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MohammadReza Molaei
Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.
mrmolaei@uk.ac.ir

Abstract

In this paper we introduce cocycles of multifunctions, and we study the concept of attractors for them by using of semibornologies. We define a kind of conjugate relation on them, and we show that a conjugacy takes an attractor to an attractor. We consider the role of semibornology on the existence of new attractors.

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1. Introduction

Cocycles in continuous or discrete cases play an important role in the modeling of nature phenomena [2, 5]. Here we extend this notion to multifunctions. We introduce the concept of attractors for cocycles of multifunctions. If in a cocycle all the multifunctions are equal, then this notion is equivalent to the concept of attractors for multifunctions, which has been studied first by Lasota and Myjak [6]. We present an equivalence relation on cocycles, and we show that if two cocycles are conjugate with a conjugacy h, then h takes an attractor of the first cocycle to an attractor of the second one. We consider the effect of semibornology on an attractor of a cocycle. We present a class of cocycles by using of top spaces.

2. Non-autonomous dynamical systems created by multifunctions

Suppose X_1 and X_2 are two sets, a relation $F_1 : X_1 \to X_2$ is called a multifunction if X_1 is the domain of F_1. If $X = \{X_i\}_{i=0}^{\infty}$ is a sequence of sets and $F = \{F_i \ : \ F_i : X_i \to X_{i+1}\}_{i=0}^{\infty}$ is a sequence of multifunctions, then we say that (X, F) is a cocycle of multifunctions. In this kind of dynamics we have many choices to continue a process. This situation occurs in the meanings of sentences. For example saying one sentence by a person can have different meanings for the others, and the new sentences of each of them, also can have different meanings for the others. To continue this process we find a cocycle of multifunctions.
When each F_i is a function, then (X, F) is a non-autonomous dynamical system [4]. So generally a cocycle of multifunctions creates a class of non-autonomous dynamical systems.
Iterated function systems [1] are also a special case of cocycle of multifunction. In a cocycle if $X_0 = X_1 = X_2 = ...$, $F_0 = F_1 = F_2 = ...$, and for each $x \in X_0$, the cardinality of $F_0(x)$ is a constant natural number, then it is an iterated function system.

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We assume that β is a semibornology for $\bigcup_{i=0}^{\infty} X_i$ [9], this means that β is a cover for $\bigcup_{i=0}^{\infty} X_i$, and it is closed under finite unions. A semibornology β is called a bornology [3] if any subset of each member of β is a member of it. For example the set of bounded subsets of a metric space is a bornology for it, and the set of closures of the subsets of a topological space is a semibornology for it.

We assume that $(A_i)_{i=0}^{\infty}$ is a sequence with $A_i \subseteq X_i$. The lower bound of $(A_i)_{i=0}^{\infty}$ is denoted by LiA_i and it is the set of $x \in \bigcup_{i=0}^{\infty} X_i$ such that for each $B \in \beta$ with $x \in B$ there is $i_0 \in N \cup \{0\}$ such that $B \cap A_i \neq \emptyset$ for all $i \geq i_0$. The upper bound of $(A_i)_{i=0}^{\infty}$ is denoted by LsA_i, and it is the set of $x \in \bigcup_{i=0}^{\infty} X_i$ such that for all $B \in \beta$ with $x \in B$ there exists $i_0 \in N \cup \{0\}$ so that $B \cap A_i \neq \emptyset$ for infinitely many $i \geq i_0$. We say that $(A_i)_{i=0}^{\infty}$ has bounded limit if $LsA_i = LiA_i$. The bounded limit of (A_i) in the case of existence is denoted by LbA_i, and it is defined by $LbA_i = LsA_i$.

Suppose $h : \bigcup_{i=0}^{\infty} X_i \to \bigcup_{i=0}^{\infty} X_i$ is a bijection with $h_i(X_i) = X_i$, where h_i is the restriction of h to X_i, and $i \in \{0, 1, 2, ...\}$. We say that h is a semibornological bijection if $h(B) \in \beta$, *and* $h^{-1}(B) \in \beta$, for all $B \in \beta$.

Theorem 2.1. *If h is a semibornological bijection, then $h(LiA_i) = Lih_i(A_i)$, $h(LsA_i) = Lsh_i(A_i)$, and $h(LbA_i) = Lbh_i(A_i)$.*

Proof. We only prove that $h(LiA_i) = Lih_i(A_i)$. The other parts can prove by similar method.

$$\begin{gathered} x \in LiA_i \Leftrightarrow \textit{ For all } h(B) \in \beta \textit{ with } x \in h(B) \textit{ there is } i_0 \textit{ such that} \\ h(B) \cap h_i(A_i) \neq \emptyset \textit{ for all } i \geq i_0. \Leftrightarrow \textit{For all } B \in \beta \textit{ with } h^{-1}(x) \in B \textit{ there is } i_0 \\ \textit{such that } h(B \cap A_i) \neq \emptyset \textit{ for all } i \geq i_0. \Leftrightarrow \textit{For all } B \in \beta \textit{ with } h^{-1}(x) \in B \textit{ there is } i_0 \\ \textit{such that } B \cap A_i \neq \emptyset \textit{ for all } i \geq i_0. \Leftrightarrow h^{-1}(x) \in LiA_i. \end{gathered}$$

Thus $h(LiA_i) = Lih_i(A_i)$.

