 (0)	LogLik: 3.25 AIC: -4.51 BIC: 0.69	2
IEMG-EFA	0.82 (0)	LogLik: 745.13 AIC: -1488.26 BIC: -1483.06	0.82 (0.01)	27.91 (14.06)	LogLik: 747.69 AIC: -1491.38 BIC: -1480.98	1.8 (0)	LogLik: 520.99 AIC: -1039.99 BIC: -1034.79	2
IEMG-LEMB	0.68 (0)	LogLik: 400.22 AIC: -798.43 BIC: -793.23	0.68 (0.01)	14.81 (5.21)	LogLik: 405.62 AIC: -807.24 BIC: -796.83	0.97 (0)	LogLik: 258.03 AIC: -514.06 BIC: -508.85	2
IEMG-IGOV	0.08 (0.01)	LogLik: 4.63 AIC: -7.26 BIC: -2.06	0.07 (0.01)	8.16 (1.53)	LogLik: 26.8 AIC: -49.6 BIC: -39.19	0.07 (0)	LogLik: 5.73 AIC: -9.46 BIC: -4.26	2
EFA-LEMB	0.57 (0)	LogLik: 255.29 AIC: -508.58 BIC: -503.38	0.59 (0.02)	12.34 (3)	LogLik: 270.34 AIC: -536.69 BIC: -526.28	0.61 (0)	LogLik: 149.55 AIC: -297.09 BIC: -291.89	2
EFA-IGOV	0.08 (0)	LogLik: 4.5 AIC: -7 BIC: -1.8	0.08 (0.03)	12.04 (3.12)	LogLik: 15.34 AIC: -26.67 BIC: -16.27	0.05 (0)	LogLik: 3.67 AIC: -5.35 BIC: -0.14	2
LEMB-IGOV	0.4 (0)	LogLik: 114.12 AIC: -226.23 BIC: -221.03	0.41 (0.02)	9.24 (2.11)	LogLik: 129.95 AIC: -255.91 BIC: -245.5	0.38 (0)	LogLik: 71.36 AIC: -140.73 BIC: -135.53	2

Table 11: Copula fitting by MLE results summary

Asset Pairs	Gumbel (4)		Frank (5)		Joe (6)		Fit
	Par	Test Stat.	Par	LogLik	Par	LogLik	
FM-IEMG	1.55 (0)	Loglik=198.4 AIC= -394.8 BIC= -389.6	4.32 (0.27)	Loglik=220.17 AIC=-438.34 BIC=-433.14	1.64 (0)	Loglik=124.47 AIC=-246.93 BIC=-241.73	2
FM-EFA	1.56 (0)	Loglik=193.33 AIC= -384.65 BIC=-379.45	4.51 (0.28)	Loglik=221 AIC=-439.99 BIC=-434.79	1.62 (0)	Loglik=109.22 AIC= -216.44 BIC=-211.24	2
FM-LEMB	1.34 (0)	LogLik: 89.66 AIC: -177.32 BIC: -172.12	3.02 (0.22)	LogLik: 106.64 AIC: -211.27 BIC: -206.07	1.37 (0)	LogLik: 54.17 AIC: -106.33 BIC: -101.13	2
FM-IGOV	1.03 (0)	LogLik: 2.05 AIC: -2.1 BIC: 3.1	0.2 (0.15)	LogLik: 0.5 AIC: 1 BIC: 6.2	1.03 (0)	LogLik: 1.49 AIC: -0.98 BIC: 4.22	2
IEMG-EFA	2.45 (0)	LogLik: 679.18 AIC: -1356.36 BIC: -1351.16	8.83 (0.71)	LogLik: 686.36 AIC: -1370.72 BIC: -1365.52	2.82 (0.05)	LogLik: 486.67 AIC: -971.33 BIC: -966.13	2
IEMG-LEMB	1.86 (0)	LogLik: 375.66 AIC: -749.32 BIC: -744.12	5.86 (0.43)	LogLik: 387.87 AIC: -773.74 BIC: -768.54	2.07 (0.03)	LogLik: 266.89 AIC: -531.78 BIC: -526.58	2
IEMG-IGOV	1.06 (0.01)	LogLik: 12.67 AIC: -23.34 BIC: -18.14	0.46 (0.16)	LogLik: 3 AIC: -4 BIC: 1.2	1.07 (0.01)	LogLik: 11.4 AIC: -20.8 BIC: -15.6	2
EFA-LEMB	1.64 (0)	LogLik: 250.57 AIC: -499.13 BIC: -493.93	4.77 (0.3)	LogLik: 259.89 AIC: -517.78 BIC: -512.58	1.8 (0.03)	LogLik: 176.4 AIC: -350.89 BIC: -345.68	2
EFA-IGOV	1.06 (0)	LogLik: 9.28 AIC: -16.55 BIC: -11.35	0.54 0.15	LogLik: 3.79 AIC: -5.58 BIC: -0.37	1.07 (0.01)	LogLik: 8.04 AIC: -14.08 BIC: -8.88	2
LEMB-IGOV	1.33 (0)	LogLik: 112.06 AIC: -222.11 BIC: -216.91	2.8 (0.22)	LogLik: 105.2 AIC: -208.39 BIC: -203.19	1.39 (0.01)	LogLik: 78.89 AIC: -155.78 BIC: -150.57	2

4.5 Bivariate exotic options valuation and sensitivity analysis

We construct 10 bivariate capped spread options and bivariate digital options written on pair combinations of the Equity Index Funds (ETFs) and Bond Index Funds (ETFs). The Equity Index funds considered are the iShares MSCI Frontier 100 ETF (FM), iShares Core MSCI Emerging Market ETF (IEMG) and iShares MSCI EAFE ETF (EFA). The Bond Index funds considered are the iShares JP Morgan Emerging Markets Local Currency Bond ETF (LEMB), iShares International Treasury Bond ETF (IGOV).We first obtain the prices through the constructed formulae and then analyse the prices sensitivity to changes in dependence parameter. We use the empirically best fitting copula which from the estimation results is the t-copula and its respective parameters in the copula in the analysis. The following values are use in the valuation of the exotic options risk free rate of 0% and assuming a ten-fold increase in standard deviation of 0.08900966, 0.1105, 0.09128075, 0.05889029 and 0.04911664 for FM, IEMG, EFA, LEMB and IGOV respectively. The current stock prices used are the following 35.22, 59.91, 71.34, 50.43, 51.08 for FM, IEMG, EFA, LEMB and IGOV respectively. The exercise price of the options are set to 50 for all exotic options. The digital option payoff is assumed to be 60.

Table 12 below shows the price, dependence delta and dependence gamma of capped and floor spread options of 10 pair combinations of equity and bond ETFs. Table 13 shows the price, dependence delta and dependence gamma of bivariate digital call and bivariate digital put of 10 pair combinations of equity and bond ETFs. To obtain the prices of the capped and floored spread options of Table 12 we use the pricing equation (25), estimating the copula function in R using the function *BiCopCDF* of the *VineCopula* library at the respective parameter values. We obtain the prices of bivariate digital call and put price in Table 13 using equation (27) and (28) and apply the function *BiCopCDF* also. The dependence delta and dependence gamma of equations (32) to (37) are obtained by *BiCopHfunc1*, *BiCopPDF*, *BiCopHfuncDeriv* of the *VineCopula* library in R as shown in the attached code in the Appendix Section.

Table 12: Bivariate Spread options prices, dependence delta and dependence gamma

		Spread Option		Spread Option	
Asset Pairs		Asset 1 Barrier (Cap)	Asset 1 Barrier (Floor)	Asset 2 Barrier (Cap)	Asset 2 Barrier (Floor)
FM-IEMG	Price	0	0	0	0
	Dep. Delta	0	0	0	0
	Dep. Gamma	0	0	0	0
FM-EFA	Price	0	0	0	0
	Dep. Delta	0	0	0	0
	Dep. Gamma	0	0	0	0
FM-LEMB	Price	0	0	0	0
	Dep. Delta	0	0	0	0
	Dep. Gamma	0	0	0	0
FM-IGOV	Price	0	0	0	0
	Dep. Delta	0	0	0	0
	Dep. Gamma	-0.00001	0	0	0
IEMG-EFA	Price	0	0.00004	0	0.00004
	Dep. Delta	0.00456	-0.00277	0	-0.00579
	Dep. Gamma	-0.05218	0.15131	-0.00003	0.66406
IEMG-LEMB	Price	-0.01993	9.49919	3.070707	6.4094
	Dep. Delta	-0.00001	-0.01009	-0.00111	0.00004
	Dep. Gamma	0.00023	-0.02915	0.00191	-0.00081
IEMG-IGOV	Price	-0.02282	8.898076	1.76339	7.11124
	Dep. Delta	-0.01073	-0.11762	-0.0006	-0.02822
	Dep. Gamma	0.01862	-1.93548	-0.04498	0.04642
EFA-LEMB	Price	0	20.91	7.76172	13.1483
	Dep. Delta	0	0	0	0
	Dep. Gamma	0	0	0	0
EFA-IGOV	Price	0	20.26	4.75232	15.50768
	Dep. Delta	0	0	0	0
	Dep. Gamma	0	0	0	0
LEMB-IGOV	Price	-0.09607	0.61928	-0.14739	0.60004
	Dep. Delta	0.58369	-0.45239	0.121074	-0.46997
	Dep. Gamma	1.54893	-0.0747	0.54599	-0.30761

Table 13: Bivariate digital options prices, dependence delta and dependence gamma

