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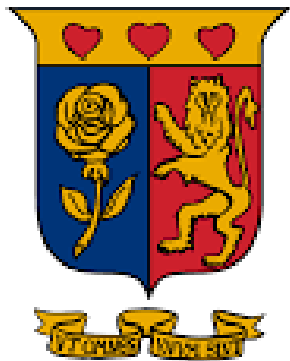
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**Determination of Optimal Public
Debt Ceiling for Kenya using
Stochastic Control**

Millicent Kithinji (096022)

A research project submitted in partial
fulfillment of the requirements for the
Master of Science in Mathematical
Finance

Strathmore Institute of Mathematical Sciences
May, 2018

Declaration

I declare that this work has not been previously submitted and approved for the award of a degree by this or any other University. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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Dedication

To mum and Dad.
For your love and immense sacrifice.

Determination of Optimal Public Debt Ceiling for Kenya using Stochastic Control

Millicent Kithinji

Abstract

Public debt is a key economic variable. It is the totality of public and publicly guaranteed debt owed by any level of government to either citizens or foreigners or both. Due to recent debt crises in developed countries such as Portugal, Italy, Ireland, Greece and Spain, debt control has become a key important fiscal policy of every government. In this study, we applied a formula proposed by (Cadenillas and Aguilar, 2015) to find out the optimal public debt ceiling for Kenya. We made modification to subjective variables in the explicit formula and used the formula to find the optimal public debt ceiling for Kenya. We illustrate that it is prudent for that government to use a fiscal policy that maintains the debt ratio under an optimal debt ceiling.

Keywords: Stochastic Optimal Control, Public debt, Debt ceiling, Hamilton-Jacobi-Bellman equation, Value function, Control process

Contents

Declaration	ii
Acknowledgement	iii
Dedication	iv
Abstract	v
List of Abbreviations	viii
1 Introduction	1
1.1 Background of the Study	1
1.2 Problem Statement	2
1.2.1 Objectives of the Study	3
2 Literature Review	4
3 Research Methodology	8
3.1 Introduction	8
3.2 The model	8
3.3 Verification theorem	12
3.4 Definition	13
3.5 Theorem	13
3.6 The explicit Solution	16
3.7 Lemma	17
4 Data Analysis and Findings	19
4.1 Data Description	19
4.2 Solution to the HJB equation	19
4.3 Finding the explicit solution for the Optimal Debt Ceiling	22
4.4 Parameters	23
5 Conclusion and Recommendation	31
5.1 Conclusion	31

5.2 Recommendation for Further study	32
List of References	33
List of Appendices	36

List of Abbreviations

CBK- Central Bank of Kenya

CBR-Central Bank Rate

EAC-East African Community

GBM-Geometric Brownian Motion

GDP-Gross Domestic Product

HJB-Hamilton-Jacobi-Bellman

LCRL- Left Continuous with Right Limits

ODE- Ordinary Differential Equation

PDE-Partial Differential Equation

SDE- Stochastic Differential Equation

SOC- Stochastic Optimal Control

CHAPTER 1

1 Introduction

1.1 Background of the Study

Public debt is a key economic variable. It is the totality of public and publicly guaranteed debt owed by any level of government to either citizens or foreigners or both. Controlling the debt/GDP ratio and keeping it below the debt ceiling is essential to both developing and developed countries. Various researchers have used different approaches to demonstrate the demerits of a high public debt to the economy. For instance; an increase in the volume of public debt can have some undesirable outcomes in the economy; such as inflation, shrinking of private investment, lowering of future growth and wages, high finance cost and repayment burden.

Due to undesirable outcomes of high debt levels; countries and regional trading blocs have developed fiscal policies to tame their respective debt/GDP ratio to a given level. For instance, in East African Community (EAC), the debt ceiling is set at 50% of GDP for all member states. For instance, as at June 2016, Kenya's gross public debt stood at 48.3 % of gross public debt percent of GDP in present value terms (National Treasury, 2016). In December, 2014 the Kenyan Legislature raised the maximum amount the national treasury can borrow from KSh.1.2 trillion to KSh.2.5 trillion. Hence the problem we study is that of determining how much debt a country can accumulate without a negative impact on economic growth. Our aim is to find out the optimal public debt ceiling for Kenya.

By optimal public debt ceiling we mean; the minimum limit of debt/GDP ratio at which the government should start implementing debt control measures. By control measures we mean that it is possible for the government to decrease the debt/GDP ratio through fiscal measures such as increasing taxes or decreasing expenditure. This implies that, the government should put fiscal adjustments measures in place when the debt ratio is above the optimal debt ceiling.