■

3. Attractors

We assume that X and F are the sets of previous section and β is a semibornology for $\bigcup_{i=0}^{\infty} X_i$.

Definition 3.1. *A non-empty subset A of $\bigcup_{i=0}^{\infty} X_i$ is called an attractor for a cocycle (X, F) if for all $\emptyset \neq B \in \beta$ with $A \subseteq B$, we have $A = LbF_i(B \cap X_i)$.*

Example 3.2. *For $i \in \{0, 1, 2, ...\}$ we take $X_i = R^{i+1}$, and we define*

$$\begin{gathered} F_i : R^{i+1} \to R^{i+2} \\ (x_1, x_2, x_3, ..., x_{i+1}) \mapsto \{(\frac{1}{2^j}x_1, \frac{1}{2^j}x_2, ..., \frac{1}{2^j}x_{i+1}, \frac{1}{2^j}x_{i+1}) \ : \ j \geq i\}. \end{gathered}$$

If $\beta = \{\bigcup_{i=0}^{\infty} B_i \ : \ B_i$ is the closure of an open set in $R^{i+1}\}$, then $\{0, (0, 0), (0, 0, 0), ...\}$ is the attractor of (X, F).

Now we assume that (X, F) and (X, G) are two cocycles of multifunctions and β is a semibornology for $\bigcup_{i=0}^{\infty} X_i$. Moreover we assume that $h : \bigcup_{i=0}^{\infty} X_i \to \bigcup_{i=0}^{\infty} X_i$ is a bijection such that $h_i(X_i) = X_i$, where h_i is the restriction of h to X_i, and $i \in \{0, 1, 2, ...\}$. We say that (X, F) is conjugate to (X, G) under the conjugacy h if h is a semibornological bijection, and for given $i \in \{0\} \cup N$, the following diagram commutes.

$$\begin{array}{ccc} X_i & \xrightarrow{F_i} & X_{i+1} \\ h_i \downarrow & & \downarrow h_{i+1} \\ X_i & \xrightarrow{G_i} & X_{i+1} \end{array} .$$

Theorem 3.3. *If A is an attractor for (X, F), and if (X, F) is conjugate to (X, G) under a conjugacy h then $h(A)$ is an attractor for (X, G).*

Proof. Suppose $B \in \beta$ with $h(A) \subseteq B$ be given. Since $h^{-1}(B) \in \beta$, and $A \subseteq h^{-1}(B)$, then $A = LbF_i(h^{-1}(B) \cap X_i)$. Theorem 2.1. implies $h(A) = Lbh_{i+1}(F_i(h^{-1}(B) \cap X_i)$. Thus

$$h(A) = LbG_i(h_i(h^{-1}(B) \cap X_i)) = LbG_i(B \cap X_i).$$

So $h(A)$ is an attractor for (X, G).

■

If (X, F) is a cocycle of multifunctions and β is a semibornology for $X = \bigcup_{i=0}^{\infty} X_i$, then we define $C(\beta)$ by

$$C(\beta) = \{D \in \beta \ : \ For\ given\ y \in D\ and\ for\ all\ x\ if\ there\ is\ j\ such\ that\ y \in F_j(x)$$
$$then\ there\ is\ B \in \beta\ with\ x \in B\ such\ that\ F_i(B \cap X_i) = D \cap X_{i+1}\ for\ all\ i\}.$$

If we denote an attractor of (X, F) with respect to a semibornology β by A^{β}, then we have the next theorem.

Theorem 3.4. *If $C(\beta)$ is a cover for X, then $C(\beta)$ is a semibornology for X. Moreover if A^{β} is an attractor and if there is $E \in C(\beta)$ such that $A^{\beta} \subseteq E$ then there is an attractor $A^{C(\beta)}$ such that $A^{\beta} \subseteq A^{C(\beta)}$.*