Asset Pairs		Digital Option	
		Call	Put
FM-IEMG	Price	0	0.68695
	Dep. Delta	-0.00005	0
	Dep. Gamma	0.000136	0
FM-EFA	Price	0	0
	Dep. Delta	0.1	0
	Dep. Gamma	0.0007	0
FM-LEMB	Price	0	25.60111
	Dep. Delta	0	0
	Dep. Gamma	0.00001	0
FM-IGOV	Price	0	16.4964
	Dep. Delta	0	0
	Dep. Gamma	0	0
IEMG-EFA	Price	59.31305	0
	Dep. Delta	-25.8089	0
	Dep. Gamma	-1070.18	0
IEMG-LEMB	Price	25.59205	0.66633
	Dep. Delta	0.17348	-3.70267
	Dep. Gamma	-0.09902	-0.05698
IEMG-IGOV	Price	16.29911	0.27514
	Dep. Delta	-0.28696	0.17313
	Dep. Gamma	1.61366	-0.97251
EFA-LEMB	Price	34.39889	0
	Dep. Delta	0	-0.00001
	Dep. Gamma	0	0.00008
EFA-IGOV	Price	43.50362	0
	Dep. Delta	0	0
	Dep. Gamma	0	0
LEMB-IGOV	Price	28.31505	10.41254
	Dep. Delta	-10.8334	4.06383
	Dep. Gamma	-13.6018	5.03736

4.6 Discussion of Results

The equity ETF EFA has the highest average return while the lowest average returns are from the bond ETF IGOV. Both bond ETFs LEMB and IGOV have negative average returns. The bond ETF IGOV has the lowest standard deviation and the equity ETF IEMG has the highest standard deviation. Generally the equity ETFs have higher standard deviation than the bond ETFs. Indicating that the equity ETFs are more risky than the bond ETFs and that the benchmark indexes of the equity ETFs are as well more variable than those of the bond ETFs. These results are show in *Table 4* and *Table 5*.

We find that the Equity and Bond ETFs prices are not normally distributed evidenced by non-normal histogram plots and QQ plots. This is confirmed by the Shapiro-Wilk test which finds that the p-values are below the significance level of 0.05, thus rejecting the null hypothesis of normal distribution. The log returns are also not normally distributed evidenced by peaked histograms and tailed QQ plots. These results are show in the pair plots of *Figure 9*, *Figure 11* and *Table 8* which displays results from the normality test.

We find that there exists positive correlation between all the equity and bond ETFs, the strongest linear correlation is between IEMG and EFA equity ETFs in prices and returns of 0.76 and 0.82 respectively. The rank transformation performed on the ETFs prices and the probability transformation performed on the log-returns do not affect the correlation structure. From the pair copula plots, however, we see that the dependence between the ETFs log-returns is not strictly a left tailed or a right tailed one, given different empirical copula plots of *Figure 12*. From the copula fitting results of the transformed log returns data we find that the t-copula is the best fitting copula for all the pair combinations of the equity and bond ETFs. Thus relying on the assumption of gaussian copula for pricing the bivariate exotic options would yield inaccurate prices for the options.

In pricing the exotic options we assume that the empirical dependence estimated by the t-copula remains static for the rest of the valuation period. All spread options written on combinations of equity ETF FM have a value of zero. The likelihood of FM being greater than IEMG, EFA, IGOV and LEMB is zero. Therefore there would be no use in writing such an option. This is also the case with the bivariate digital call written on combinations of FM, indicating that the likelihood of the FM price to be above the strike price at maturity is zero. And buying such an option would not be meaningful.

All options written on EFA combinations, except bivariate put and EFA-capped spread, are tradable displaying positive option values. However the options written on EFA combinations are insensitive to changes in the t-copula parameter. The dependence gamma as well is inelastic.

Generally from the analysis we find that the dependence delta is negative, indicating increase in correlation/dependence will generally lead to a decrease in the bivariate exotic options prices. Pointing towards high downside correlation experienced in financial asset returns, and in this case with use of ETF data, explains systemic effects and cross asset market downside dependence.

Emerging and Developed market equity ETFs IEMG and EFA underlying combinations for bivariate digital call options are the most sensitive (-26.8) to changes in t-copula parameter. Indicating these markets have higher downside correlation and an option written on the two would lose value with an increase in correlation. Similarly, the Emerging and Developed market bond ETFs, LEMB and IGOV are also correlated on the downside and an increase in the correlation would reduce the value of the bivariate digital call written on these options with a sensitivity of -10.8. Generally the dependence gamma follows the sign of the dependence delta in most of the pair underlying combinations, and increase in correlation results in a decrease in the dependence delta. We could therefore expect that the graph of the bivariate option prices against the copula parameter to be a concave downward function.

For Emerging market equity ETF and Developed market and bond ETFs, IEMG-IGOV, we find that the dependence delta is much lower on aggregate (but still negative) than that of the other emerging and developed market equity ETFs, and emerging and developed market bond ETFs. This shows that the bivariate exotic options written on emerging market equity ETF and developed market bond ETF are less sensitive to changes in the copula parameter and exhibit lower downside dependence. These bivariate exotic options written on emerging and developed market ETFs would be best for correlation/dependence hedging.

We find that the constructed capped and floored bivariate spread options correspond to gap options. As described in McDonald (2006), a gap option has a strike price, K_1, and a trigger price, K_2. The trigger price determines whether or not the gap option will have a nonzero payoff. The strike price determines the amount of the nonzero payoff. The strike price may be greater than or less than the trigger price. If the strike price is equal to the trigger price, then the gap option is an ordinary option. A gap call option has a nonzero payoff which may be positive or negative if the final stock price exceeds the trigger price. If we graph the payoff of a gap call option as a function of its final stock price, then we have that there is a gap where S_T=K_2. There are no negative payoffs because the trigger price is greater than the strike price. But if trigger price is less than the strike price for a gap call option, then negative payoffs are possible. For gap options with a given strike increasing the payment trigger reduces the premium. The reason is that when the payment trigger is above the strike, the option holder will have to make a payment to the option writer in some cases. The negative values obtains for the bivariate spread options correspond to gap options and reflect the possibility that the option buyer will end up making a payment at maturity to the option seller.

Chapter Five: Conclusion

The objective of this thesis was to construct and value bivariate exotic options (bivariate capped/floored spreads and bivariate digital options) constructed on 10 pairs of Frontier, Emerging and Developed market equity and bond ETFs. We sought also to analyze the options price sensitivity to changes in the copula parameter of the fitted copula, its correlation/dependence delta. We also analyze the dependence gamma which is the sensitivity of the options dependence delta to changes in the fitted copula parameter.

The findings reveal that equity ETFs had higher returns than bond ETFs during the study period. However the equity ETFs are more risky since they had higher standard deviations compared to the bond ETFs. The log-returns of both the equity ETFs and bond ETFs were found to be normally distributed. High and strong correlations was found in emerging and developed markets equity ETFs than emerging and developed markets bond ETFs, generally all correlation between the ETFs was positive.

When valuing the constructed bivariate cap/floor spread option and the bivariate digital option we find that the t-copula captures best the dependence between the 10 pair combinations of the underlyings. We price using this empirical t-copula. All spread options written on combinations of equity ETF FM have a value of zero. The likelihood of FM being greater than IEMG, EFA, IGOV and LEMB is zero.

We find that the dependence delta is negative, indicating increase in correlation/dependence will generally lead to a decrease in the bivariate exotic options prices. Emerging and Developed market equity ETFs IEMG and EFA underlying combinations for bivariate digital call options is the most sensitive to changes in copula parameter. This shows that emerging market and developed market equity ETFs are more prone to downside dependence and a change in correlation would result in a huge loss of option values. Similarly, the emerging market and developed market bond ETFs have negative but lower dependence delta compared to the developed and emerging market equity ETFs. The emerging market equity ETF IEMG and developed market bond ETF IGOV have much lower dependence delta than the emerging and developed market equity and bond ETFs mentioned above. IEMG-IGOV pair combination is less sensitive to changes in the copula parameter and exhibit lower downside dependence. These bivariate exotic options written on emerging equity and developed market bond ETFs would be best for correlation/dependence hedging.

The dependence gamma is found on aggregate to also be negative. We conclude therefore that the bivariate exotic option price and copula parameter graph would be a concave downward function. This may however be an inelastic function as exhibited by the options written on EFA combinations which are insensitive to changes in the t-copula parameter. The dependence gamma as well is inelastic.

Negative values of constructed spread options are obtained; this indicates that the spread option construction is similar to that of a gap option, which has a different trigger condition other than the strike price. And with the negative option premium values reflecting the possibility that the option buyer will end up making a payment at maturity to the option seller.

In conclusion this study finds that changes in copula parameter significantly affect the price of the bivariate exotic options depending on the strength of the dependence of the underlyings. With high dependence the bivariate exotic option will be more sensitive to changes in the copula parameter. Option with this kind of high sensitivity are ideal for weak or negative dependence speculation. Where the dependence is weak we find that the bivariate exotic option prices are insensitive to change to changes in the dependence structure. The direction of the dependence delta is found to be informed by nature of the dependence. The weakly dependent asset combinations would be ideal for dependence hedging. Generally the dependence gamma follows the sign and strength of the dependence delta. Assumption of a Gaussian copula would have resulted in inaccurate prices.

This study could be extended to incorporate a mean reverting OU process for the stock price model and a dynamic copula with non-static copula parameter for example as that of Zhang and Guegan(2008). Different results and much accurate prices are expected if the assumption of geometric Brownian motion and static copula parameter are relaxed. Also further research in this area would be to address the problem of dependency under non-normality, where the log-returns process is assumed to be non-normal, thus working with non-gaussian marginal probability distributions and using pair copulae techniques to price bivariate or multivariate exotic derivatives. It could also be extended to incorporate an extra payoff condition say for the spread options for asset 1 high barrier to have an extra condition for asset 2 low barrier, then for the three payoff conditions trivariate copula to be used to assess the dependence.