The study on optimal debt ceiling using stochastic control has not been studied before in Kenya. Hence, the study will contribute to the literature of public debt.

In this study, we adopted a theoretical model proposed by (Cadenillas and Aguilar, 2015). In the model, they developed an explicit formula for determining optimal public debt ceiling for a given country. We chose to use their model because it is country- specific since it uses macroeconomic variables for each country. Hence, unlike the common practice by regional blocs to set a unified debt ceiling, the model is consistent with observation by (Wyplosz, 2005)that each country should have a different debt ratio. We therefore apply the explicit formula to determine the optimal public debt ratio ceiling in Kenya and the corresponding value function.

1.2 Problem Statement

Due to recent debt crises in developed countries such as Portugal, Italy, Ireland , Greece and Spain, debt control has become a key fiscal policy of every government. Due to the need for debt control, different regional trading blocs such as East African Community, European Economic community,and various countries have set their maximum debt ratio to a given percentage. Hence, each member country should maintain their debt ratio below the given debt ratio ceiling. These debt control measures have been put in place in recognition that governments all over the world borrow. This is because the government revenue raised through taxes and fees is not enough to cover government expenditure hence meet the budget deficit, borrowing is inevitable. However, high public debt has undesirable effect on the economy and hence should be controlled.

Despite the existing debt ceilings in various jurisdictions, the debt ceilings have not been developed using a theoretical framework. Hence, the recent framework by Cadenillas and Aguilar is a ground breaking research in public debt management using stochastic control.

Our contribution to Optimal debt ceiling literature is application of the Stochastic control model proposed by (Cadenillas and Aguilar, 2015) to determine the optimal debt ceiling for Kenya with an aim of advising the public debt policy.

1.2.1 Objectives of the Study

The main objective of the study was to determine the optimal public debt ceiling in Kenya using stochastic control approach. We seek to address the question:

- What is the optimal public debt ceiling for Kenya ?

Specific Objective

1. Determine the the optimal debt ceiling for Kenya.
2. Determine the Value function (the minimum cost incurred by the government at a given debt ratio when all admissible controls are considered.)

CHAPTER 2

2 Literature Review

Stochastic optimal control was first introduced to economics and finance literature by Merton (1971) who studied the optimal portfolio selection problem in continuous time. In his framework, Merton modeled the portfolio as a controlled stochastic process and found the optimal investment strategy that maximizes a given objective. He used PDEs known as Hamilton-Jacobi-Bellman equations to obtain more precise solutions in continuous time portfolio optimization.

Since then, Stochastic optimal control has been used in studies that seek to determine the optimal debt ratio. For instance (Wei-han and Zhou, 2014) derived a formula for optimal public debt ratio for public and private sectors using Bellman's techniques. The model considered capital productivity, return on asset, interest rate, and regime switch on the market where the goal of the controller was to maximize the utility of wealth at Maturity by selecting an the optimal debt ratio.

In Stochastic control problems, the behavior of a dynamical stochastic process is influenced to obtain a given goal. The notion of control means that behavior of a dynamic system is influenced with an aim of obtaining a given goal. If the aim is optimizing a given objective function that relies on the control inputs into a system, then we have an optimal control problem. The fundamental components of a Stochastic Control problem are:

- **State Process, X .** This process defines the nature of the physical system of interest. It captures the minimal necessary information needed to describe the problem. Typically, the process is influenced by the control. Usually, its time dynamics is prescribed through an ordinary or stochastic differential equation. The progression of the state process is influenced by a control.
- **Control Process, Z .** This is a stochastic process, selected by the "agent" to influence the state of the system.

- **Admissible Control, \mathcal{A} .** These are controls that meets admissibility conditions. The conditions either be technical or physical, for instance, continuity or smoothness conditions, and, budget constraints.
- **Objective Function.** This is the function to be maximized (or minimized). The objective function is usually denoted as $J(x, Z)$. It represents the expected total cost of the system when the control process is implemented.
- **Value function, V .** This function defines the value of the minimum cost or reward. It is usually denoted by V and is obtained by optimizing the cost or reward over all admissible controls for a given initial state. The aim of a Stochastic Control problem is to describe the value function and find a control whose cost or reward achieves the minimum value over all admissible controls. The major problem in optimal control is to find the minimizing control process (Ross and Soner, 2004).

$$\text{Value function} = V(x) = \inf_{Z \in \mathcal{A}} J(x; Z.)$$

Most of the literature on the study of Stochastic control use dynamic programming or the Hamilton-Jacobi-Bellman (HJB) framework rather than stochastic maximum principles. The HJB is central to optimal control theory.

