Strathmore UNIVERSITY

STRATHMORE INSTITUTE OF MATHEMATICAL SCIENCES

MASTER IN DATA SCIENCE AND ANALYTICS

END OF SEMESTER EXAMINATION

DSA 8405 PROBABILITY AND STOCHASTIC PROCESSES

DATE: 11th December 2024 TIME: 3 Hours

Instructions

1. This examination consists of **Four** questions.
2. Answer **Question ONE (COMPULSORY)** and any other **TWO** questions.

Question 1 (20 Marks)

(a) Explain the meaning of a *covariance stationary Stochastic process* (2 Marks)

(b) Find the generating function of the sequence defined by Let $\{a_k\} = \left\{\frac{1}{k!}\right\}$ (2 Marks)

(b) Let the probability density function be defined by Poisson distribution given by

$$Pr\{X = k\} = p_k = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0,1,2, \dots, \lambda > 0$$

(i) Determine the probability generating function of X (2 Marks)

(ii) Hence find its mean and variance (3 Marks)

(c) Classify the states of the following Markov chains.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.3 & 0.0 & 0.7 & 0.0 \\ 0.1 & 0.2 & 0.0 & 0.7 \end{pmatrix} \end{array}$$

(3 Marks)

(d) A camera store stocks a particular model camera that can be ordered weekly. On Saturday night the store places an order that is delivered in time for the opening of the store on Monday. If the number of cameras on hand at the end of the week is less than one (no cameras in stock), the store orders up to 3. Otherwise, the store does not order, that is, if there are any cameras in stock, no order is placed. The possible states of the process are integers 0, 1, 2, 3, representing the possible number of cameras on hand at the end of the week. The corresponding transition matrix is given by

$$\begin{pmatrix} 0.0800 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0. \\ 0.080 & 0.184 & `0.368 & 0.368 \end{pmatrix}$$

Given that there is one camera left in stock at the end of the week, determine the probability that

(i) there will be no cameras left in stock in 2 weeks later (2 Marks)

(ii) there will be two cameras in stock at the end of the week

(2 Marks)

(f) customers arrive at a store in groups consisting of 1 or 2 individuals with equal probability and the arrival of groups is in accordance with a Poisson process with mean rate $\lambda = 0.5$. Let x_i be the number of individuals arriving at a store. (2 Marks)

(i) Find mean number of customers arriving in four minutes

(ii) Determine the variance of the number of customers arriving in four minutes (2 Marks)

Question 2 (20 Marks)

(a) Consider the process $X(t) = A\cos\lambda t + B\sin\lambda t$ where A and B are uncorrelated random variables with mean 0 and variance 1, whereas λ is a positive constant. Show that $X(t)$ is a weakly stationary process (10 Marks)

(b) Consider the process $\{X(t) : t \in T\}$ with $\Pr\{X(t) = n\} = \begin{cases} \dfrac{e^{-at}(at)^n}{n!} & , a > 0, n = 0,1,2,.. \\ o & , otherwise \end{cases}$

Show that $X(t)$ is not stationary. (10 Marks)

Question 3 (20 Marks)

(a) Show that the following double transition matrix is an irreducible and ergodic Markov chain

$$\begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

(10 Marks)

(b) Part time students admitted to a Master program in a University are considered to be first year students until they complete 15 credits successfully. Then they are classified as 2nd year students and may begin to take more advanced courses and work on their thesis required for graduation.

Past records indicate that at the end of each year 10% of the first-year students (F) drop out of the program (D) and 30% become 2nd year students (S). Also, 10% of the 2nd year students drop out of the program 40% graduate (G) each year. Students that graduate or drop out never return to the program.

(i) Formulate the transition matrix, hence classify the states (2 Marks)
(ii) Determine the probability that a 1st year student graduates within 2 years. (4 Marks)

(iii) Suppose at the end of each year, the faculty examines the progress each 2nd year student has made in writing the required thesis. Past records show that 30% of the 2nd year have their thesis approved (A) and 10% of the students are dropped from the program for insufficient progress (D), never to return. The remaining students continue to work on their thesis. Determine the probability that a 2nd year student completes the thesis required within 4 years. (4 Marks)

Question 4 (20 Marks)

Let Z(t) represent the population size at a time t and $P_n(t)$ be the probability that a population is of size n at a time t. Further let $P_n(t) = \Pr[Z(t) = n]$. Let Δt represent a small interval of time over which this population is being studied, $\lambda_n(t) + 0\Delta t$ is the probability that a birth occurs within the time interval Δt and $\mu_n(t) + 0\Delta t$ is the probability that from a population of size n a death occurs. The difference differential equations for the birth-death process are given by

$$P_n'(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t), \text{ for } n \geq 1$$

and $P_n'(t) = -\mu_0 P_0(t) - \lambda_0 P_0(t) + \mu_1 P_1(t)$ for $n = 0$,

suppose that $\lambda_n = \lambda\left[\frac{1+an}{1+\lambda at}\right]$ and the death component is zero so that the difference differential equations for the birth process for a population size n at a time t is given by

$$P_n'(t) = -\left(\lambda_n\left[\frac{1+an}{1+\lambda at}\right]\right)P_n(t) + \lambda\left[\frac{1+an}{1+\lambda at}\right]P_{n-1}(t), \text{ for } n \geq 1$$

and

$$P_n'(t) = -\left[\frac{\lambda}{1+\lambda at}\right]P_0(t), \text{ for } n = 0, \text{ where } a \text{ is an arbitrary parameter}$$

(a) Suppose the initial conditions are $p_n(0) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$ Show that the generating function $G(s,t)$ is given by

$$G(s,t) = (1+\lambda at)^{\frac{-1}{a}}\left[1 - \frac{\lambda at}{1+\lambda at}s\right]^{\frac{-1}{a}}$$ (16 Marks)

(b) Find the probability that the population is of size n at time t, (4 Marks)