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Appendix

I. One parameter copula families

(a) Gaussian copula

The Gaussian copula belongs to the family of elliptical copulas. Elliptical copulas are simply the copulas of elliptically contoured distributions. An advantage of elliptical copula is that one can specify different levels of correlation between the marginals.

The Gaussian copula is defined by

$$C_\rho\left(u,v\right)=\phi_\rho(\phi_X^{-1}\left(u\right),\phi_Y^{-1}(v)) \tag{30}$$

Where $\phi_{X,Y}(\cdot,\cdot,\rho)$ is the joint distribution of two standard normal distributed random variables with correlation $\rho\in(-1,1)$, ϕ is the standard normal distribution and ϕ^{-1} is the quantile function. The density of the bivariate Gaussian copula is

$$c\left(u,v\right)=\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{\rho^2\left(x^2+y^2\right)-2\rho xy}{2(1-\rho^2)}\right\} \tag{31}$$

Where $x=\phi_X^{-1}\left(u\right)$ and $y=\phi_Y^{-1}\left(v\right)$. For the Gaussian copula the h-function is as follows:

$$h\left(u,v,\rho\right)=\frac{\partial C(u,v,\rho)}{\partial v}=\phi_{X,Y}\left(\frac{\phi_X^{-1}\left(u\right)-\rho\phi_Y^{-1}\left(v\right)}{\sqrt{1-\rho^2}}\right) \tag{32}$$

(b) T − copula

The t-copula is also an elliptical copula. Unlike the Gaussian copula the t-copula has a second parameter, the degrees of freedom denoted by $\nu>0$. The density of the bivariate t-copula with parameters $\rho\in(-1,1)$ is given by

$$c\left(u_1,u_2;\rho,v\right)=\frac{1}{2\pi\sqrt{1-\rho^2}}\frac{1}{dt\left(x,v\right)dt(y,v)}\left(1+\frac{x^2+y^2-2\rho xy}{v(1-\rho^2)}\right)^{-\frac{v+2}{2}} \tag{33}$$

Where

$$dt\,(x,v)=\frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\ \left(\frac{v}{2}\right)\sqrt{\pi v}}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}$$

is the density of the univariate t-distribution with ν degrees of freedom and $\Gamma()$ is the gamma function. Here $x=t_v^{-1}\,(u)\,,\quad u\in(0,1)$ with $t_v^{-1}()$being the quantile function of the univariate t-distribution with ν degrees of freedom. The h-function corresponding to the t-copula as described in Schepsmeier and Stober(2014) is

$$h\,(u_1,u_2;\rho,v)=t_{v+1}\left(\frac{t_v^{-1}\,(u_1)-\rho t_v^{-1}\,(u_2)}{\sqrt{\frac{(v+\left(t_v^{-1}(u_2)\right)^2)(1-\rho^2)}{v+1}}}\right)=t_{v+1}\left(\frac{x_1-\rho x_2}{\sqrt{\frac{(v+x_2^2)(1-\rho^2)}{v+1}}}\right)\quad(34)$$

(c) **Frank copula**

The Frank copula is defined as

$$C_\theta\,(u,v)=-\frac{1}{\theta}\ln\left\{1-\frac{(1-e^{-\theta u})(1-e^{-\theta v})}{1-e^{-\theta}}\right\}\quad(35)$$

The density function for the Frank copula is

$$c\,(u,v;\theta)=\theta\left(1-e^{-\theta}\right)e^{-\theta(u+v)}\left[\left(1-e^{-\theta}\right)-\left(1-e^{-\theta u}\right)\left(1-e^{-\theta v}\right)\right]^{-2}\quad(36)$$

And the h-function (conditional CDF) as in in Schepsmeier and Stober(2014)is

$$h\,(u,v;\theta)=-\frac{e^{\theta}(e^{\theta u}-1)}{e^{\theta v+\theta u}-e^{\theta v+\theta}-e^{\theta u+\theta}+e^{\theta}}\quad(37)$$

(d) **Clayton copula**

The clayton copula is given by

$$C_\theta\,(u,v)=\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}=A(u,v,\theta)^{-1/\theta}\quad(38)$$

Where

$$A(u,v,\theta) = (u^{-\theta} + v^{-\theta} - 1)$$

And its corresponding density

$$c\,(u,v;\theta) = \frac{(1+\theta)(uv)^{-1-\theta}}{A(u,v,\theta)^{\frac{1}{\theta}+2}} \tag{39}$$

where $0 < \theta < \infty$ controls the degree of dependence. If $\theta \to \infty$ the Clayton copula converges to the monotonicity copula with perfect positive dependence, $\theta = 0$ corresponds to independence.

The h-function of the clayton copula can be calculated as

$$h\,(u,v,\theta) = v^{-\theta-1} A(u,v,\theta)^{-1-\ \frac{1}{\theta}} \tag{40}$$

The Clayton copula is an asymmetric Archimedean copula and exhibits greater dependence in the negative tail than in the positive. It is suitable for describing dependencies in the left tail.

(e) **Gumbel copula**

The Gumbel copula is defined as

$$C_\theta\,(u,v) = \exp\left\{-(-\log\ u)^\theta + (-\log\ v^\theta)^{\frac{1}{\theta}}\right\} = \exp\left[-(t_u + t_v)^{\frac{1}{\theta}}\right] \tag{41}$$

Where $t_u = (-\log\ u)^\theta$ and $t_v = (-\log\ v)^\theta$, $\theta \geq 1$ is the parameter of dependence. For $\theta \to \infty$ the Gumbel copula converges to the comonotonic copula with perfect dependence, in contrast $\theta = 1$ means independence. The h-function is given as the first derivative of C with respect to v is

$$h\,(u,v;\theta) = -\frac{e^{-(t_u+t_u)^{\frac{1}{\theta}}}(t_u + t_v)^{\frac{1}{\theta}-1} t_v}{v\ln(v)} \tag{42}$$

And its density

$$c\,(u,v;\theta) = C(u,v;\theta)\frac{1}{uv}(t_u+t_v)^{-2+\frac{2}{\theta}}(\ln\ u\cdot\ln\ v)^{\theta-1}\left\{1+(\theta-1)((t_u+t_v)^{-\frac{1}{\theta}}\right\} \tag{43}$$

Gumbel copulas are often used to model extreme distributions. They are asymmetric Archimedean copula, exhibiting greater dependence in the positive than in the negative tail.

II. Theorem 1.1: Girsanov Theorem

Two probability measures $\mathbb{F}$ and $\mathbb{Q}$ on *(Ω, F_T)* are said to be *equivalent* if, for any event $A \in F_T$, the equality $\mathbb{F}\,(A) = 0$ holds if and only if $\mathbb{Q}\,(A) = 0$. In other words, $\mathbb{F}$ and $\mathbb{Q}$ are equivalent on *(Ω, F_T)* if they have the same set of null events in the σ -field F_T . Given *a one-dimensional standard Brownian motion on a probability space (Ω, F_T,$\mathbb{Q}$). We define the process X by setting the wiener process of the geometric brownian motion to $W^{\mathbb{F}} = W^{\mathbb{Q}} + \lambda T$ for $t \in [0,T\,]$. Let the probability measure $\mathbb{F}$ be equivalent to $\mathbb{Q}$ on (Ω, F_T), be defined through the formula*

$$\frac{d\mathbb{F}}{d\mathbb{Q}} = \exp\left(-\lambda {W_T}^{\mathbb{Q}} - \frac{1}{2}\lambda^2 T\right) = \eta_T$$

This is the *Radon-Nikodým density* of $\mathbb{F}$ with respect to $\mathbb{Q}$ is defined as the unique F_T - measurable random variable η_T such that we have, for any event $A \in F_T$.
By virtue of the above proposition, setting $W^{\mathbb{F}} = W^{\mathbb{Q}} - \sigma(T-t)$, the *Radon-Nikodým density is:*

$$\frac{d\mathbb{F}}{d\mathbb{Q}} = \exp\left(\sigma {W_T}^{\mathbb{Q}} - \frac{1}{2}\sigma^2(T-t)\right) = \eta_T$$

III. Multidimensional Girsanov Theorem Application

Assuming that the stochastic differential equations for stock 1 and stock 2 follow a geometric brownian motion and are given under real-world probability measure $\mathbb{P}$ as:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dw_1^{\mathbb{P}}$$
$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dw_1^{\mathbb{P}}$$

where $\mathbb{E}(dw_1 dw_2) = \rho_{1,2}dt$. By Ito's lemma, we have:

$$dY = (\mu_1 - \mu_2 + {\sigma_2}^2 - \rho_{1,2}\sigma_1\sigma_2)Ydt + \sigma_1 Y dw_1^{\mathbb{P}} - \sigma_2 Y dw_2^{\mathbb{P}}$$
$$dY = \mu_3 Y dt + \Gamma Y \ d\hat{w}^{\mathbb{P}}$$

Where $\mu_3 = \mu_1 - \mu_2 + {\sigma_2}^2 - \rho_{1,2}\sigma_1\sigma_2$ and $d\hat{w}$ is a standard brownian motion under $\mathbb{P}$ with $d\hat{w}^{\mathbb{P}} \sim N(0, dt)$ and Γ is given as $\Gamma = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 - 2\rho_{1,2}\sigma_1\sigma_2}$.
Using the change of measure by girsanov theorem , we have that under the risk neutral measure $\mathbb{Q}$ the stochastic differential equations above become

$$dS_1 = rS_1 dt + \sigma_1 S_1 dw^{\mathbb{Q}} \tag{44}$$
$$dS_2 = rS_2 dt + \sigma_2 S_2 dw^{\mathbb{Q}} \tag{45}$$

