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A Comparative Study of Crank-Nicolson scheme and Monte-Carlo Option Pricing

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DECLARATION

I declare that this work has not been previously submitted and approved for the award of a degree by this or any other University. To the best of my knowledge and belief, the Research Project contains no material previously published or written by another person except where due reference is made in the Research Project itself.

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Table of Contents

ABSTRACT	IV
CHAPTER 1: INTRODUCTION	1
BACKGROUND OF THE STUDY	1
RESEARCH OBJECTIVES.....	2
RESEARCH QUESTIONS.....	2
SIGNIFICANCE OF THE STUDY.....	2
PROBLEM STATEMENT	3
CHAPTER 2: LITERATURE REVIEW	4
CHAPTER 3: METHODOLOGY AND DATA	7
INTRODUCTION.....	7
CRANK- NICOLSON SCHEME	7
CONVERSION OF THE BLACK-SCHOLES PDE INTO HEAT EQUATION	7
CRANK-NICOLSON SCHEME GRID.....	8
MONTE-CARLO METHODS	10
BLACK-SCHOLES FORMULA	10
RESEARCH DESIGN.....	11
NATURE OF STUDY	11
POPULATION AND SAMPLING.....	11
DATA COLLECTION AND PROCEDURE.....	11
MEAN ABSOLUTE PERCENTAGE ERROR (MAPE).....	11
CHAPTER 4: FINDINGS AND DATA ANALYSIS	12
ANALYSIS OF THE MONTE-CARLO SIMULATION	12
ANALYSIS OF THE ANTITHETIC VARIATE TECHNIQUE.....	13
ANALYSIS OF CRANK-NICOLSON RESULTS.....	15
CHAPTER 5: CONCLUSION AND RECOMMENDATIONS	17
APPENDIX	18
BIBLIOGRAPHY	20

ABSTRACT

This study examines the rate of convergence and the accuracy of the two primary option pricing methods used currently by professionals; Monte-Carlo and Crank-Nicolson scheme using the Black-Scholes price as the benchmark price. We also introduce the Antithetic variates to the Monte Carlo, to check how much the technique improves the accuracy of the model. A model that converges faster and is accurate will be important in the valuation of large number of options, this will be beneficial to the current and potential investors dealing with large number of options, usually this is the case in practice. Similarly, by control variates technique, we can use our result to improve the accuracy of pricing options that do not have closed form solution such as American options or other exotic options.

Key words: Monte-Carlo, Crank-Nicolson, Black-Scholes, Antithetic variates, Control variates.

CHAPTER 1: INTRODUCTION

Background of the Study

Numerical methods have become very important in finance based on the following arguments. To start with, the underlying models that describe the prices of securities and relevant state variable have become more sophisticated. In addition, the types of securities and their derivatives have become more complicated. Most importantly, the restructuring of derivatives has made the risk management process more complex and mandatory to control the financial atmosphere. For instance, due to regulatory requirements, many financial institutions have to report Value-at Risk which invokes many complicated assumptions.

A wide range of numerical methods are available for the purposes of addressing the issues but for the purpose of this study, we focus on Monte Carlo method and Crank Nicolson model because these are the two primary numerical methods currently used by professionals (Fadugba, Nwozo, & Babalola, 2012).

Crank Nicolson finite difference method is the central difference of implicit and explicit method. According to (Fadugba, 2013), Crank Nicolson finite difference is a popular choice for pricing options in that options will satisfy the Black-Scholes partial differential equation or appropriate variants of it. The difference between each option contract is in determining the boundary conditions that it satisfies. Crank Nicolson method can also be applied to American options and exotic options

According to (Fadugba, 2013) the basis of Monte Carlo simulation is the strong law of large numbers, stating that the arithmetic mean of independent, identically distributed random variables converges towards their mean almost surely. Monte Carlo simulation method uses the risk valuation result. The expected payoff in a risk neutral world is calculated using a sampling procedure. The law of large numbers ensures that this estimate converges to the correct value as the number of draws increases. The central limit theorem provides information about the likely magnitude of the error in the estimate after a finite number of draws.

There are derivatives which cannot be priced using the closed form solutions because of the complexity of the derivatives; one of the ways to solve that problem is by using numerical valuation. In this study, we will focus on numerical valuation of option pricing by comparing the Crank-Nicolson model and Monte-Carlo simulation method.

Research Objectives

To use the Black-Scholes price as the benchmark in determining which of the two primary numerical models; Monte-Carlo and Crank-Nicolson scheme has a faster convergence rate and is more accurate. In addition, to find out how much the technique of Antithetic covariates may be used to improve the accuracy of the Monte-Carlo method.

