UNIVERSITY EXAMINATION
Strathmore Institute of Mathematical Sciences
Bachelor of Business Science (Financial Engineering)
BSM 3114 - Differential Equations with Financial Applications

Date: Monday 31^{st} July 2023 Time: 2 Hours

Instruction

1. Answer **QUESTION ONE** and any other **TWO QUESTIONS**

QUESTION ONE [30 Marks]

a) Determine the order and degree of the following differential equations:]

   i. $y' + (y'')^2 = (x + y'')^2$ [2 Mark]

   ii. $\left[3 + \frac{d^2y}{dx^2}\right]^{1.5} = J^2 \left[\frac{d^2y}{dx^2}\right]^2$ [2 Mark]

   iii. $y = 7\left[\frac{dy}{dx}\right]^2 + 4x\left[\frac{dx}{dy}\right]$ [2 Mark]

b) Solve using the integrating factor method the differential equation: . [4 Marks]

$$\frac{dy}{dx} + yCot(x) = Cos(x)$$

c) Given that $y_1 = e^x$ is a solution to $(x-1)y'' - xy' + y = 0$. Further, use reduction of order to find the second solution and in turn the general solution. . [4 Marks]

d) Express the differential equation $u''u'u = 1 + t^2$ as a system of first order linear differential equations. [2 Marks]

e) Find all functions y_1 and y_2 such satisfies the following system

$$y_1' = y_1 - 3y_2$$
$$y_2' = y_1 + 5y_2$$

. [5 Marks]

f) Given a homogeneous system of two linear first order differential equations in two unknowns. If its corresponding matrix gives a repeated eigenvalue Σ. Write its general solution:

i. When two linearly independent eigenvectors exists. [1 Mark]

ii. When only one linearly independent eigenvector exists. [1 Mark]

g) By making a substitution of the form $y = vx$ where v is a function of x, solve the initial value problem $xy' = y\,[ln(x) - ln(y)]$, $y(1) = 4$.
. [4 Marks]

h) Sketch a phase plane diagram for the system. [3 Marks]

$$x_1' = -x_1$$
$$x_2' = 4x_2$$

i) Solve by separation of variables the differential equation $y' = e^{2x+y}$ given that $x = 0$ when $y = 0$. [3 Marks]

QUESTION TWO [20 Marks]

a) A Bernoulli differential equation (named after James Bernoulli) is of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$. Determine whether the equation is linear or nonlinear when $n = 0$ and $n = 1$. Further, show that the substitution $u = y^{n-1}$ transforms the Bernoulli equation into the linear equation; [5 Marks]

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

b) Let $P(t)$ be the performance level of someone learning a skill as a function of training time t. The graph of P is called a learning curve and can be defined by

$$\frac{dP}{dt} = k(M - P(t))$$

for some positive constant k. Is the equation linear or non linear?, solve it using the integrating factor method. [5 Marks]

c) Find the constant $\lambda \in \mathbb{R}$ such that $y(t) = x^{\lambda}$ is a solution to the differential equation $x^2y'' + xy' = y$. [3 Marks]

d) Two workers are hired for an assembly line. Paul processed 25 units during the first hour and 45 units during the second hour. Betty processed 35 units during the first hour and 50 units during the second hour. Using the model $P(t) = M + Ce^{-kt}$ where $P(t)$ is the performance level of someone learning a new skill as a function of training time t. Determine the number of units that can be produced in one hour by each worker. [7 Marks]

QUESTION THREE [20 Marks]

a) Solve the following systems:

i. $\mathbf{x}' = A\mathbf{x}$ given that $A = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$ [4 Marks]

ii. $\mathbf{y}' = B\mathbf{y}$ given that $B = \begin{bmatrix} -2 & 6 \\ -3 & 4 \end{bmatrix}$ [5 Marks]

iii. $\mathbf{t}' = C\mathbf{t}$ given that $C = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$ if $\mathbf{t}(0) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ [5 marks]

b) Find the general solution of the following system of differential equations:
. [6 marks]

$$\begin{aligned} y_1' &= 3y_1 - 7y_2 - 3y_3 \\ y_2' &= y_1 - 4y_2 - 2y_3 \\ y_3' &= y_1 + 2y_2 + 2y_3 \end{aligned}$$

QUESTION FOUR [20 Marks]

a) Use the method of integrating factor to solve $x\frac{dy}{dx} - 2y = x^4 Sin(x)$ given that . [5 Marks]

b) Find the general solution to $2t^2y'' + ty' - 3y = 0$ given that $y_1 = t^{-1}$ is one solution. [5 Marks]

c) By diving through by x^2 and making the substitution $y = vx$ where v is also a function of x, solve the IVP $xyy' + 4x^2 + y^2 = 0$ given that $y(2) = -7$. [5 Marks]

d) Given $y' - (4x - y + 1)^2 = 0$ with the condition $y(0) = 2$, make the substitution $v = 4x - y$ and find find its solution. [5 Marks]

QUESTION FIVE [20 Marks]

a) Given

$$\frac{da}{dS} = b^2 - ab - b$$
$$\frac{db}{dS} = 2a^2 + ab - 7a.$$

Find the equilibrium point(s), phase plane diagram and the relationship between the variables a and b. [8 Marks]

b) Consider the non-linear system

$$x' = -x + xy$$
$$y' = xy - 2y.$$

Determine:

i. The critical point(s) [2 Marks]
ii. The Jacobian matrix/matrices [2 Marks]
iii. The eigenvalues of ii above. [2 Marks]
iv. The corresponding eigenvectors. [2 Marks]
v. The trajectories of the system [2 Marks]

c) List the two methods that can be used to analyse a non linear system of the form [2 Marks]

$$\frac{dp}{dR} = F(p, q)$$
$$\frac{dq}{dR} = G(p, q)$$