By multidimensional Girsanov theorem, under measure $\mathbb{P}$. We can write the Brownian process $dw_1^{\mathbb{P}}$ in terms of independent brownian motions $dw_2^{\mathbb{P}}$ and dv which remains a brownian motion under $\mathbb{P}$ and $\mathbb{Q}$ independent of $dw_2^{\mathbb{P}}$ and $dw_2^{\mathbb{Q}}$ under both measures. Hence $dw_1^{\mathbb{P}}$ is defined by

$$dw_1^{\mathbb{P}} = \rho_{1,2}dw_2^{\mathbb{P}} + \sqrt{1 - \rho_{1,2}^2}dv$$

We therefore have that

$$dY = ({\sigma_2}^2 - \rho_{1,2}\sigma_1\sigma_2)Ydt + \sigma_1 Y dw_1^{\mathbb{P}} - \sigma_2 Y dw_2^{\mathbb{P}}$$
$$dY = ({\sigma_2}^2 - \rho_{1,2}\sigma_1\sigma_2)Ydt + \sigma_1 Y(\rho_{1,2}dw_2^{\mathbb{P}} + \sqrt{1 - \rho_{1,2}^2}dv) - \sigma_2 Y dw_2^{\mathbb{P}}$$
$$dY = ({\sigma_2}^2 - \rho_{1,2}\sigma_1\sigma_2)Ydt + (\sigma_1\rho_{1,2} - \sigma_2)Ydw_2^{\mathbb{P}} + \sigma_1 Y\sqrt{1 - \rho_{1,2}^2}dv$$

By girsanov theorem setting $dw_2^{\mathbb{Q}} = dw_2^{\mathbb{P}} - \sigma_2 dt$. We have that

$$dY = (\sigma_1\rho_{1,2} - \sigma_2)Ydw_2^{\mathbb{Q}} + \sigma_1 Y\sqrt{1 - \rho_{1,2}^2}dv$$
$$dY = \sigma_1 Y(\rho_{1,2}dw_2^{\mathbb{Q}} + \sqrt{1 - \rho_{1,2}^2}dv) - \sigma_2 Y dw_2^{\mathbb{Q}}$$
$$dY = Y(\sigma_1 dw_1^{\mathbb{Q}} - \sigma_2 dw_2^{\mathbb{Q}})$$

$$dY = \Gamma Y d\hat{w}^{\mathbb{Q}} \tag{46}$$

Where $d\hat{w}^{\mathbb{Q}}$ is a standard brownian motion under $\mathbb{Q}$ with $d\hat{w}^{\mathbb{Q}} \sim N(0, dt)$ and Γ is given as $\Gamma = \sqrt{\sigma_1{}^2 + \sigma_2{}^2 - 2\rho_{1,2}\sigma_1\sigma_2}$.

IV. Proof of Proposition 1.1

We apply chain rule decomposition to obtain the pair copula first derivative with respect to the dependence parameter as shown below.

$$\begin{aligned}\frac{\partial C_\theta(u,v)}{\partial \theta} &= \frac{\partial C_\theta(u,v)}{\partial h_v} \times \frac{\partial h_v}{\partial \theta} \\ &= \frac{\partial C(u,v)/\partial u}{\partial h_v/\partial u} \times \frac{\partial h_v}{\partial \theta} \\ &= \frac{h_u}{c(u,v)} \times \frac{\partial h_v}{\partial \theta} \\ &= \frac{h_u h_{v,\theta}}{c(u,v)}\end{aligned}$$

V. Proof of Proposition 1.2

We apply Faa di Bruno formula of chain rule decomposition in higher derivatives to obtain the pair copula second derivative with respect to the dependence parameter as shown below.

$$\frac{\partial^2 C_\theta(u,v)}{\partial \theta^2} = \frac{\partial C(u,v)}{\partial h_v} \times \frac{\partial^2 h_v}{\partial \theta^2} + \left(\frac{\partial h_v}{\partial \theta}\right)^2 \times \frac{\partial^2 C(u,v)}{\partial h_v^2}$$

where from proposition 1.1,

$$\frac{\partial C_\theta(u,v)}{\partial u} = \frac{\partial C_\theta(u,v)}{\partial h_v} \times \frac{\partial h_v}{\partial u}$$

and chain rule second derivative we have,

$$\frac{\partial^2 C_\theta(u,v)}{\partial u^2} = \frac{\partial C(u,v)}{\partial h_v} \times \frac{\partial^2 h_v}{\partial u^2} + \left(\frac{\partial h_v}{\partial u}\right)^2 \times \frac{\partial^2 C(u,v)}{\partial h_v^2} \tag{47}$$

and from equation (47) we have that

$$\frac{\partial^2 C_\theta(u,v)}{\partial h_v^2} = \frac{\frac{\partial^2 C_\theta(u,v)}{\partial u^2} - \frac{\partial C(u,v)}{\partial h_v} \times \frac{\partial^2 h_v}{\partial u^2}}{\left(\frac{\partial h_v}{\partial u}\right)^2}$$
$$= \frac{\frac{\partial h_u}{\partial u} - \frac{h_u}{c(u,v)} \times \frac{\partial c(u,v)}{\partial u}}{c(u,v)^2}$$
$$= \frac{h_{u,u} - \frac{h_u}{c(u,v)} \frac{\partial c(u,v)}{\partial u}}{c(u,v)^2}$$

We thus have that the pair copula second derivative with respect to the dependence parameter becomes

$$\frac{\partial^2 C_\theta\left(u,v\right)}{\partial \theta^2} = \frac{h_u h_{v,\theta,\theta}}{c(u,v)} + \left(h_{v,\theta}\right)^2 \left(\frac{h_{u,u} - \frac{h_u}{c(u,v)} \frac{\partial c(u,v)}{\partial u}}{c(u,v)^2}\right)$$

where h_u is the h-function with respect to u, $h_{u,u} = \frac{\partial h_u}{\partial u}$, $h_{v,\theta} = \frac{\partial h_v}{\partial \theta}$, $h_{v,\theta,\theta} = \frac{\partial^2 h_v}{\partial \theta^2}$ and $c(u,v)$ is the copula density.

VI. R Codes

(a) **Plots**

```
par(mfrow=c(3,1))
tsFMprice<- ts(FM, frequency = 264, start = 2012)
tsIEMGprice<- ts(IEMG, frequency = 264, start = 2012)
tsEFAprice<- ts(EFA, frequency = 264, start = 2012)
tsLEMBprice<- ts(LEMB, frequency = 264, start = 2012)
tsIGOVprice<- ts(IGOV, frequency = 264, start = 2012)
#Daily Price Plots
par(mfrow=c(3,1))
plot(tsFMprice, main="Daily Prices for Frontier Market Equity ETF", xlab="Years",
ylab="Price", type="l")
plot(tsIEMGprice,main="Daily Prices for Emerging Market Equity ETF", xlab="Years",
ylab="Price", type="l" )
plot(tsEFAprice, main="Daily Prices for Developed Market Equity ETF", xlab="Years",
ylab="Price", type="l")
par(mfrow=c(3,1))
plot(tsLEMBprice, main="Daily Prices for Emerging Market Bond ETF", xlab="Years",
ylab="Price", type="l")
plot(tsIGOVprice, main="Daily Prices for Developed Market Bond ETF", xlab="Years",
ylab="Price", type="l")
par(mfrow=c(3,1))
tsFMlogreturns<- ts(FMlogreturns, frequency = 264, start = 2012)
tsIEMGlogreturns<- ts(IEMGlogreturns, frequency = 264, start = 2012)
tsEFAlogreturns<- ts(EFAlogreturns, frequency = 264, start = 2012)
tsLEMBlogreturns<- ts(LEMBlogreturns, frequency = 264, start = 2012)
tsIGOVlogreturns<- ts(IGOVlogreturns, frequency = 264, start = 2012)
```

#Daily logreturns plot

par(mfrow=c(3,1))

plot(tsFMlogreturns, main="Daily Log Returns for Frontier Market Equity ETF", xlab="Years",

ylab="Log returns", type="l")

plot(tsIEMGlogreturns,main="Daily Log Returns for Emerging Market Equity ETF", xlab="Years",

ylab="Log returns", type="l")

plot(tsEFAlogreturns, main="Daily Log Returns for Developed Market Equity ETF", xlab="Years",

ylab="Log returns", type="l")

par(mfrow=c(3,1))

plot(tsLEMBlogreturns, main="Daily Log Returns for Emerging Market Bond ETF", xlab="Years",

ylab="Log returns", type="l")

plot(tsIGOVlogreturns, main="Daily Log Returns for Developed Market Bond ETF", xlab="Years",

ylab="Log returns", type="l")

par(mfrow=c(3,1))

tsFMgrossreturns<- ts(FMgrossreturns, frequency = 264, start = 2012)

tsIEMGgrossreturns<- ts(IEMGgrossreturns, frequency = 264, start = 2012)

tsEFAgrossreturns<- ts(EFAgrossreturns, frequency = 264, start = 2012)

tsLEMBgrossreturns<- ts(LEMBgrossreturns, frequency = 264, start = 2012)

tsIGOVgrossreturns<- ts(IGOVgrossreturns, frequency = 264, start = 2012)

#Multivariate plot

seriesprices<-cbind(tsFMprice, tsIEMGprice, tsEFAprice, tsLEMBprice, tsIGOVprice)

ts.plot(seriesprices, xlab="Years", ylab="Price",type="l",ylim=c(20,90),col=1:5, main="Prices of Equity and Bond ETFs")

legend(2012,90,c("FM", "IEMG", "EFA", "LEMB", "IGOV"),lty=1, col=1:5)