Research Questions

1. Which of the two models; Monte-Carlo method and Crank-Nicolson scheme converges faster to the Black Scholes price?
2. Which of the two models; Monte-Carlo method and Crank-Nicolson scheme is more accurate, with Black Scholes price being the benchmark?
3. How much does the method of Antithetic covariate improve the Accuracy of the Monte-Carlo method?

Significance of the Study

A model that converges faster and is accurate will be important in the valuation of large number of options, this will be beneficial to the current and potential investors dealing with large number of options, usually this is the case in practice.

Using the control variate technique, we can use our result to improve the accuracy of pricing options that do not have closed form solution such as American options or other exotic options (Hull, 2012). For instance, we calculate the price of the European call using Black-Scholes option price c_{EBSM} and Monte-Carlo option price c_{EM} . We also calculate the price of the American call using the Monte-Carlo c_{AM} . The error when calculating the European call option using Monte Carlo $c_{EM} - c_{EBSM}$ is assumed to be equal to the error when calculating the American call option using the Monte-Carlo method. The price of the American call option is then given by (Hull, 2012):

$$c_{AM} + (c_{EM} - c_{EBSM})$$

Problem Statement

Many options, for instance American options and exotic options, either there are no closed form solutions or if the closed form solution exists, they are complicated and difficult to evaluate accurately using the conventional methods (Lai & Spanier, 2000). In such cases finite difference, Monte Carlo and Lattice methods are valuable. In practice, we normally want to calculate the price of book of options and this should be done fast and accurately. This is the reason why a method with high speed and high accuracy is desired.

While a lot of studies have been done on the comparative studies of numerical methods, some of the findings are contrasting; it is still not clear between Crank-Nicolson and Monte Carlo methods which one has a faster convergence rate or is more accurate in pricing European options. Again none of these studies goes further to investigate how much the Antithetic covariates can improve the accuracy of the two methods. This study seeks to fill these gaps of knowledge. It seeks to reconcile the contrasting findings and also to introduce the use of Antithetic covariates in this comparison.

The literature of option pricing models dates back to Black and Scholes (1973) in which they derived a closed form solution for pricing European vanilla option. In a Black-Scholes market, the option is priced under the risk-neutral valuation with the assumption that the share price follows a geometric Brownian motion. For many other options, for instance American options and exotic options, either there are no closed form solutions or if the closed form solution exists, they are complicated and difficult to evaluate accurately using the conventional methods (Lai & Spanier, 2000). In such cases Monte Carlo methods, Finite difference schemes and Lattice approaches (Binomial and Trinomial) have been suggested (Lai & Spanier, 2000). Monte Carlo simulation was proposed by (Boyle, 1977), Finite Differences are discussed in Schwartz (1979) while the binomial models are discussed by Cox, Ross and Rubenstein (1979) and Trinomial by (Kamrad & Ritchken, 1991).

Numerous studies have examined the suitability of various option pricing techniques to different option styles. For instance, it is clear from previous studies, such as (Feng, 2012), that as the number of steps becomes large; the binomial model converges to the Black-Scholes. Nwozo and Fadugba (2012) examined three numerical methods for option pricing. They examined binomial model, finite differences and Monte Carlo method. They analyzed the advantages of each of the model and noted that binomial models are good for pricing options with early exercise date but cannot be used in more complex situations and also not flexible. Finite difference method converge faster and are more accurate while Monte Carlo works very well in pricing European options (Fadugba, Nwozo, & Babalola, The Comparative study of Finite Difference Method and Monte Carlo Method for Pricing European Option, 2012). Trinomial has the advantage over binomial as it gives another degree of freedom but is more complex (Ritchken, 1995). Zvan, Vetzal and Forsyth (1997) proposed a way of pricing contingent claims with general algebraic constraints on the solution. The constraints included barriers and early exercise features. They further compared the rate of convergence of the implicit with that of explicit method. The results showed that the use of implicit method leads to convergence in fewer time step compared to explicit scheme (Zvan, Vetzal, & Forsyth, 1997).

From the above analysis, two techniques stand out for pricing European options; Monte-Carlo methods and Finite difference and again according to (Fadugba, Nwozo, & Babalola, The Comparative study of Finite Difference Method and Monte Carlo Method for Pricing

European Option, 2012) these are the two primary numerical methods that are predominantly used by professionals. We chose specifically Crank-Nicolson which is the centered of the two Finite difference schemes; Explicit and Implicit. We also chose Black-Scholes model as the most suitable benchmark against which we obtain the rate of convergence of the two models. The reason for this was as follows, according to (Smith, 1976) the analysis are quite robust with respect to the derivation of Black-Scholes formula. This has been shown by the subsequent modifications of the Black-Scholes Model by Merton (1973b) and others. Klar (2002) also adds that, even though Black-Scholes has known flaws, no one has come up with something better without increasing complexity of the model.