There are two principal approaches for solving the SOC problem namely, the Pontryagin's maximum principle (Pontryagin, 1959) and Bellman's dynamic programming (Bellman, 2010). Pontryagin's maximum principle asserts that any optimal control along with the optimal state trajectory will evaluate the Hamiltonian system. The HJB System consists of a collection of adjoint equations and the maximum conditions. Bellman's dynamic programming divides the dynamic optimization problem into simpler sub-problems. Bellman's dynamic programming rides on the optimality principle and defines the relationships in a set of optimal control problems whose initial time and states are different via the HJB equation. The two methods are similar in that it is possible to deduce the Hamilton system from the HJB equation, and vice versa (Yong and Zhou, 1999).

Despite the fact that both methods yield the same results, Bellman's dynamic programming takes advantage of the recursive nature of the problem and defines the value function of the objective function, as a function of the state process for the SOC (Wei-han and Zhou, 2014). In our study, the dynamic programming equation (HJB) is a second order linear inhomogeneous second ordinary differential equation (rather than a PDE). This means that in the inaction region the value function depends only parametrically on the variable associated to the purely controlled state (Ferrari, 2016).

Our problem is focused on finding the optimal debt ceiling for Kenya. This is motivated by the rising surge in public debt that has received increased attention from policy makers, practitioners and scholars. Much of this attention has been focused on understanding optimal debt strategy and structure of public debt. However, little attention has been given to the study of debt ceiling. Debt ceiling is defined as the level of debt ratio at which fiscal adjustments are not necessary. As a result, if a country's debt ratio is above that level, it is prudent for the government to take intervention measures to regulate the debt. However, if the debt ratio is below the debt ceiling, the debt is under control hence no need for fiscal adjustments (Cadenillas and Aguilar, 2015).

The existing literature on public debt management addresses the key issues, but has no theoretical framework to address debt control problem. Apparently, the existing Public debt ratio ceiling are determined by empirical and statistical analysis but without a backing of the theoretical framework. Existing theoretical models that focus on public debt management include (Barrow 1974, 1999), (Rogo, 1989) and (Stein 2006, 2012). However, the models do not examine the public debt ceiling problem.

A theoretical model for studying debt ceiling was introduced for the first time in the literature by (Cadenillas and Aguilar, 2015). The model was proposed to be used by any government to regulate its debt by fixing a ceiling on its debt ratio. In the model, the Debt ratio follow the dynamics of a geometric Brownian motion when the debt ratio is below the debt ceiling. This is consistent with the stochastic

debt ratio dynamics as illustrated in various macroeconomic literature. An explicit debt ceiling formula was derived. The formula is used to derive an optimal debt ceiling for each country. This is because the economic parameters in the formula are specific to each country.

(Ferrari, 2016) modelled the government debt reduction problem over an infinite time horizon as a singular stochastic control problem. In his model, he showed that it was prudent for government to enforce a fiscal policy that maintains the debt/GDP ratio under an inflation dependent ceiling. His model used a similar government cost function with the Cadenillas and Aguilar model cost function. However, the Markovian formulation of the singular control problem is fully two-dimensional, whereas that of Cadenillas and Aguilar is one dimensional. Hence, his optimal debt ceiling is a curve while Cadenillas and Aguilar optimal debt ceiling is a constant. We however, leave the multidimensional singular control approach for future research.

In our study, we apply the Cadenillas and Aguilar model with slight modifications in the parameters used in the formula. For instance, instead of using the discount rate as used in the model we applied the risk-free rate which is the Central Bank Rate (CBR). CBR the lowest rate of interest the Central Bank of Kenya charges on loans to banks. Hence, to capture the autonomy of the Central bank in determining the lending rate we used the CBR rate.

CHAPTER 3

3 Research Methodology

3.1 Introduction

The research adopted optimization methodology as proposed by (Cadenillas and Aguilar, 2015) to determine the optimal level of Kenya's Public debt ratio ceiling. First, the optimal debt ceiling was obtained by solving the value function for the HJB equation. The value function took a form of a second order linear nonhomogeneous constant coefficient ODE rather than a PDE. The solution to the ODE was obtained explicitly hence finding the optimal debt ceiling. Secondly, the verification theory was used to verify that the value function obtained was optimal.

3.2 The model

We set up the stochastic optimal control problem and present the explicit solution with theorem and proof by following the outline in (Cadenillas and Aguilar, 2015).

State Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathcal{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$ where \mathcal{F}_t is the flow of information over time. A levy process, W on $(\Omega, \mathcal{F}, \mathcal{P})$ taking values in \mathbb{R}^d is a d-dimensional \mathcal{F}_t -Brownian motion if:

1. W is \mathcal{F}_t -adapted
2. $\forall 0 \leq s \leq t, W_t - W_s$, the increments are independent of \mathcal{F}_s ,
3. $W_0 = 0$, a.s
4. $\forall 0 \leq s \leq t, W_t - W_s \sim N(0, t - s)$ increments are normally distributed,
5. The sample paths of W are continuous with probability of one.

Let $X = \{X_t, t \in [0, \infty)\}$ represent the state variable which is a country's debt ratio defined as:

$$X_t = \frac{\text{gross public debt at time, } t}{\text{gross domestic product (GDP) at time, } t} \quad (1)$$

$X = \{X_t, t \in [0, \infty)\}$ is an \mathbb{F} -adapted stochastic process that follows the Brownian motion process given below:

$$X_t = x + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s - Z_t, \quad (2)$$

or, equivalently,

$$dX_t = \mu X_t dt + \sigma X_t dW_t - dZ_t,$$

Given that,

$\mu = (r - g) \in \mathbb{R}$ and $\sigma \in (0, \infty)$ are constants, μ is the drift of the process and σ is the volatility, W is a Brownian motion, $r \in [0, \infty)$ is the real interest rate on debt, $g \in \mathbb{R}$ the rate of economic growth.

The initial debt ratio is given by $x \in (0, \infty)$. It is possible for the country to have an initial debt ratio of $x = 0$. This is possible in cases where the the country has not accrued or incurred any debt in a given period.

Control process

This is a stochastic process, chosen by the agent i.e. the government to influence the state of the system. The control Z is given as $Z = \{Z_t, t \in [0, \infty)\}$. The control process takes values in the control space, $\mathbb{Z} \subset \mathbb{R}^p$.

Admissible control

Controls which only meets certain admissibility criterion can be considered by the agent. The control, Z is adapted, non-negative, non-decreasing with left continuous

with right limits (LCRL) sample paths i.e $Z : [0; 1)\Omega \mapsto [0, \infty)$, is a an admissible singular control, ($Z \in \mathcal{A}$) if $J(x; Z) \leq \infty$. $\mathcal{A}(x) = \mathcal{A}$ is the set of all admissible controls. A control is admissible ($Z \in \mathcal{A}(x)$) if there exists a unique solution to the state equation and if the transversality condition below holds:

Conventionally, $Z_0 = 0$.

$\forall Z \in \mathcal{A}(x)$,

$$\lim_{T \rightarrow \infty} E_x[e^{-\lambda T} X_T^{2n} = 0] \quad (3)$$

The transversality condition implies that in case there is a flexibility at maturity time, T , then the marginal benefit of taking advantage of that flexibility at the optimum must be zero.

Definition: Left Continuous with Right Limits, LCRL

A real valued stochastic process ($X_{t \in [0, T]}$) on $(\Omega, \mathcal{F}, \mathcal{P})$ is a LCRL stochastic process if $\forall t \in [0, T]$:

1. Left limit of the process as s approaches t from the below (Left hand side) exists, i.e. $\lim_{s \rightarrow t, s \geq t} X_s = X_{t-}$. Right limit of the process as s approaches t from the above (right hand side) exists, i.e., $\lim_{s \rightarrow t, s \leq t} X_s = X_{t+}$.
2. $X_{t-} = X_t$

This means that, only the left continuity is needed hence allowing jumps. A continuous stochastic process therefore means that the process is LCRL.

Objective function

The government's goal is to minimize the cost function $J(x; Z)$ given that the control $Z \in \mathcal{A}$. The cost function is given by:

$$J(x; Z) = E_x[\int_0^\infty e^{-\lambda t} h(X_t) dt + \int_0^\infty e^{-\lambda t} k dZ_t]$$

Where; $k \in (0, \infty)$ denotes the proportional marginal cost of debt reduction, $\lambda \in (0, \infty)$ represents the government's discount rate, and h represents the cost function of having a debt, which is positive and convex, with $h(0) \geq 0$. $\int_0^\infty e^{-\lambda t} k dZ_t$ in J , denotes the cumulative discounted cost associated with the specific and deliberate goal of reducing debt.

(Cadenillas and Aguilar, 2015) denoted the economic cost of having debt, $h \rightarrow [0, \infty)$ is as:

$$h(x) = \alpha x^{2n} + \beta \quad (4)$$

Here, α is a strictly non-negative constant representing characteristics of debt, β is a positive constant scale parameter, $n \geq 1$ denotes debt aversion.