(b) **Pair Plots**

```
seriespricesdata<-cbind(FM,IEMG,EFA,LEMB,IGOV)
seriespricesdata
library(psych)
cor(seriespricesdata)
pairs.panels(seriespricesdata)
library(clusterSim)
FMunipobs=pobs(FM)
IEMGunipobs=pobs(IEMG)
EFAunipobs=pobs(EFA)
LEMBunipobs= pobs(LEMB)
IGOVunipobs=pobs(IGOV)
u_seriespricespobsdata<-cbind(FMunipobs,IEMGunipobs,EFAunipobs,LEMBunipobs,
IGOVunipobs)
u_seriespricespobsdata
colnames(u_seriespricespobsdata)=c("FM","IEMG","EFA","LEMB","IGOV")
library(psych)
cor(u_seriespricespobsdata)
pairs.panels(u_seriespricespobsdata)
serieslogreturnsdata<-cbind(FMlogreturns,IEMGlogreturns,EFAlogreturns,
LEMBlogreturns,IGOVlogreturns)
serieslogreturnsdata
cor(serieslogreturnsdata)
library(psych)
cor(serieslogreturnsdata)
pairs.panels(serieslogreturnsdata)
library(clusterSim)
```

```
FMlogreturnsunipobs=pobs(FMlogreturns)
IEMGlogreturnsunipobs=pobs(IEMGlogreturns)
EFAlogreturnsunipobs=pobs(EFAlogreturns)
LEMBlogreturnsunipobs= pobs(LEMBlogreturns)
IGOVlogreturnsunipobs=pobs(IGOVlogreturns)
u_seriespricespobsdata<-cbind(FMlogreturnsunipobs,IEMGlogreturnsunipobs,
EFAlogreturnsunipobs,LEMBlogreturnsunipobs,
IGOVlogreturnsunipobs)
u_serieslogreturnspobsdata
colnames(u_seriespricespobsdata)=c(”FMlogreturns”,”IEMGlogreturns”,
”EFAlogreturns”,”LEMB”,”IGOVlogreturns”)
library(psych)
cor(u_serieslogreturnspobsdata)
pairs.panels(u_serieslogreturnspobsdata)
```

(c) **Normality Tests**

```
shapiro.test(FM)
shapiro.test(IEMG)
shapiro.test(EFA)
shapiro.test(LEMB)
shapiro.test(IGOV)
library(car)
par(mfrow=c(2,3))
qqPlot(FM)
qqPlot(IEMG)
qqPlot(EFA)
qqPlot(LEMB)
qqPlot(IGOV)
```

shapiro.test(FMlogreturns)

shapiro.test(IEMGlogreturns)

shapiro.test(EFAlogreturns)

shapiro.test(LEMBlogreturns)

shapiro.test(IGOVlogreturns)

par(mfrow=c(2,3))

qqPlot(FMlogreturns, ylab="FM Log Returns")

qqPlot(IEMGlogreturns, ylab="IEMG Log Returns")

qqPlot(EFAlogreturns, ylab="EFA Log Returns")

qqPlot(LEMBlogreturns, ylab="LEMB Log Returns")

qqPlot(IGOVlogreturns, ylab="IGOV Log Returns")

(d) **Pair copula selection and estimations (extract for FM-IEMG, IEMG-LEMB and LEMB-IGOV)**

FM_IEMG=BiCopSelect(u_FMlogreturns, u_IEMGlogreturns, familyset = NA, selectioncrit = "AIC",

indeptest = FALSE, level = 0.05, weights = NA, rotations = FALSE,

se = TRUE, presel = TRUE, method = "mle")

summary(FM_IEMG)

est_FM_IEMG_1=BiCopEst(u_FMlogreturns, u_IEMGlogreturns, family=1, method = "mle", se = TRUE)

summary(est_FM_IEMG_1)

est_FM_IEMG_2=BiCopEst(u_FMlogreturns, u_IEMGlogreturns, family=2, method = "mle", se = TRUE)

summary(est_FM_IEMG_2)

est_FM_IEMG_3=BiCopEst(u_FMlogreturns, u_IEMGlogreturns, family=3, method = "mle", se = TRUE)

summary(est_FM_IEMG_3)

est_FM_IEMG_4=BiCopEst(u_FMlogreturns, u_IEMGlogreturns, family=4, method = "mle", se = TRUE) summary(est_FM_IEMG_4)

est_FM_IEMG_5=BiCopEst(u_FMlogreturns, u_IEMGlogreturns, family=5, method = ”mle”, se = TRUE)

summary(est_FM_IEMG_5)

est_FM_IEMG_6=BiCopEst(u_FMlogreturns, u_IEMGlogreturns, family=6, method = ”mle”, se = TRUE)

summary(est_FM_IEMG_6)

IEMG_LEMB=BiCopSelect(u_IEMGlogreturns, u_LEMBlogreturns, familyset = NA, selectioncrit = ”AIC”,

indeptest = FALSE, level = 0.05, weights = NA, rotations = FALSE,

se = TRUE, presel = TRUE, method = ”mle”)

summary(IEMG_LEMB)

est_IEMG_LEMB_1=BiCopEst(u_IEMGlogreturns, u_LEMBlogreturns, family=1, method = ”mle”, se = TRUE)

summary(est_IEMG_LEMB_1)

est_IEMG_LEMB_2=BiCopEst(u_IEMGlogreturns, u_LEMBlogreturns, family=2, method = ”mle”, se = TRUE)

summary(est_IEMG_LEMB_2)

est_IEMG_LEMB_3=BiCopEst(u_IEMGlogreturns, u_LEMBlogreturns, family=3, method = ”mle”, se = TRUE)

summary(est_IEMG_LEMB_3)

est_IEMG_LEMB_4=BiCopEst(u_IEMGlogreturns, u_LEMBlogreturns, family=4, method = ”mle”, se = TRUE)

summary(est_IEMG_LEMB_4)

est_IEMG_LEMB_5=BiCopEst(u_IEMGlogreturns, u_LEMBlogreturns, family=5, method = ”mle”, se = TRUE)

summary(est_IEMG_LEMB_5)

est_IEMG_LEMB_6=BiCopEst(u_IEMGlogreturns, u_LEMBlogreturns, family=6, method = ”mle”, se = TRUE) summary(est_IEMG_LEMB_6)

LEMB_IGOV=BiCopSelect(u_LEMBlogreturns, u_IGOVlogreturns, familyset = NA, selectioncrit = ”AIC”,

indeptest = FALSE, level = 0.05, weights = NA, rotations = FALSE,

se = TRUE, presel = TRUE, method = ”mle”)

summary(LEMB_IGOV)

est_LEMB_IGOV_1=BiCopEst(u_LEMBlogreturns, u_IGOVlogreturns, family=1, method = ”mle”, se = TRUE)

summary(est_LEMB_IGOV_1)

est_LEMB_IGOV_2=BiCopEst(u_LEMBlogreturns, u_IGOVlogreturns, family=2, method = ”mle”, se = TRUE)

summary(est_LEMB_IGOV_2)

est_LEMB_IGOV_3=BiCopEst(u_LEMBlogreturns, u_IGOVlogreturns, family=3, method = ”mle”, se = TRUE)

summary(est_LEMB_IGOV_3)

est_LEMB_IGOV_4=BiCopEst(u_LEMBlogreturns, u_IGOVlogreturns, family=4, method = ”mle”, se = TRUE)

summary(est_LEMB_IGOV_4)

est_LEMB_IGOV_5=BiCopEst(u_LEMBlogreturns, u_IGOVlogreturns, family=5, method = ”mle”, se = TRUE)

summary(est_LEMB_IGOV_5)

est_LEMB_IGOV_6=BiCopEst(u_LEMBlogreturns, u_IGOVlogreturns, family=6, method = ”mle”, se = TRUE)

summary(est_LEMB_IGOV_6)