Here, we review the various comparative studies that have investigated the problem of pricing option with Monte Carlo and Crank-Nicolson. To begin with, Kumar (2011) did a comparative study of the Crank-Nicolson and the higher order Compact scheme (fourth order). They compared the two with Monte Carlo price as the benchmark. They obtained that the results using the higher order are closer to the Monte Carlo results than the Crank-Nicolson. They noted that Monte Carlo simulation suffer from severe drawbacks like computational costs and a certain amount of uncertainty in pricing. In contrast, the usages of numerical partial differential equation (PDE) approaches have less computational cost and also provide a unique answer.

Almendral and Oosterlee (2005) conducted a study on numerical valuation on jump diffusion. They chose the jump-diffusion approach with constant coefficients and found numerically the value of European Vanilla options. They did this by solving the partial integral-differential equations for two models: the classical Merton's model and Kou's model. Fadugba et al (2012) did a comparative study of finite difference method and Monte Carlo method for pricing European option. They compared the convergence of the two methods to the analytic Black-Scholes price of the call option. They found that the Crank-Nicolson method converges faster and it is more accurate for pricing the European option.

Kwon and Lewis (2000) did a comparative study for finite difference method and Monte Carlo method for barrier options. They found that Monte Carlo method had larger errors than finite difference especially when the barrier level is close to the stock price.

In some cases, even when they increase the number of sample paths, they could not get convergence of errors. Fadugba et al (2012) studied the stability and accuracy of Crank-Nicolson and implicit method. It was obtained that Crank-Nicolson method is more accurate

and converges faster than implicit method. Klar & Jacobsen (2002) priced European call options using two methods: Black Scholes equation and Monte Carlo method. They concluded that Monte Carlo method is so imprecise so they should only be used when all the alternatives are worse.

In contrast to the above findings that Monte Carlo methods are not accurate, Grant, Vora and Weeks (1997) found that Monte-Carlo method produces accurate estimates for option values. Nwozo and Fadugba (2012) also assert that Monte Carlo method works very well in pricing European options.

Due to the contrasting findings, there is still need to do further studies on the comparative studies of the Crank-Nicolson and Monte Carlo methods to reconcile them. Again none of the above studies introduces the use of Antithetic covariates to improve the accuracy of the two methods. This is the extra edge that this particular study provides.

Introduction

This chapter is divided into two parts. The first part covers the mathematical construction of the various the two methods; Crank-Nicolson scheme and the Monte-Carlo Methods. The second part outlines the most appropriate methods of design, sample selection, data collection and analysis.

Crank- Nicolson Scheme

We first need to convert the Black-Scholes Partial differential equation (PDE) into heat equation by transformation of variables. We then use the heat equation to approximate the derivatives of the option (smooth function). We then use these derivatives in the Crank-Nicolson Scheme to calculate the price of the option.

Conversion of the Black-Scholes PDE into Heat Equation

Black-Scholes PDE for a function $V(t, S)$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} rs + \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

$$\text{Define } S = Ke^x \quad x = \log\left(\frac{S}{K}\right) \quad \tau = \frac{(T-t)\sigma^2}{2}$$

Consider $V(S, t) = V(x, \tau)$

$$\frac{\partial V}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial V}{\partial \tau} \quad \frac{\partial V}{\partial S} = \frac{1}{S} \frac{\partial V}{\partial x} \quad \frac{\partial^2 V}{\partial S^2} = -\frac{1}{S^2} \frac{\partial V}{\partial x} + \frac{1}{S^2} \frac{\partial^2 V}{\partial x^2}$$

Putting these in the PDE

$$-\frac{\sigma^2}{2} \left(\frac{\partial V}{\partial \tau} - \frac{\partial^2 V}{\partial x^2} + \left(1 - \frac{2r}{\sigma^2}\right) \frac{\partial V}{\partial x} - \frac{2r}{\sigma^2} V \right) = 0$$

$$\text{set } a = \frac{2r}{\sigma^2} - 1$$

$$\frac{\partial V}{\partial \tau} - \frac{\partial^2 V}{\partial x^2} - a \frac{\partial V}{\partial x} + (1 + a)V$$