If the government never intervenes then, $Z = 0$ and the debt ratio X would be a Brownian motion given as follows:

$$X_t = x \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}$$

In this case, the objective function is:

$$\begin{aligned} J(x; Z) &= E_x\left[\int_0^\infty e^{-\lambda t} h(X_t) dt + 0\right] \\ &= E_x\left[\int_0^\infty e^{-\lambda t} (\alpha X_t^{2n} + \beta) dt\right] \\ &= \int_0^\infty e^{-\lambda t} \alpha E_x[X_t^{2n}] dt + \frac{\beta}{\lambda} \\ &= \alpha \int_0^\infty e^{-\lambda t} x^{2n} \exp\{(\sigma^2 n(2n-1) + 2\mu n)t\} dt + \frac{\beta}{\lambda} \\ &= \alpha x^{2n} \int_0^\infty \exp\{-(\lambda - (\sigma^2 n(2n-1) + 2\mu n)t)\} dt + \frac{\beta}{\lambda} \\ J(x; Z) &= \begin{cases} \frac{\alpha x^{2n}}{\lambda - \sigma^2 n(2n-1) - 2\mu n} + \frac{\beta}{\lambda} & \text{if } \lambda - \sigma^2 n(2n-1) - 2\mu n > 0 \\ \infty & \lambda - \sigma^2 n(2n-1) - 2\mu n \leq 0 \end{cases} \end{aligned}$$

Hence, the total cost no-intervention policy is finite iff:

$$\lambda > \sigma^2 n(2n - 1) + 2\mu n \quad (5)$$

Value Function, V

This function defines the value of the minimum cost or reward. It is usually denoted by V . It is obtained by optimizing the cost or reward over all admissible controls for a given initial state. The aim of a Stochastic Control problem is to describe the value function and find a control, Z^* whose cost or reward achieves the smallest value, $V(x) = J(x, Z^*)$ over all admissible controls. i.e. For $V : (0, \infty) \rightarrow \mathbb{R}$, the value function of optimal stopping time is defined as:

$$V(x) = \inf_{Z \in \mathcal{A}} J(x; Z)$$

3.3 Verification theorem

Proposition

The value function is positive, non-decreasing and convex. Further, $V(0+) = \frac{\beta}{\lambda}$.

Let $\psi : (0, \infty) \rightarrow \mathbb{R}$ be a $C^2(0, \infty)$ function. The operator \mathcal{L} is defined as:

$$\mathcal{L}\psi(x) = \frac{1}{2}\sigma^2 x^2 \psi''(x) + \mu x \psi'(x) - \lambda \psi(x).$$

Consider the HJB equation below:

$$\forall x > 0 : \min\{\mathcal{L}v(x) + h(x), k - v'(x)\} = 0 \quad (6)$$

Equation (6) is a variational inequality and is defined over the function $V : (0, \infty) \rightarrow \mathbb{R}$

The solution of equation(6), v defines the continuation region, $\mathcal{C} = \mathcal{C}^v$ and intervention region, $\Sigma = \Sigma^v$ by:

$$\mathcal{C} = \mathcal{C}^v : \{x \in (0, \infty) : \mathcal{L}v(x) + h(x) = 0 \text{ and } k - v'(x) > 0\} \quad (7)$$

$$\Sigma = \Sigma^v = \{x \in (0, \infty) : \mathcal{L}v(x) + h(x) \geq 0 \text{ and } k - v'(x) = 0\} \quad (8)$$

The regions \mathcal{C} and Σ form a partition of $(0, \infty)$. For instance, if v solves the Hamiltonian, then $\mathcal{C} \cup \Sigma = (0, \infty)$ and $\mathcal{C} \cap \Sigma = \emptyset$. The control process associated with v is defined as follows:

3.4 Definition

Let v satisfy the Hamiltonian equation (6). An \mathcal{F}_t -adapted, positive, and an increasing control process Z^v , with $Z_0^v = 0$ whose sample paths are LCRL, is associated with the function v when the 3 conditions are met:

1. $X_t^v = x + \int_0^t \mu X_s^v ds + \int_0^t \sigma X_s^v dW_s - Z_t^v, \forall t \in [0, \infty), P - a.s.,$
2. $X_t^v \in \bar{\mathcal{C}}, \forall t \in (0, \infty), P - a.s.,$
3. $\int_0^\infty I_{\{X_t^v \in \mathcal{C}\}} dZ_t^v = 0, P - a.s$

Where I_A represents the indicator function of the occurrence $A \subset [0, \infty)$.

An associate control process Z_v is admissible if $J(x, Z_v) < \infty$.

A sufficient condition for an optimal policy is given by:

3.5 Theorem

We state the theorem and proof as outlined by (Cadenillas and Aguilar, 2015).