(e) **Exotic Options Pricing and Sensitivity Analysis (extract of LEMB-IGOV)**

```
S_FM=35.22
S_IEMG=59.91
S_EFA=71.34
S_LEMB=50.43
S_IGOV=51.08
sd(FMlogreturns)
sd(IEMGlogreturns)
sd(EFAlogreturns)
sd(LEMBlogreturns)
sd(IGOVlogreturns)
sd_FM=sd(FMlogreturns)*10
sd_IEMG=sd(IEMGlogreturns)*10
sd_EFA=sd(EFAlogreturns)*10
sd_LEMB=sd(LEMBlogreturns)*10
sd_IGOV=sd(IGOVlogreturns)*10
var_FM=sd_FM∧2
var_IEMG=sd_IEMG∧2
var_EFA=sd_EFA∧2
var_LEMB=sd_LEMB∧2
var_IGOV=sd_IGOV∧2
T=0.5
k_1=50
k_2=50
#FM-IEMG
omega_FM_IEMG=sqrt(var_FM + var_IEMG - 2*cor(FMlogreturns, IEMGlogreturns)*sd_FM*sd_IEMG)
e1_FM_IEMG=(log(S_FM/S_IEMG)+ (0.5*omega_FM_IEMG∧2)*T)/
(omega_FM_IEMG*sqrt(T))
e1_FM_IEMG
u_e1_FM_IEMG=pnorm(e1_FM_IEMG)
u_e1_FM_IEMG
e2_FM_IEMG=e1_FM_IEMG- (omega_FM_IEMG*sqrt(T))
u_e2_FM_IEMG=pnorm(e2_FM_IEMG)
u_e2_FM_IEMG
```

```
#IEMG-LEMB
omega_IEMG_LEMB=sqrt(var_IEMG + var_LEMB - 2*cor(IEMGlogreturns, LEMBlo-
greturns)*sd_IEMG*sd_LEMB)
e1_IEMG_LEMB=(log(S_IEMG/S_LEMB)+ (0.5*omega_IEMG_LEMB∧2)*T)/
(omega_IEMG_LEMB*sqrt(T))
e1_IEMG_LEMB
u_e1_IEMG_LEMB=pnorm(e1_IEMG_LEMB)
u_e1_IEMG_LEMB
e2_IEMG_LEMB=e1_IEMG_LEMB- (omega_IEMG_LEMB*sqrt(T))
u_e2_IEMG_LEMB=pnorm(e2_IEMG_LEMB)
u_e2_IEMG_LEMB
d1_IEMG_low=(log(S_IEMG/k_2)+(0.5*var_IEMG)*T)/(sd_IEMG*sqrt(T))
d1_IEMG_low
d2_IEMG_low=d1_IEMG_low - (sd_IEMG*sqrt(T))
d2_IEMG_low
u_d1_IEMG_call_low=pnorm(d1_IEMG_low)
u_d2_IEMG_call_low=pnorm(d2_IEMG_low)
u_d1_IEMG_put_low=pnorm(-d1_IEMG_low)
u_d2_IEMG_put_low=pnorm(-d2_IEMG_low)
#LEMB-IGOV
d1_LEMB=(log(S_LEMB/k_1)+(0.5*var_LEMB)*T)/(sd_LEMB*sqrt(T))
d1_LEMB
d2_LEMB=d1_LEMB - (sd_LEMB*sqrt(T))
d2_LEMB
u_d1_LEMB_call=pnorm(d1_LEMB)
u_d2_LEMB_call=pnorm(d2_LEMB)
u_d1_LEMB_put=pnorm(-d1_LEMB)
u_d2_LEMB_put=pnorm(-d2_LEMB)
omega_LEMB_IGOV=sqrt(var_LEMB + var_IGOV - 2*cor(LEMBlogreturns, IGOVlo-
greturns)*sd_LEMB*sd_IGOV)
e1_LEMB_IGOV=(log(S_LEMB/S_IGOV)+ (0.5*omega_LEMB_IGOV∧2)*T)/
(omega_LEMB_IGOV*sqrt(T))
e1_LEMB_IGOV
u_e1_LEMB_IGOV=pnorm(e1_LEMB_IGOV)
u_e1_LEMB_IGOV
```

```
e2_LEMB_IGOV=e1_LEMB_IGOV- (omega_LEMB_IGOV*sqrt(T))
u_e2_LEMB_IGOV=pnorm(e2_LEMB_IGOV)
u_e2_LEMB_IGOV
d1_IGOV_low=(log(S_IGOV/k_2)+(0.5*var_IGOV)*T)/(sd_IGOV*sqrt(T))
d1_IGOV_low
d2_IGOV_low=d1_IGOV_low - (sd_IGOV*sqrt(T))
d2_IGOV_low
u_d1_IGOV_call_low=pnorm(d1_IGOV_low)
u_d2_IGOV_call_low=pnorm(d2_IGOV_low)
u_d1_IGOV_put_low=pnorm(-d1_IGOV_low)
u_d2_IGOV_put_low=pnorm(-d2_IGOV_low)


#HIGHCAPSPREAD (LEMB-IGOV)
library(VineCopula)
c_highcapspread_LEMB_IGOV_t=S_LEMB*BiCopCDF(u_e1_LEMB_IGOV, u_d1_LEMB_put,
family=2, par=0.41, par2 =9.24)-S_IGOV*BiCopCDF(u_e2_LEMB_IGOV, u_d2_LEMB_put,
family=2, par=0.41, par2 =9.24)
c_highcapspread_LEMB_IGOV_t
BiCopCDFfirstderiv_c1_LEMB_IGOV_t=(BiCopHfunc1(u_e1_LEMB_IGOV,u_d1_LEMB_put,
family=2, par=0.41, par2 = 9.24)/BiCopPDF(u_e1_LEMB_IGOV,u_d1_LEMB_put , fam-
ily=2, par=0.41, par2 = 9.24))*BiCopHfuncDeriv(u_e1_LEMB_IGOV ,u_d1_LEMB_put,
family=2, par=0.41, par2 = 9.24, deriv = ”par”)
BiCopCDFfirstderiv_c2_LEMB_IGOV_t=(BiCopHfunc1(u_e2_LEMB_IGOV, u_d2_LEMB_put,
family=2, par=0.41, par2 =9.24)/BiCopPDF(u_e2_LEMB_IGOV, u_d2_LEMB_put, fam-
ily=2, par=0.41, par2 =9.24))*BiCopHfuncDeriv(u_e2_LEMB_IGOV,u_d2_LEMB_put,
family=2, par=0.41, par2 =9.24, deriv = ”par”)
c_highcapspread_LEMB_IGOV_t_delta=S_LEMB*BiCopCDFfirstderiv_c1_LEMB_IGOV_t
- S_IGOV*BiCopCDFfirstderiv_c2_LEMB_IGOV_t
c_highcapspread_LEMB_IGOV_t_delta
BiCopCDFderiv_c1_LEMB_IGOV_t=(BiCopHfuncDeriv(u_d1_LEMB_put,u_e1_LEMB_IGOV,
family=2, par=0.41, par2 =9.24, deriv = ”par”)-(BiCopHfunc1(u_e1_LEMB_IGOV,
u_d1_LEMB_put, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e1_LEMB_IGOV,u_d1_LEMB_put , family=2, par=0.41, par2 = 9.24))*Bi-
CopDeriv(u_e1_LEMB_IGOV,u_d1_LEMB_put, family=2, par=0.41, par2 = 9.24, deriv
```

= ”u1”))/
(BiCopPDF(u_e1_LEMB_IGOV,u_d1_LEMB_put , family=2, par=0.41, par2 = 9.24))∧2
BiCopCDFsecondderiv_c1_LEMB_IGOV_t=BiCopHfuncDeriv2(u_e1_LEMB_IGOV ,u_d1_LEMB_put, family=2, par=0.41, par2 = 9.24, deriv = ”par”)*(BiCopHfunc1
(u_e1_LEMB_IGOV,u_d1_LEMB_put, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e1_LEMB_IGOV,u_d1_LEMB_put , family=2, par=0.41, par2 = 9.24))+
BiCopCDFderiv_c1_LEMB_IGOV_t*(BiCopHfuncDeriv
(u_e1_LEMB_IGOV ,u_d1_LEMB_put, family=2, par=0.41, par2 = 9.24, deriv = ”par”))∧2
BiCopCDFderiv_c2_LEMB_IGOV_t=(BiCopHfuncDeriv(u_d2_LEMB_put,u_e2_LEMB_IGOV, family=2, par=0.41, par2 =9.24, deriv = ”par”)-(BiCopHfunc1
(u_e2_LEMB_IGOV,u_d2_LEMB_put, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e2_LEMB_IGOV,u_d2_LEMB_put , family=2, par=0.41, par2 = 9.24))*BiCopDeriv(u_e2_LEMB_IGOV,u_d2_LEMB_put, family=2, par=0.41, par2 = 9.24, deriv
= ”u1”))/
(BiCopPDF(u_e2_LEMB_IGOV,u_d2_LEMB_put , family=2, par=0.41, par2 = 9.24))∧2
BiCopCDFsecondderiv_c2_LEMB_IGOV_t=BiCopHfuncDeriv2(u_e2_LEMB_IGOV ,u_d2_LEMB_put, family=2, par=0.41, par2 = 9.24, deriv = ”par”)*(BiCopHfunc1
(u_e2_LEMB_IGOV,u_d2_LEMB_put, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e2_LEMB_IGOV,u_d2_LEMB_put , family=2, par=0.41, par2 = 9.24))+
BiCopCDFderiv_c2_LEMB_IGOV_t*(BiCopHfuncDeriv(u_e2_LEMB_IGOV ,u_d2_LEMB_put, family=2, par=0.41, par2 = 9.24, deriv = ”par”))∧2
c_highcapspread_LEMB_IGOV_t_gamma=S_LEMB*BiCopCDFsecondderiv_c1_LEMB_IGOV_t
- S_IGOV*BiCopCDFsecondderiv_c2_LEMB_IGOV_t
c_highcapspread_LEMB_IGOV_t_gamma

#HIGHFLOORSPREAD (LEMB-IGOV)

c_highfloorspread_LEMB_IGOV_t=S_LEMB*BiCopCDF(u_e1_LEMB_IGOV, u_d1_LEMB_call, family=2, par=0.41, par2 =9.24)-S_IGOV*BiCopCDF(u_e2_LEMB_IGOV, u_d2_LEMB_call, family=2, par=0.41, par2 =9.24)
c_highfloorspread_LEMB_IGOV_t
BiCopCDFhighfloorfirstderiv_c1_LEMB_IGOV_t=(BiCopHfunc1
(u_e1_LEMB_IGOV,u_d1_LEMB_call, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e1_LEMB_IGOV,u_d1_LEMB_call , family=2, par=0.41, par2 = 9.24))*Bi-