If we define $u(\tau, x) = \frac{1}{K} e^{\left(\frac{a^2}{4} + 1 + a\right)\tau + \frac{a}{2}x} v(\tau, x)$

$$\frac{\partial u}{\partial \tau} = \frac{1}{K} \left(\frac{a^2}{4} + 1 + a \right) u(\tau, x) + \frac{1}{K} e^{\left(\frac{a^2}{4} + 1 + a\right)\tau + \frac{a}{2}x} \frac{\partial V}{\partial \tau}$$

$$\frac{\partial u}{\partial x} = \frac{a}{2K} e^{\left(\frac{a^2}{4} + 1 + a\right)\tau + \frac{a}{2}x} v(\tau, x) + \frac{1}{K} e^{\left(\frac{a^2}{4} + 1 + a\right)\tau + \frac{a}{2}x} \frac{\partial V}{\partial x}$$

$$\frac{\partial^2 u}{\partial^2 x} = \frac{a^2}{4} \frac{1}{K} e^{\left(\frac{a^2}{4}+1+a\right)\tau+\frac{a}{2}x} v(\tau, x) + \frac{a}{2} \frac{1}{K} e^{\left(\frac{a^2}{4}+1+a\right)\tau+\frac{a}{2}x} \frac{\partial V}{\partial x} + \frac{1}{K} e^{\left(\frac{a^2}{4}+1+a\right)\tau+\frac{a}{2}x} \frac{\partial^2 V}{\partial^2 x}$$

$$\frac{a^2}{4} u(\tau, x) + e^{\left(\frac{a^2}{4}+1+a\right)\tau+\frac{a}{2}x} \left(a \frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial^2 x} \right)$$

$$\begin{aligned} & \frac{a^2}{4} u(t, S) + e^{\left(\frac{a^2}{4}+1+a\right)\tau+\frac{a}{2}x} \left(\frac{\partial u}{\partial \tau} + (1+a)u \right) \\ &= \left(\frac{a^2}{4} + 1 + a \right) u(\tau, x) + e^{\left(\frac{a^2}{4}+1+a\right)\tau+\frac{a}{2}x} \frac{\partial V}{\partial \tau} = \frac{\partial u}{\partial \tau} \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial u(x, \tau)}{\partial \tau} &= \frac{\partial^2 u(x, \tau)}{\partial^2 x} \\ u(\tau, x) &= \frac{1}{K} e^{\left(\frac{a^2}{4}+1+a\right)\tau+\frac{a}{2}x} v(\tau, x) = \frac{1}{K} e^{\left(\frac{a^2}{4}+1+a\right)\tau+\frac{a}{2}x} v(t, S) \end{aligned}$$

$$V(s, t) = K e^{-\left(\frac{a^2}{4}+1+a\right)\tau-\frac{ax}{2}} u(x, \tau)$$

Crank-Nicolson scheme grid

We discretize the time and state space variables and obtain a two-dimensional grid. We need to solve the heat equation on the grid $\tau \in [0, \frac{1}{2}\sigma^2 T]$ and $x \in R$. The time space is discretized as t_0, t_1, \dots, t_m with equal steps of $\Delta\tau$ while the state space is discretized as x_0, x_1, \dots, x_{max} with intervals of Δx . We denote $u_{l,i}$ = the value at time t_l and state x_i . We apply the following heat equation as derived above.

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial^2 x}$$

We then approximate each of the following

$$\frac{\partial u}{\partial \tau} = \frac{u_{l,i+1} - u_{l,i}}{\Delta T}$$

$$\frac{\partial^2 u}{\partial^2 x} = \frac{(u_{l,i+1} - 2u_{l,i} + u_{l,i-1} + u_{l+1,i+1} - 2u_{l+1,i} + u_{l+1,i-1})}{2(\Delta x)^2}$$

The Crank-Nicolson scheme is given by:

$$u_{l+1,i} = \frac{\Delta T}{2(\Delta x)^2} (u_{l,i+1} - 2u_{l,i} + u_{l,i-1} + u_{l+1,i+1} - 2u_{l+1,i} + u_{l+1,i-1}) + u_{l,i}$$

For a double barrier option, the following boundary conditions are satisfied:

$$u_{0,i} = e^{\left(\frac{2r}{\sigma^2}-1\right)x_i} \max(e^{x_i} - 1, 0)$$

$$u_{l,i} = 0 \text{ for}$$

$$u_{0,i} = 0 \text{ if } x_i = 0$$

Range for τ

$$\tau \in [0, \frac{1}{2}\sigma^2 T]$$

$$0 < \tau < \frac{1}{2}\sigma^2$$

The price of an option is then given by:

$$V(s, t) = K e^{-\left(\frac{\sigma^2}{4}+1+a\right)\tau - \frac{ax}{2}} u_{l,i}$$

Monte-Carlo Methods

We assume that the stock price follows a geometric Brownian motion, we then simulate 1000 parts of the stock price. We use this to calculate the expected payoff of a European call option and discount at a risk free rate by the Risk neutral valuation. This is done as follows:

We assume a Black-Scholes market where:

$$dS_t = S_t r dt + \sigma S_t dW_t$$

$$S_T = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T}$$

$$dB_t = r B_t dt$$

Where W_t is the standard Brownian motion under the risk-neutral measure Q .