Let $v \in \mathcal{C}^2(0, \infty)$ be a non-decreasing, convex function on $(0, \infty)$, with $v(0+) = \frac{\beta}{\lambda}$. If v satisfies the HJB equation (6) $\forall x \in (0, \infty)$, and there exists $d \in (0, \infty)$ s.t. the continuation region, \mathcal{C} associated with v is $\mathcal{C}^v = (0, d)$. It implies that $\forall Z \in \mathcal{A}(x)$:

$$v(x) \leq J(x; Z)$$

Additionally, the control, Z^v related to v satisfies

$$v(x) \leq J(x; Z^v)$$

This means that, $\hat{Z} = Z^v$ is optimal control and $V = v$ is the value function for the objective function.

Proof. Given that v is twice continuously differentiable, v' and v'' are bounded functions, we apply a suitable Ito's lemma to yield:

$$\begin{aligned} v(x) = & E_x[e^{-\lambda T}v(X_T)] + E_x\left[\int_0^T e^{-\lambda t}v'(X_t)dZ_t^c\right] - E_x\left[\int_0^T e^{-\lambda t}X_t\sigma v'(X_t)dW_t\right] \\ & - E_x\left[\int_0^T e^{-\lambda t}\left\{\frac{1}{2}\sigma^2X_t^2v''(X_t) + \mu X_tv'(X_t) - \lambda v(X_t)\right\}dt\right] \\ & - E_x\left[\sum_{t \in \Delta, 0 \leq t \leq T} e^{-\lambda t}\{v(X_{t+}) - v(X_t)\}\right] \end{aligned}$$

v satisfies the Hamiltonian (6), hence we have $\mathcal{L}v(x) + h(x) \geq 0$ and $v'(x) \leq k$ $\forall x \in (0, \infty)$. Therefore:

$$\int_0^T e^{-\lambda t}\left\{\frac{1}{2}\sigma^2X_t^2v''(X_t) + \mu X_tv'(X_t) - \lambda v(X_t)\right\}dt \leq \int_0^T e^{-\lambda t}h(X_t)dt \quad (9)$$

$$\int_0^T e^{-\lambda t}v'(X_t)dZ_t^c \leq \int_0^T e^{-\lambda t}kdZ_t^c \quad (10)$$

and

$$v(X_{t+}) - v(X_t) \leq k(Z_{t+} - Z_t), \forall t \in \Delta \quad (11)$$

Hence

$$\begin{aligned}
v(x) &\leq E_x[e^{-\lambda T}v(X_T)] + E_x \int_0^T e^{-\lambda t} h(X_t) dt + E_x[e^{-\lambda t} k dZ_t^c] \\
&+ E_x[\sum_{t \in \Delta, 0 \leq t \leq T} e^{-\lambda t} k \{Z_{t+} - Z_t\}] - E_x[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t] \\
&= E_x[e^{-\lambda T}v(X_T)] + E_x[\int_0^T e^{-\lambda t} h(X_t) dt + e^{-\lambda t} k dZ_t] \\
&\quad - E_x[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t]
\end{aligned}$$

Given $Z \in \mathcal{A}(x)$. From the transversality condition (3), and the linear growth of v on the intervention region $\Sigma^v = [d, \infty)$;

$$\lim_{T \rightarrow \infty} E_x[e^{-\lambda T} v X_T] = 0$$

Thus, the first part of the theorem is proved.

The second part of the theorem is proved as below:

We let the process X^v be generated by Z^v . Given that $C^v = (0, d)$. Given that the function $\mathcal{L}v(x) + h(x)$ is continuous, $\forall x \in (0, \infty)$, using (9), we obtain $\mathcal{L}v(x) + h(x) = 0, \forall x \leq d$.

$$\int_0^T e^{-\lambda t} \left\{ \frac{1}{2} \sigma^2(X_t^v) + \mu X_t^v v'(X_t^v) - \lambda v(X_t^v) \right\} dt = - \int_0^T e^{-\lambda t} h(X_t^v) dt$$

$$v(x) \leq E_x[e^{-\lambda T}v(X_T)] + E_x \int_0^T e^{-\lambda t} h(X_t) dt + E_x[e^{-\lambda t} k dZ_t^v] - E_x[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t] \quad (12)$$

From the transversality condition (3), on intervention region, $\Sigma = [d, \infty)$ we have:

$$\lim_{T \rightarrow \infty} E_x[e^{-\lambda T} v(X_T^V)] = 0$$

and

$$E_x[\int_0^T e^{-\lambda t} X_t \sigma v'(X_t) dW_t] = 0$$

Letting $T \rightarrow \infty$, we obtain:

$$v(x) = E_x \left[\int_0^\infty e^{-\lambda t} h(X_t^v) dt + \int_0^\infty e^{-\lambda t} k dZ_t^v \right] = J(x, Z^v)$$

This proves that Z^v is an admissible control. Hence, the theorem is proved!