CopHfuncDeriv(u_e1_LEMB_IGOV ,u_d1_LEMB_call, family=2, par=0.41, par2 = 9.24, deriv = "par")
BiCopCDFhighfloorfirstderiv_c2_LEMB_IGOV_t=(BiCopHfunc1
(u_e2_LEMB_IGOV, u_d2_LEMB_call, family=2, par=0.41, par2 =9.24)/
BiCopPDF(u_e2_LEMB_IGOV, u_d2_LEMB_call, family=2, par=0.41, par2 =9.24))*BiCopHfuncDeriv
(u_e2_LEMB_IGOV,u_d2_LEMB_call, family=2, par=0.41, par2 =9.24, deriv = "par")
c_highfloorspread_LEMB_IGOV_t_delta=
S_LEMB*BiCopCDFhighfloorfirstderiv_c1_LEMB_IGOV_t -
S_IGOV*BiCopCDFhighfloorfirstderiv_c2_LEMB_IGOV_t
c_highfloorspread_LEMB_IGOV_t_delta
BiCopCDFhighfloorderiv_c1_LEMB_IGOV_t=(BiCopHfuncDeriv
(u_d1_LEMB_call,u_e1_LEMB_IGOV, family=2, par=0.41, par2 =9.24, deriv = "par")-
(BiCopHfunc1
(u_e1_LEMB_IGOV,u_d1_LEMB_call, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e1_LEMB_IGOV,u_d1_LEMB_call , family=2, par=0.41, par2 = 9.24))*
BiCopDeriv(u_e1_LEMB_IGOV,u_d1_LEMB_call, family=2, par=0.41, par2 = 9.24, deriv = "u1"))/
(BiCopPDF(u_e1_LEMB_IGOV,u_d1_LEMB_call , family=2, par=0.41, par2 = 9.24))∧2
BiCopCDFhighfloorsecondderiv_c1_LEMB_IGOV_t=BiCopHfuncDeriv2(u_e1_LEMB_IGOV ,u_d1_LEMB_call, family=2, par=0.41, par2 = 9.24, deriv = "par")*(BiCopHfunc1
(u_e1_LEMB_IGOV,u_d1_LEMB_call, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e1_LEMB_IGOV,u_d1_LEMB_call , family=2, par=0.41, par2 = 9.24))+
BiCopCDFhighfloorderiv_c1_LEMB_IGOV_t*(BiCopHfuncDeriv
(u_e1_LEMB_IGOV ,u_d1_LEMB_call, family=2, par=0.41, par2 = 9.24, deriv = "par"))∧2
BiCopCDFhighfloorderiv_c2_LEMB_IGOV_t=(BiCopHfuncDeriv
(u_d2_LEMB_call,u_e2_LEMB_IGOV, family=2, par=0.41, par2 =9.24, deriv = "par")-
(BiCopHfunc1
(u_e2_LEMB_IGOV,u_d2_LEMB_call, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e2_LEMB_IGOV,u_d2_LEMB_call , family=2, par=0.41, par2 = 9.24))*BiCopDeriv(u_e2_LEMB_IGOV,u_d2_LEMB_call, family=2, par=0.41, par2 = 9.24, deriv = "u1"))/
(BiCopPDF(u_e2_LEMB_IGOV,u_d2_LEMB_call , family=2, par=0.41, par2 = 9.24))∧2
BiCopCDFhighfloorsecondderiv_c2_LEMB_IGOV_t=BiCopHfuncDeriv2(u_e2_LEMB_IGOV ,u_d2_LEMB_call, family=2, par=0.41, par2 = 9.24, deriv = "par")*(BiCopHfunc1

(u_e2_LEMB_IGOV,u_d2_LEMB_call, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e2_LEMB_IGOV,u_d2_LEMB_call , family=2, par=0.41, par2 = 9.24))+
BiCopCDFhighfloorderiv_c2_LEMB_IGOV_t*(BiCopHfuncDeriv
(u_e2_LEMB_IGOV ,u_d2_LEMB_call, family=2, par=0.41, par2 = 9.24, deriv = "par"))∧2
c_highfloorspread_LEMB_IGOV_t_gamma=S_LEMB*BiCopCDFhighfloorsecondderiv_c1_LEMB_IGOV_t
- S_IGOV*BiCopCDFhighfloorsecondderiv_c2_LEMB_IGOV_t
c_highfloorspread_LEMB_IGOV_t_gamma

#LOWCAPSPREAD AND LOWFLOOR SPREAD (LEMB-IGOV)

c_lowcapspread_LEMB_IGOV_t=S_LEMB*BiCopCDF(u_e1_LEMB_IGOV, u_d1_IGOV_put_low,
family=2, par=0.41, par2 =9.24)-S_IGOV*BiCopCDF(u_e2_LEMB_IGOV, u_d2_IGOV_put_low,
family=2, par=0.41, par2 =9.24)
c_lowcapspread_LEMB_IGOV_t
c_lowcapspread_LEMB_IGOV_t=S_LEMB*BiCopCDF(u_e1_LEMB_IGOV, u_d1_IGOV_call_low,
family=2, par=0.41, par2 =9.24)-S_IGOV*BiCopCDF(u_e2_LEMB_IGOV, u_d2_IGOV_call_low,
family=2, par=0.41, par2 =9.24)
c_lowcapspread_LEMB_IGOV_t
BiCopCDFlowcapfirstderiv_c1_LEMB_IGOV_t=(BiCopHfunc1(u_e1_LEMB_IGOV,u_d1_IGOV_put_low,
family=2, par=0.41, par2 = 9.24)/BiCopPDF(u_e1_LEMB_IGOV,u_d1_IGOV_put_low ,
family=2, par=0.41, par2 = 9.24))*BiCopHfuncDeriv(u_e1_LEMB_IGOV ,u_d1_IGOV_put_low,
family=2, par=0.41, par2 = 9.24, deriv = "par")
BiCopCDFlowcapfirstderiv_c2_LEMB_IGOV_t=(BiCopHfunc1(u_e2_LEMB_IGOV, u_d2_IGOV_put_low,
family=2, par=0.41, par2 =9.24)/BiCopPDF(u_e2_LEMB_IGOV, u_d2_IGOV_put_low,
family=2, par=0.41, par2 =9.24))*BiCopHfuncDeriv(u_e2_LEMB_IGOV,u_d2_IGOV_put_low,
family=2, par=0.41, par2 =9.24, deriv = "par")
c_lowcapspread_LEMB_IGOV_t_delta=S_LEMB*BiCopCDFlowcapfirstderiv_c1_LEMB_IGOV_t
- S_IGOV*BiCopCDFlowcapfirstderiv_c2_LEMB_IGOV_t
c_lowcapspread_LEMB_IGOV_t_delta
BiCopCDFlowfloorfirstderiv_c1_LEMB_IGOV_t=(BiCopHfunc1(u_e1_LEMB_IGOV,u_d1_IGOV_call_low,
family=2, par=0.41, par2 = 9.24)/BiCopPDF(u_e1_LEMB_IGOV,u_d1_IGOV_call_low ,
family=2, par=0.41, par2 = 9.24))*BiCopHfuncDeriv(u_e1_LEMB_IGOV ,u_d1_IGOV_call_low,
family=2, par=0.41, par2 = 9.24, deriv = "par")

BiCopCDFlowfloorfirstderiv_c2_LEMB_IGOV_t=(BiCopHfunc1(u_e2_LEMB_IGOV, u_d2_IGOV_call_low, family=2, par=0.41, par2 =9.24)/BiCopPDF(u_e2_LEMB_IGOV, u_d2_IGOV_call_low, family=2, par=0.41, par2 =9.24))*BiCopHfuncDeriv(u_e2_LEMB_IGOV,u_d2_IGOV_call_low, family=2, par=0.41, par2 =9.24, deriv = "par")

c_lowfloorspread_LEMB_IGOV_t_delta=S_LEMB*BiCopCDFlowfloorfirstderiv_c1_LEMB_IGOV_t - S_IGOV*BiCopCDFlowfloorfirstderiv_c2_LEMB_IGOV_t

c_lowfloorspread_LEMB_IGOV_t_delta

BiCopCDFlowcapderiv_c1_LEMB_IGOV_t=(BiCopHfuncDeriv(u_d1_IGOV_put_low,u_e1_LEMB_IGOV, family=2, par=0.41, par2 =9.24, deriv = "par")-(BiCopHfunc1(u_e1_LEMB_IGOV,u_d1_IGOV_put_low, family=2, par=0.41, par2 = 9.24)/BiCopPDF(u_e1_LEMB_IGOV,u_d1_IGOV_put_low , family=2, par=0.41, par2 = 9.24))*BiCopDeriv(u_e1_LEMB_IGOV,u_d1_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "u1"))/(BiCopPDF(u_e1_LEMB_IGOV,u_d1_IGOV_put_low , family=2, par=0.41, par2 = 9.24))∧2

BiCopCDFlowcapsecondderiv_c1_LEMB_IGOV_t=BiCopHfuncDeriv2(u_e1_LEMB_IGOV ,u_d1_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "par")*(BiCopHfunc1 (u_e1_LEMB_IGOV,u_d1_IGOV_put_low, family=2, par=0.41, par2 = 9.24)/ BiCopPDF(u_e1_LEMB_IGOV,u_d1_IGOV_put_low , family=2, par=0.41, par2 = 9.24))+ BiCopCDFlowcapderiv_c1_LEMB_IGOV_t*(BiCopHfuncDeriv (u_e1_LEMB_IGOV ,u_d1_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "par"))∧2

BiCopCDFlowcapderiv_c2_LEMB_IGOV_t=(BiCopHfuncDeriv(u_d2_IGOV_put_low,u_e2_LEMB_IGOV, family=2, par=0.41, par2 =9.24, deriv = "par")-(BiCopHfunc1(u_e2_LEMB_IGOV,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24)/BiCopPDF(u_e2_LEMB_IGOV,u_d2_IGOV_put_low , family=2, par=0.41, par2 = 9.24))*BiCopDeriv(u_e2_LEMB_IGOV,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "u1"))/(BiCopPDF(u_e2_LEMB_IGOV,u_d2_IGOV_put_low , family=2, par=0.41, par2 = 9.24))∧2