We then calculate the price of the call option using the following risk-neutral valuation formula

$$c_t = \frac{1}{1000} \left(\sum_i^N \max(0, S_T - K) \right) e^{-r(T-t)}$$

Black-Scholes formula

Black and Scholes (1973) derived a closed form solution for pricing European vanilla option. In a Black-Scholes market, the option is priced under the risk-neutral valuation with the assumption that the share price follows a geometric Brownian motion. The following is the Black-Scholes formula for European call option:

$$c_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)$$

And that of European put option is given by:

$$p_t = K e^{-r(T-t)} \Phi(-d_2) - S_t \Phi(-d_1)$$

where:

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T - t}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T - t}{\sigma\sqrt{T-t}}$$

Research Design

This study takes on a comparative study approach. This is because it is trying to compare two numerical option pricing method with a benchmark being Black-Scholes price

Nature of study

The study is quantitative in nature as it involves use of empirical data which to calculate option prices using each of the three approaches; Black-Scholes model, Monte-Carlo method and Crank-Nicolson scheme.

Population and sampling

The data used for this study was simulated using Matlab. With the same set of assumptions we calculate the option price using Black Scholes, Basic Monte-Carlo, Modified Monte-Carlo (using Antithetic) and Crank-Nicolson. The set assumptions were as follows initial share price of 80, 90, 95, 100, 105, 110, 115 and 120, stock volatility of 20%, time to maturity of one year, risk-free rate of 5%.

Data Collection and Procedure

The data we need for each selected company are the current share price, volatility, maturity, risk-free rate and the strike price of the call options. We use these to calculate the price of the option using the three methods; Black-Scholes model, Monte-Carlo method and Crank-Nicolson scheme.

Mean Absolute percentage error (MAPE)

This is an error statistic that we will use to determine the accuracy of each method. Here we calculate the error as a percentage of the Black-Scholes price, the actual price. This is calculated as follows:

$$MAPE = \left| \frac{C_{Model} - C_B}{C_B} \right|$$

C_{Model} – the price given by the model

The model with the lower value of MAPE is the more accurate.

Analysis of the Monte-Carlo Simulation

We calculate the stock price using the Geometric Brownian motion at different initial stock price. We generate 1000 random numbers, plug them in the equation, obtain the stock price at maturity, calculate the payoff, and discount the payoff at the risk-free rate. We then obtain the average and this becomes the Monte-Carlo price. This can be summarized as follows:

$$c_1 = e^{-rT} \frac{1}{1000} \sum \max[S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma Z\sqrt{T}} - K, 0]$$

Where:

$S_0 = (80, 120)$ – the stock price at time 0

$K = 100$ – The exercise price

$r = 0.05$ – Risk – free rate

$\sigma = 0.2$ – Stock price volatility

$T = 0.25$ – Maturity

$n = 1000$ – the number of Simulations

We calculated the option prices at different initial share prices using the two methods. For each initial share price, 1000 simulations were made. The table below shows the variation of the option price with the stock price for both the Monte-Carlo method and the Black-Scholes. Using the BSM price as the actual, we also calculate the Mean Absolute percentage errors (MAPE).

Table 1: Comparison between BSM Price and Monte-Carlo price

Share price	80	85	90	95	100	105	110	115	120
BSM price	0.0564	0.268	0.8975	2.271	4.615	7.922	11.98	16.54	21.35
Monte-Carlo price	0.0503	0.2703	1.019	2.181	4.451	7.383	12.03	16.79	21.53
MAPE	10.80%	0.86%	13.54%	3.96%	3.55%	6.80%	0.42%	1.51%	0.84%