3.6 The explicit Solution

Definition

Given that v is a function satisfying the HJB equation (6), whose continuation region is given by \mathcal{C} . The public debt ratio ceiling, b , given that $\mathcal{C} \neq \emptyset$ is

$$b = \sup\{x \in (0, \infty) \mid x \in \mathcal{C}\}$$

Furthermore, if v is the value function, b is the optimal debt ceiling.

To determine the optimal debt ceiling, we obtain the value function. The HJB equation (6) in $\mathcal{C} = (0; b)$ means that:

$$\frac{1}{2}\sigma^2 x^2 v''(x) + \mu x v'(x) - \lambda v(x) = -\alpha x^{2n} - \beta$$

The HJB equation (6) in $\Sigma = [b, \infty)$ means that:

$$v(x) = v(b) + k(x - b)$$

We obtain the differential equation

$$\frac{1}{2}\sigma^2 x^2 v''(x) + \mu x v'(x) - \lambda v(x) = -\alpha x^{2n} - \beta, \text{ if } x < b \quad (13)$$

and

$$v'(x) = k, \text{ if } x \geq b \quad (14)$$

The solution of differential equation in the continuation and intervention regions is given by:

$$v(x) = \begin{cases} \mathcal{A}x^{\gamma_1} + Bx^{\gamma_2} + \alpha \zeta x^{2n} + \frac{\beta}{\lambda}, & \text{if } x < b \\ kx + D, & \text{if } x \geq b \end{cases}$$

Here

$$\bar{\mu} = \mu - \frac{1}{2}\sigma^2 \quad (15)$$

$$\gamma^1 = \frac{-\bar{\mu} - \sqrt{\bar{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} < 0 \quad (16)$$

$$\gamma^2 = \frac{-\bar{\mu} + \sqrt{\bar{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} > 0 \quad (17)$$

$$\varsigma = \frac{1}{\lambda - \sigma^2 n(2n - 1) - 2\mu n} \quad (18)$$

Since v is twice continuously differentiable and $V(0+) = \frac{\beta}{\lambda}$; constants A, B, b, and D can be found from the following system of initial conditions:

$$V(0+) = \frac{\beta}{\lambda}; \quad (19)$$

$$v(b^+) = v(b^-); \quad (20)$$

$$v'(b^+) = v'(b^-); \quad (21)$$

$$v''(b^+) = v''(b^-) \quad (22)$$

The parameters satisfy the following conditions:

3.7 Lemma

The results below are valid:

1. $\varsigma > 0$,
2. $\lambda > \mu$
3. $\gamma^2 > 2n$
4. $\lambda - \sigma^2 n(\gamma^2 - 1) - \gamma_2 \mu n > 0$
5. $2\lambda - \bar{\mu} - \sqrt{\bar{\mu}^2 + 2\lambda\sigma^2} > 0$

Since $\gamma_1 < 0$, condition (19) implies $A = 0$. Therefore, the value function is given by:

$$v(x) = \begin{cases} Bx^{\gamma_2} + \alpha\zeta x^{2n} + \frac{\beta}{\lambda}, & \text{if } x < b \\ kx + D, & \text{if } x \geq b \end{cases}$$

For optimality, we obtain the first and second derivatives as follows:

$$v'(x) = \begin{cases} B_{\gamma_2} x^{\gamma_2-1} + \alpha\zeta x^{2n-1} + \frac{\beta}{\lambda}, & \text{if } x < b \\ k, & \text{if } x \geq b \end{cases}$$

$$v''(x) = \begin{cases} B_{\gamma_2}(\gamma_2 - 1)x^{\gamma_2-2} + \alpha\zeta 2n(2n - 1)x^{2n-2} + \frac{\beta}{\lambda}, & \text{if } x < b \\ 0, & \text{if } x \geq b \end{cases}$$

From the initial conditions (19)-(22), we solve for the constants b, B and D explicitly, as a function of the parameters $(k, n, \lambda, \mu, \sigma, \alpha, \beta)$.

CHAPTER 4

4 Data Analysis and Findings

4.1 Data Description

In the study, we obtained public debt statistics from the Ministry of Finance, Kenya. The country's GDP statistics was obtained from the world bank data portal while the interest rate statistics was obtained from Central Bank. The sample is a data set of 37 annual observations covering the period between 1980 and 2016.