Proof. Let $D_1, D_2 \in C(\beta)$ and $y \in D_1 \cup D_2$ be given. Without loss of generality we assume that $y \in D_1$. We choose $z \in D_2$, then there exist $a \in X$ and $l \in N \cup \{0\}$ such that $z \in F_l(a)$. If $y \in F_j(x)$ for some j, then there exist $B_1, B_2 \in \beta$ such that $x \in B_1$ and $a \in B_2$ and $F_i(B_j \cap X_i) = D_j \cap X_{i+1}$ for all $i \in N \cup \{0\}$. For given $i \in N \cup \{0\}$, $F_i((B_1 \cup B_2) \cap X_i) = F_i(B_1 \cap X_i) \cup F_i(B_2 \cap X_i) = (D_1 \cap X_{i+1}) \cup (D_2 \cap X_{i+1}) = (D_1 \cup D_2) \cap X_{i+1}$. Since $x \in B_1 \cup B_2 \in \beta$, then $D_1 \cup D_2 \in C(\beta)$. So it is a semibornology for X. If $x \in A^{\beta}$, then for all $B, E \in \beta$ with $x \in B$, and $A^{\beta} \subseteq E$ there is i_0 such that for infinitely $i \geq i_0$, $B \cap F_i(X \cap E) \neq \emptyset$, and there is $k \in N$ so that for all $i \geq i_0 + k$, $B \cap F_i(X \cap E) \neq \emptyset$. Since $C(\beta) \subseteq \beta$, then for all $B, E \in C(\beta)$ with $x \in B$, and $A^{\beta} \subseteq E$ there is i_0 such that for infinitely $i \geq i_0$, $B \cap F_i(X \cap E) \neq \emptyset$, and there is $k \in N$ so that for all $i \geq i_0 + k$, $B \cap F_i(X \cap E) \neq \emptyset$. Thus $x \in A^{C(\beta)}$.

■

Theorem 3.5. *Suppose $x \in A^{\beta} \cap X_j$ for some j. If $C(\beta)$ is a cover for X, then $F_j(x) \subseteq A^{C(\beta)}$.*

Proof. Let $y \in F_j(x)$, $E \in C(\beta)$ with $y \in E$, and $D \in C(\beta)$ with $A^{C(\beta)} \subseteq D$ be given. Then there is $B \in \beta$ with $x \in B$ such that $F_i(B \cap X_i) = E \cap X_{i+1}$ for all $i \in N \cup \{0\}$. Since $D \in C(\beta)$, then there is $S \in \beta$ such that $F_i(S \cap X_i) = D \cap X_{i+1}$ for all $i \in N \cup \{0\}$. We know that $x \in A^{\beta}$

so there is i_0 such that for infinitely $i \geq i_0$, $B \cap F_{i-1}(S \cap X_{i-1}) \neq \emptyset$, and there is $k \in N$ so that for all $i \geq i_0 + k$, $B \cap F_{i-1}(S \cap X_{i-1}) \neq \emptyset$. We have

$$E \cap F_i(D \cap X_i) = E \cap X_{i+1} \cap F_i(D \cap X_i) = F_i(B \cap X_i) \cap F_i(F_{i-1}(S \cap X_{i-1})) \supseteq F_i(B \cap X_i \cap F_{i-1}(S \cap X_{i-1})).$$

Thus for infinitely $i \geq i_0$, $E \cap F_i(D \cap X_i) \neq \emptyset$, and for all $i \geq i_0 + k$, $E \cap F_i(D \cap X_i) \neq \emptyset$. Hence $y \in A^{C(\beta)}$.

■

4. Cocycle of multifunctions created by a family of diffeomorphisms on top spaces

We begin this section by recalling the definition of a top space [7]. A smooth manifold T is called a top space if it has a binary smooth operator

$$\begin{array}{rcl} m : T \times T & \longrightarrow & T \\ (a, b) & \mapsto & ab \end{array}$$

with the following conditions.
(i) (T, m) is a semigroup;
(ii) For all $a \in T$, there is a unique $e(a) \in T$ such that $ae(a) = e(a)a = a$;
(iii) For all $a \in T$ there is $a^{-1} \in T$ such that $aa^{-1} = a^{-1}a = e(a)$, and the mapping

$$\begin{array}{rcl} in : T & \longrightarrow & T \\ a & \mapsto & a^{-1} \end{array}$$

is a smooth mapping.
Clearly each Lie group is a top space, but the converse may not be true. We can use of Rees matrix semigroups [10] to construct top spaces which are not Lie groups. In fact if G is a Lie group, M and N are two smooth manifolds, and $p : M \times N \to G$ is a smooth mapping then $N \times G \times M$ with the product $(n, a, s)(k, b, l) = (n, ap(s, k)b, l)$ is a top space [8].

Now we assume that $T = \{T_i\}_{i=0}^{\infty}$ is a family of top spaces. We take

$$f = \{f_i \mid f_i : T_i \to T_i \ is\ a\ diffeomorphism\}_{i=0}^{\infty},\ and$$

$$g = \{g_i \mid g_i : T_i \to T_{i+1}\ is\ a\ smooth\ map\}_{i=0}^{\infty}.$$

For given i we define a multifunction $F_i : T_i \to T_{i+1}$ by $F_i(t) = \{g_{i,j}(t) \mid j \in Z\}$ where

$$g_{i,j}(t) = \begin{cases} g_i(f_i^{j-1}(t))...g_i(f_i(t))g_i(t) & if\ j > 0 \\ g_i(f_i^{-j}(t))...g_i(f_i^{-1}(t)) & if\ j < 0 \\ e_{i+1}(g_i(t)) & if\ j = 0 \end{cases}.$$

If $F = \{F_i\}_{i=0}^{\infty}$, then (T, F) is a cocycle of multifunctions.

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MohammadReza Molaei
e-mail: mrmolaei@uk.ac.ir
Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.