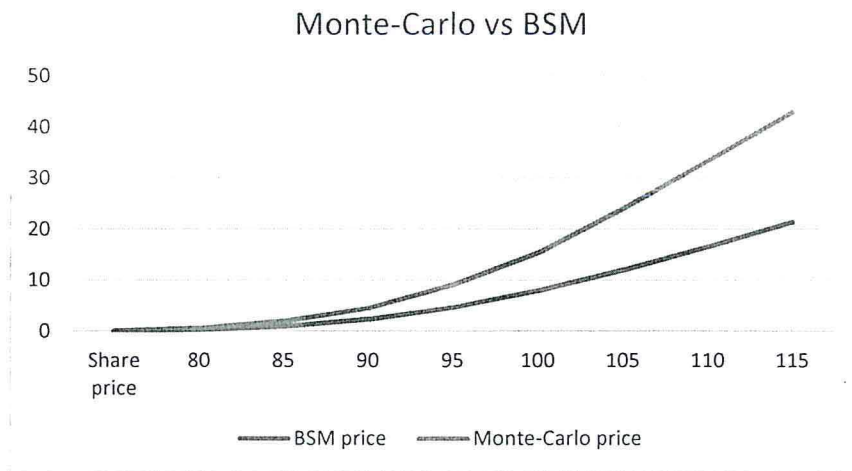


Figure 1: Graphical Representation of BSM vs. Monte-Carlo Price

Discussion of the Results

From the above graphical representation, the Monte-Carlo prices are higher than the BSM prices for all the values of the initial share prices. Monte-Carlo method tends to generally overestimate the price of the call option.

Analysis of the Antithetic variate technique

The method involves calculating the option price using two estimates of the share price. Each estimate is calculated using the same set of random numbers. In approximating the second estimate the random numbers are negated. The payoffs for each are then obtained and discounted to the present. The Antithetic price is obtained by getting the average of the Monte-Carlo prices. This can be illustrated below:

$$c_1 = e^{-rT} \frac{1}{1000} \sum \max[S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma Z\sqrt{T}} - K, 0]$$

$$c_2 = e^{-rT} \frac{1}{1000} \sum \max[S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma(-Z)\sqrt{T}} - K, 0]$$

Where:

$S_0 = (80, 120)$ – the stock price at time 0

$K = 100$ – The exercise price

$r = 0.05$ – Risk – free rate

$\sigma = 0.2$ – Stock price volatility

$T = 0.25$ – Maturity

$n = 1000$ – the number of Simulations

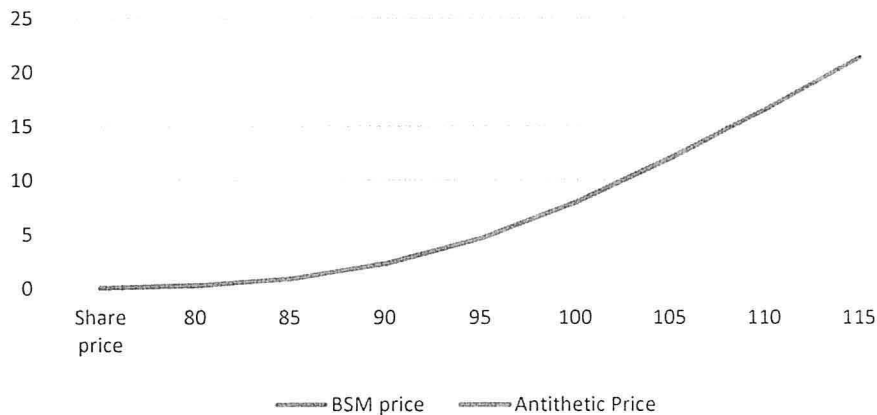
The Antithetic price is given by:

$$c_a = \frac{c_1 + c_2}{2}$$

We calculated the option prices at different initial share prices using the two methods. For each initial share price, 1000 simulations were made. The table below shows the variation of the option price with the stock price for both the Antithetic variate of Monte-Carlo method and the Black-Scholes. Using the BSM price as the actual, we also calculate the Mean Absolute percentage errors (MAPE).

Share price	80	85	90	95	100	105	110	115	120
BSM price	0.0564	0.268	0.8975	2.271	4.615	7.922	11.98	16.54	21.35
Antithetic Price	0.0472	0.2937	0.8545	2.279	4.615	7.968	12.07	16.53	21.35
MAPE	0.92%	2.57%	4.30%	0.84%	0.00%	4.60%	9.00%	1.00%	0.00%

Antithetic Monte- Carlo price vs BSM Price



Discussion of the Results:

From the above graphical representation, we observe that the Antithetic variate price converges (they are on the same curve) to the BSM price.