BiCopCDFlowcapsecondderiv_c2_LEMB_IGOV_t=BiCopHfuncDeriv2(u_e2_LEMB_IGOV ,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "par")*(BiCopHfunc1 (u_e2_LEMB_IGOV,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24)/ BiCopPDF(u_e2_LEMB_IGOV,u_d2_IGOV_put_low , family=2, par=0.41, par2 = 9.24))+ BiCopCDFlowcapderiv_c2_LEMB_IGOV_t*(BiCopHfuncDeriv (u_e2_LEMB_IGOV ,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "par"))∧2

c_lowcapspread_LEMB_IGOV_t_gamma=
S_LEMB*BiCopCDFlowcapsecondderiv_c1_LEMB_IGOV_t -

S_IGOV*BiCopCDFlowcapsecondderiv_c2_LEMB_IGOV_t
c_lowcapspread_LEMB_IGOV_t_gamma
BiCopCDFlowfloorderiv_c1_LEMB_IGOV_t=(BiCopHfuncDeriv(u_d1_IGOV_call_low,u_e1_LEMB_IGOV,
family=2, par=0.41, par2 =9.24, deriv = ”par”)-(BiCopHfunc1
(u_e1_LEMB_IGOV,u_d1_IGOV_call_low, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e1_LEMB_IGOV,u_d1_IGOV_call_low , family=2, par=0.41, par2 = 9.24))*Bi-
CopDeriv(u_e1_LEMB_IGOV,u_d1_IGOV_call_low, family=2, par=0.41, par2 = 9.24, de-
riv = ”u1”))/(BiCopPDF(u_e1_LEMB_IGOV,u_d1_IGOV_call_low , family=2, par=0.41,
par2 = 9.24))∧2
BiCopCDFlowfloorsecondderiv_c1_LEMB_IGOV_t=BiCopHfuncDeriv2(u_e1_LEMB_IGOV
,u_d1_IGOV_call_low, family=2, par=0.41, par2 = 9.24, deriv = ”par”)*(BiCopHfunc1
(u_e1_LEMB_IGOV,u_d1_IGOV_call_low, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e1_LEMB_IGOV,u_d1_IGOV_call_low , family=2, par=0.41, par2 = 9.24))+
BiCopCDFlowfloorderiv_c1_LEMB_IGOV_t*(BiCopHfuncDeriv
(u_e1_LEMB_IGOV ,u_d1_IGOV_call_low, family=2, par=0.41, par2 = 9.24, deriv =
”par”))∧2
BiCopCDFlowfloorderiv_c2_LEMB_IGOV_t=(BiCopHfuncDeriv(u_d2_IGOV_call_low,u_e2_LEMB_IGOV,
family=2, par=0.41, par2 =9.24, deriv = ”par”)-(BiCopHfunc1(u_e2_LEMB_IGOV,u_d2_IGOV_call_low,
family=2, par=0.41, par2 = 9.24)/BiCopPDF(u_e2_LEMB_IGOV,u_d2_IGOV_call_low ,
family=2, par=0.41, par2 = 9.24))*BiCopDeriv(u_e2_LEMB_IGOV,u_d2_IGOV_call_low,
family=2, par=0.41, par2 = 9.24, deriv = ”u1”))/(BiCopPDF(u_e2_LEMB_IGOV,u_d2_IGOV_call_low
, family=2, par=0.41, par2 = 9.24))∧2
BiCopCDFlowfloorsecondderiv_c2_LEMB_IGOV_t=BiCopHfuncDeriv2(u_e2_LEMB_IGOV
,u_d2_IGOV_call_low, family=2, par=0.41, par2 = 9.24, deriv = ”par”)*(BiCopHfunc1
(u_e2_LEMB_IGOV,u_d2_IGOV_call_low, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_e2_LEMB_IGOV,u_d2_IGOV_call_low , family=2, par=0.41, par2 = 9.24))+
BiCopCDFlowfloorderiv_c2_LEMB_IGOV_t*(BiCopHfuncDeriv
(u_e2_LEMB_IGOV ,u_d2_IGOV_call_low, family=2, par=0.41, par2 = 9.24, deriv =
”par”))∧2
c_lowfloorspread_LEMB_IGOV_t_gamma=S_LEMB*BiCopCDFlowfloorsecondderiv_c1_LEMB_IGOV_t
- S_IGOV*BiCopCDFlowfloorsecondderiv_c2_LEMB_IGOV_t
c_lowfloorspread_LEMB_IGOV_t_gamma

#BIVARIATE DIGITAL OPTIONS(LEMB-IGOV)

c_putdigital_LEMB_IGOV_t=D*BiCopCDF(u_d2_LEMB_put, u_d2_IGOV_put_low, family=2, par=0.41, par2 =9.24)
c_putdigital_LEMB_IGOV_t
c_calldigital_LEMB_IGOV_t=D*BiCopCDF(u_d2_LEMB_call, u_d2_IGOV_call_low, family=2, par=0.41, par2 =9.24)
c_calldigital_LEMB_IGOV_t
BiCopCDFputdigitalfirstderiv_c2_LEMB_IGOV_t=(BiCopHfunc1(u_d2_LEMB_put, u_d2_IGOV_put_low, family=2, par=0.41, par2 =9.24)/BiCopPDF(u_d2_LEMB_put, u_d2_IGOV_put_low, family=2, par=0.41, par2 =9.24))*BiCopHfuncDeriv(u_d2_LEMB_put,u_d2_IGOV_put_low, family=2, par=0.41, par2 =9.24, deriv = "par")
c_putdigital_LEMB_IGOV_t_delta=D*BiCopCDFputdigitalfirstderiv_c2_LEMB_IGOV_t
c_putdigital_LEMB_IGOV_t_delta
BiCopCDFcalldigitalfirstderiv_c2_LEMB_IGOV_t=(BiCopHfunc1(u_d2_LEMB_call, u_d2_IGOV_call_low, family=2, par=0.41, par2 =9.24)/BiCopPDF(u_d2_LEMB_call, u_d2_IGOV_call_low, family=2, par=0.41, par2 =9.24))*BiCopHfuncDeriv(u_d2_LEMB_call,u_d2_IGOV_call_low, family=2, par=0.41, par2 =9.24, deriv = "par")
c_calldigital_LEMB_IGOV_t_delta=D*BiCopCDFcalldigitalfirstderiv_c2_LEMB_IGOV_t
c_calldigital_LEMB_IGOV_t_delta
BiCopCDFputdigitalderiv_c2_LEMB_IGOV_t=(BiCopHfuncDeriv(u_d2_IGOV_put_low,u_d2_LEMB_put, family=2, par=0.41, par2 =9.24, deriv = "par")-(BiCopHfunc1(u_d2_LEMB_put,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24)/BiCopPDF(u_d2_LEMB_put,u_d2_IGOV_put_low , family=2, par=0.41, par2 = 9.24))*BiCopDeriv(u_d2_LEMB_put,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "u1"))/(BiCopPDF(u_d2_LEMB_put,u_d2_IGOV_put_low , family=2, par=0.41, par2 = 9.24))∧2
BiCopCDFputdigitalsecondderiv_c2_LEMB_IGOV_t=BiCopHfuncDeriv2 (u_d2_LEMB_put,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "par")*(BiCopHfunc1 (u_d2_LEMB_put,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24)/ BiCopPDF(u_d2_LEMB_put,u_d2_IGOV_put_low , family=2, par=0.41, par2 = 9.24))+ BiCopCDFputdigitalderiv_c2_LEMB_IGOV_t*(BiCopHfuncDeriv (u_d2_LEMB_put ,u_d2_IGOV_put_low, family=2, par=0.41, par2 = 9.24, deriv = "par"))∧2
c_putdigital_LEMB_IGOV_t_gamma=D*BiCopCDFputdigitalsecondderiv_c2_LEMB_IGOV_t
c_putdigital_LEMB_IGOV_t_gamma
BiCopCDFcalldigitalderiv_c2_LEMB_IGOV_t=(BiCopHfuncDeriv(u_d2_IGOV_call_low,u_d2_LEMB_call, family=2, par=0.41, par2 =9.24, deriv = "par")-(BiCopHfunc1(u_d2_LEMB_call,u_d2_IGOV_call_low,

family=2, par=0.41, par2 = 9.24)/BiCopPDF(u_d2_LEMB_call,u_d2_IGOV_call_low ,
family=2, par=0.41, par2 = 9.24))*BiCopDeriv(u_d2_LEMB_call,u_d2_IGOV_call_low,
family=2, par=0.41, par2 = 9.24, deriv = "u1"))/(BiCopPDF(u_d2_LEMB_call,u_d2_IGOV_call_low
, family=2, par=0.41, par2 = 9.24))∧2
BiCopCDFcalldigitalsecondderiv_c2_LEMB_IGOV_t=BiCopHfuncDeriv2
(u_d2_LEMB_call,u_d2_IGOV_call_low, family=2, par=0.41, par2 = 9.24, deriv = "par")*(BiCopHfunc1
(u_d2_LEMB_call,u_d2_IGOV_call_low, family=2, par=0.41, par2 = 9.24)/
BiCopPDF(u_d2_LEMB_call,u_d2_IGOV_call_low , family=2, par=0.41, par2 = 9.24))+
BiCopCDFcalldigitalderiv_c2_LEMB_IGOV_t*(BiCopHfuncDeriv
(u_d2_LEMB_call ,u_d2_IGOV_call_low, family=2, par=0.41, par2 = 9.24, deriv = "par"))∧2
c_calldigital_LEMB_IGOV_t_gamma=D*BiCopCDFcalldigitalsecondderiv_c2_LEMB_IGOV_t
c_calldigital_LEMB_IGOV_t_gamma