Strathmore Institute of Mathematical Sciences

Master of Science in Statistical Science

End of Semester Examination

STA 8202: Stochastic Processes

Date: 15th December 2021 **Duration: 3 Hours**

Instructions:

1. Answer **Question 1** and **any other two** questions.
2. Show all your workings clearly in the answer sheet

Question 1 (30 Marks)

(a) Explain the meaning of the following terms as used in stochastic processes:

(i) A sample path [1 Mark].

(ii) A point process [1 Mark].

(iii) Branching Process [1 Mark].

(iv) A Martingale [1 Mark].

(v) A *birth-death* process [1 Mark].

(b) Consider a steady state in performance measure of a simple queuing system *(M/M/1)* with arrival rate λ and service rate μ.
Let $P_n(t) = P$ [n customers in the system at time t]. Obtain:

(i) Expected number of customers in the system, $E(n)$. [3 Marks]

(ii) Average waiting time in the system including service time, $E(w)$. [2 Marks].

(c) A gambler bets *1 US dollar* on the first game. If he wins he gets a net profit of *1 US dollar* but if he loses he suffers a loss of *negative 1 US dollar* and will bet *2 US dollars* in the next play. If he wins he will get *4 US dollars* and hence will have made a net profit of *1 US dollar* but if he loses he will bet *4 US dollars* on the next game and so on.

(i) Develop the gambler's net profit table if the gambler loses 5 times in a row and then wins. [2 Marks].

(ii) Using the martingale technique, develop the net-winning process of the gambler. [3 Marks].

(d) Let $\{X_n, n \geq 0\}$ be a Markov chain with three states $0, 1, 2$ and with transition matrix given by

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix} \end{array}$$

and the initial distribution $P_r\{X_0 = i\} = \frac{1}{3}$, $i = 0, 1, 2$. Obtain:

(i) $P_r\{X_1 = 1/X_0 = 2\}$

[1 Mark]

(ii)

$$P_r\{X_2 = 2, X_1 = 1/X_0 = 2\}$$

[2 Marks]

(iii)

$$P_r\{X_2 = 2, X_1 = 1, X_0 = 2\}$$

[2 Marks]

(iv) Find the probability of obtaining the trajectory $(2, 1, 0, 0, 2)$. [3 Marks].

(e) Suppose that in the *pre-colonial* era, Nairobi University (UoN), Kenyatta University (KU), and Moi University (MoU) admitted only male students. Assume that, at that time, 80 percent of the sons of UoN men went to UoN and the rest went to MoU, 40 percent of the sons of MoU men went to MoU, and the rest split evenly between UoN and KU; and of the sons of KU men, 70 percent went to KU, 20 percent to UoN, and 10 percent to MoU.

(i) Find the probability that the grandson of a man from UoN went to UoN. [3 Marks]

(ii) Modify the above by assuming that the son of a UoN man always went to UoN and write down the probability matrix. [2 Marks]

(iii) Using the probability matrix in **(ii)** above, find the probability that the grandson of a man from UoN went to UoN.

[2 Marks]

Question 2 (15 Marks)

(a) State **three** conditions that determine the performance measures of a queuing system. [3 Marks]

(b) Assuming *M/M/1 Queuing Model*, show that:

(i) Expected waiting time in the queue is

$$E(w) = \frac{\lambda}{\mu(\mu - \lambda)}$$

[3 Marks]

(ii) Expected length of the queue is

$$E(m) = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

where μ and λ are the average number of customers being served per unit time and the average number of customers arriving per unit time respectively. [3 Marks]

(c) At one-man barber shop, customers arrive according to Poisson distribution at a mean arrival rate of 5 % per hour and the hair-cutting is exponentially distributed with the average hair-cut taking 10 minutes. It is assumed that because of his excellent reputation, customers are always willing to wait. Calculate the following:

(i) Average number of customers in the shop and the average number of customers waiting to be served. [2 Marks]

(ii) The percentage of time an arrival can walk right in without having to wait. [2 Marks]

(iii) The percentage of customers who have to wait prior to getting in the barber's chair. [2 Marks]

Question 3 (15 Marks)

A Markov chain with non-negative integers as its state space has transition matrix given by:

$$\begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} E_0 & E_1 & E_2 & E_3 & E_4 & \cdots \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & & \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots & & \\
0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & \cdots & & \\
0 & 0 & \frac{7}{8} & 0 & \frac{1}{8} & \cdots & & \\
0 & 0 & 0 & \frac{15}{16} & 0 & \frac{1}{16} & 0 & \cdots \\
 & \cdots & & & & & &
\end{bmatrix}
\end{array}$$

(a) Write down the formula for the transition probabilities. [3 Marks]

(b) Let the stationary vector be $[V_0, V_1, V_2, V_3, \cdots]$. Write a difference-differential equation giving V_x in terms of V_{x-1} and V_{x+1}.

[4 Marks]

(c) Find $P_{33}^{(2)}$ [3 Marks]

(d) Let $G(s) = \sum_{j=0}^{\infty} V_j s^j$. Show that $G(s) = p(s)G(\frac{1}{2}s)$, where $p(s)$ is a polynomial in s. [5 Marks]

Question 4 (15 Marks)

(a) Consider the difference-differential equations of the zero growth rate *birth-death* process given by:

$$P_n'(t) = -2\lambda_n P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\lambda P_{n+1}(t), \ n \geq 1$$

$$P_0'(t) = \lambda P_1(t), \quad n = 0$$

where $P_n(t)$ is the probability that the population is of size n at a time , t, and λ represents a birth in the population. Let $\lambda_n = n\lambda$.
Derive the mean and variance of the zero growth rate process using the probability generating function $(p.g.f)$ technique where λ implies birth.
Use the initial condition

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

[10 Marks].

(b) Consider a Markov chain with 3 states $S = \{1, 2, 3\}$ that has the following transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(i) Draw the state transition diagram for this chain. [2 marks]

(ii) If it's known that $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, find $P(X_1 = 3, X_2 = 2, X_3 = 1)$. [3 marks]

Question 5 (15 Marks)

(a) Assume that a man's profession can be classified as professional, skilled labourer, or unskilled labourer. Assume that, of the daughters of professional men, 80 percent are professional, 10 percent are skilled labourers, and 10 percent are unskilled labourers. In the case of daughters of skilled labourers, 60 percent are skilled labourers, 20 percent are professional, and 20 percent are unskilled. Finally, in the case of unskilled labourers, 50 percent of the daughters are unskilled labourers, and 25 percent each are in the other two categories. Assume that every man has at least one daughter, and form a Markov chain by following the profession of a randomly selected daughter of a given family through several generations.

(i) Set up the matrix of transition probabilities. [3 marks]

(ii) Find the probability that a randomly selected grand-daughter of an unskilled labourer is a professional woman. [3 marks]

(b) Consider a homogeneous Markov chain with state space $S = \{V_1, V_2, V_3\}$ as shown below.

$$P_{ij} = \begin{array}{c c} & \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

(i) Obtain a *3-step* transition matrix. [3 Marks].

(ii) Classify the states of the given Markov chain. [6 Marks].