Analysis of Crank-Nicolson Results

We discretize the time and state space variables and obtain a two-dimensional grid. We need to solve the heat equation on the grid $\tau \in [0, \frac{1}{2}\sigma^2 T]$ and $x \in R$. The time space is discretized as t_0, t_1, \dots, t_m with equal steps of $\Delta\tau$ while the state space is discretized as $x_0, x_1 \dots x_{max}$ with intervals of Δx . We denote $u_{l,i}$ = the value at time t_l and state x_i . The Crank-Nicolson scheme is given by:

$$u_{l+1,i} = \frac{\Delta T}{2(\Delta x)^2} (u_{l,i+1} - 2u_{l,i} + u_{l,i-1} + u_{l+1,i+1} - 2u_{l+1,i} + u_{l+1,i-1}) + u_{l,i}$$

Share price	80	85	90	95	100	105	110	115	120
x	-0.223	-0.163	-0.1054	-0.051	0	0.0487	0.0953	0.14	0.1823
$u(x, 1)$	0.005059	0.0234	0.0968	0.2613	0.5508	1.014	1.5833	2.2725	3.005

We start by calculating $u_{0,i}$ using the formula below:

$$u_{0,i} = e^{\left(\frac{2r}{\sigma^2}-1\right)x_i} \max(1 - e^{x_i}, 0)$$

The following are the results for $u(x, 1)$ at various stock prices

The price of the option is then given by:

$$V(S, 0) = Ke^{-\left(\frac{\sigma^2}{4}+1+a\right)\tau - \frac{ax}{2}} u(x, 1)$$

$$\text{Where: } x = \log\left(\frac{S}{K}\right) \quad \tau = \frac{(T-t)\sigma^2}{2} = 0.02 \quad a = \frac{2r}{\sigma^2} - 1 = 1.5$$

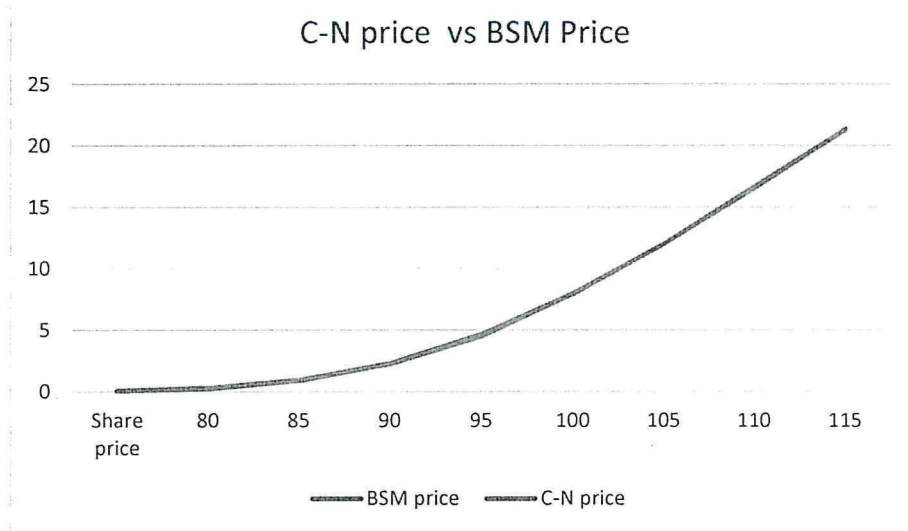
These put together give the price of the option as:

$$V(S, 0) = 8.1268e^{-0.75x} u(x, 1)$$

For each initial share price, 1000 simulations were made. The table below shows the variation of the option price with the stock price for both the Crank-Nicolson method and the Black-Scholes. Using the BSM price as the actual, we also calculate the Mean Absolute percentage errors (MAPE).

Share price	80	85	90	95	100	105	110	115	120
BSM price	0.0564	0.268	0.8975	2.271	4.615	7.922	11.98	16.54	21.35
C-N Price	0.0486	0.215	0.851	2.207	4.476	7.944	11.98	16.63	21.30
MAPE	13.83%	19.78%	5.18%	2.82%	3.01%	0.28%	0.00%	0.54%	0.23%

Table 2: Results from Crank- Nicolson Price



The Antithetic variate makes the Monte-Carlo price more accurate than the basic Monte-Carlo method

Comparison of Crank-Nicolson and the Crude (Basic) Monte-Carlo

In this case, Crank Nicolson proved to be more accurate due to its smaller value of MAPE of as compared to the crude Monte- Carlo.

Comparison of Crank-Nicolson and the Antithetic Monte-Carlo

When we modify the Monte-Carlo approach and introduce the Antithetic variate, we end up with a lower MAPE, lower than that of Crank-Nicolson.

CHAPTER 5: CONCLUSION AND RECOMMENDATIONS

From the above findings, we note that the Crank-Nicolson scheme is more accurate compared to the crude Monte-Carlo method (without the Antithetic variate). This finding is consistent with Fadugba et al (2012) who also found out that Crank Nicolson finite difference method converges faster and more accurate, it is fairly robust and good for pricing European put and call options. However, if we introduce the Antithetic variate to the Monte-Carlo Method, it outperforms the Crank-Nicolson as it leads to the smallest value of MAPE. In addition to that, Crank Nicolson method requires sophisticated algorithms for solving large sparse linear systems of equations, somewhat problematic for path dependent options and is relatively difficult to code.

We therefore recommend the use of Monte-Carlo with Antithetic variate as opposed to Crank-Nicolson as it is more accurate and easier to code.

Areas of Further Study

The Crank Nicolson method is just one of the many finite difference methods that there are, another comparative study can be done involving Monte-Carlo and other finite difference methods. Secondly, this study focused on the plain Vanilla options, a further comparative study could be done on American options and Exotic options.

APPENDIX

```
classdef comparativestudyofBSM
    %UNTITLED2 Summary of this class goes here
    % Detailed explanation goes here

    properties
        So;
        K;
        r;
        Sigma;
        T;
        N = 1000;
        Z;
        St;
        Payoff;
        BSM_value;
        Anthithetic_Montecarlo_Price;
        Montecarlo_Price;
    end

    methods
        function f = set.So(f,value)
            f.So= value;
        end
        function f = set.K(f,value)
            f.K = value;
        end
        function f = set.Sigma(f,value)
            f.Sigma= value;
        end
        function f = set.T(f,value)
            f.T = value;
        end
        function f = set_Z(f)
            f.Z = norminv(rand(1,f.N),0,1);
        end
        function f = St_Calc(f)
            f.St = f.So*exp((f.r-0.*f.Sigma^2)*f.T + f.Sigma*f.Z*sqrt(f.T));
        end
        function f= Payoff_Calc(f)
            f.Payoff = (sum(max((f.St-f.K),0))/f.N)*exp(-f.r*f.T);
            f.Montecarlo_Price = f.Payoff;
        end
        function f = BSM_Calc(f)
            d1=((log(f.So/f.K)+(f.r+0.5*f.Sigma^2))*f.T)/(f.Sigma*sqrt(f.T));
            d2 = d1- f.Sigma*sqrt(f.T);
            f.BSM_value = f.So*normcdf(d1,0,1)-f.K*exp(-f.r*f.T)*normcdf(d2,0,1);
        end
        function f = Anthithetic_Montecarlo_Price_Calc(f)
            f.St = f.So*exp((f.r-0.*f.Sigma^2)*f.T + f.Sigma*-f.Z*sqrt(f.T));
            Price_neg_Z = (sum(max((f.St-f.K),0))/f.N)*exp(-f.r*f.T);
            f.Anthithetic_Montecarlo_Price = (Price_neg_Z+f.Payoff)/2;
        end
    end
end
```

```

end

end

AnalysisVar = comparativestudyofBSM;
AnalysisVar.So = 100;
AnalysisVar.K = 100;
AnalysisVar.r = 0.05;
AnalysisVar.Sigma = 0.2;
AnalysisVar.T = 0.25;
AnalysisVar.N = 10000;
AnalysisVar = AnalysisVar.Set_Z;
AnalysisVar = AnalysisVar.St_Calc;
AnalysisVar = AnalysisVar.Payoff_Calc;
AnalysisVar.Montecarlo_Price
AnalysisVar = AnalysisVar.BSM_Calc;
AnalysisVar.BSM_value

So_s = 80:120;
n=1;
for So = So_s
    AnalysisVar.So =So;
    AnalysisVar = AnalysisVar.BSM_Calc;
    AnalysisVar = AnalysisVar.St_Calc;
    AnalysisVar = AnalysisVar.Payoff_Calc;
    AnalysisVar = AnalysisVar.Anthithetic_Montecarlo_Price_Calc;
    Montecarlo(n) = AnalysisVar.Montecarlo_Price;
    BSM_value(n) = AnalysisVar.BSM_value;
    Anthithetic_Montecarlo_Price(n) = AnalysisVar.Anthithetic_Montecarlo_Price;
    n=n+1;
end
plot(So_s,BSM_value,'DisplayName','BSM');
hold on;
plot (So_s,Montecarlo,'DisplayName','Monte Carlo');
hold on;
plot(So_s,Anthithetic_Montecarlo_Price,'DisplayName','Anthithetic Monte Carlo');
title('Comparative study of BSM','Interpreter','latex');
xlabel('$S_{0}$','Interpreter','latex');
ylabel('$Price$','Interpreter','latex');
legend('show');

```

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