4.2 Solution to the HJB equation

We determine the optimal debt ceiling, by obtaining the value function. The Hamilton-Jacobi-Bellman equation (6) in the region $\mathcal{C} = (0; b)$ means that:

$$\frac{1}{2}\sigma^2 x^2 v''(x) + \mu x v'(x) - \lambda v(x) = -\alpha x^{2n} - \beta \quad (23)$$

Equation (23) is a second order linear in-homogeneous constant coefficient ODE. The solution to this class of ODE is of the form $X = X_c + X$ where X_c is a complementary function. The complementary solution can be easily found from the roots of the characteristic polynomial. X is any specific function that satisfies the inhomogeneous equation. We will use the Method of Undetermined Coefficients (sometimes referred as Judicious Guessing) to solve the ODE. In this method an appropriate ansatz is used to determine the general form of the particular solution X based on the inhomogeneous term g in the given equation. We solve for X and the complementary function X_c as follows.

$$\text{Let } g = -\alpha x^{2n} - \beta$$

$$\begin{aligned}
X &= -Px^{2n} + Q \\
X' &= -2nPx^{2n-1} \\
X'' &= -2n(2n-1)Px^{2n-2}
\end{aligned}$$

Replacing the derivatives in the ODE we obtain:

$$\frac{1}{2}\sigma^2 x^2(-2n(2n-1)Px^{2n-2}) + \mu x(-2nPx^{2n-1}) - \lambda(-Px^{2n} + Q) = -\alpha x^{2n} - \beta \quad (24)$$

Simplifying and equating like terms together in equation(24), we have:

$$\begin{aligned}
\left\{\frac{1}{2}\sigma^2(-2n(2n-1)P)\right\}x^{2n} + \{\mu 2nQ\}x^{2n} + \{\lambda P\}x^{2n} - \lambda Q &= -\alpha x^{2n} - \beta \\
\frac{1}{2}\sigma^2(-2n(2n-1)P - 2\mu nP + \lambda P) &= -\alpha \\
\implies P &= \frac{\alpha}{\lambda - \sigma^2 n(2n-1) - 2\mu n} \\
-\lambda Q &= -\beta \\
\implies Q &= \frac{\beta}{\lambda}
\end{aligned}$$

Hence,

$$X = \frac{\alpha}{\lambda - \sigma^2 n(2n-1) - 2\mu n} + \frac{\beta}{\lambda}$$

We now solve the complementary function X_c . The corresponding homogeneous equation has the characteristic equation $\frac{1}{2}\sigma^2 + \mu - \lambda = 0$. So to get the complementary function we find the roots of the equation.

$$\frac{1}{2}\sigma^2 + \mu - \lambda = 0$$

The two roots are given by:

$$\begin{aligned}
X &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\
r1 &= \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2\lambda}}{2a} \\
r2 &= \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\lambda}}{2a}
\end{aligned}$$

Since $b^2 > 4C$, the complementary solution of X_c is given by :

$$X_c = C_1 \exp\left(\frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}x\right) + C_2 \exp\left(\frac{-\mu - \sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}x\right)$$

Hence, the solution of the ODE equation is:

$$v(x) = C_1 \exp\left(\frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}x\right) + C_2 \exp\left(\frac{-\mu - \sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}x\right) + \frac{\alpha}{\lambda - \sigma^2 n(2n - 1) - 2\mu n} + \frac{\beta}{\lambda}$$

The HJB equation associated with the intervention region, $\Sigma = [b, \infty)$ means that:

$$v(x) = v(b) + k(x - b) \quad (25)$$

The general solution of (23) and (25) is

$$v(x) = \begin{cases} Ax^{\gamma^1} + Bx^{\gamma^2} + \alpha\varsigma x^{2n} + \frac{\beta}{\lambda}, & \text{if } x < b \\ kx + D, & \text{if } x \geq b \end{cases}$$

Where,

$$\bar{\mu} = \mu - \frac{1}{2}\sigma^2 \quad (26)$$

$$\gamma^1 = \frac{-\bar{\mu} - \sqrt{\bar{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} < 0 \quad (27)$$

$$\gamma^2 = \frac{-\bar{\mu} + \sqrt{\bar{\mu}^2 + 2\lambda\sigma^2}}{\sigma^2} > 0 \quad (28)$$

$$\varsigma = \frac{1}{\lambda - \sigma^2 n(2n - 1) - 2\mu n} \quad (29)$$

4.3 Finding the explicit solution for the Optimal Debt Ceiling

To determine, the optimal debt ceiling, we use the optimality criterion where the value function is differentiated twice to find the minimum expected cost.

Given the value function as:

$$v(x) = \begin{cases} Bx^{\gamma_2} + \alpha\zeta x^{2n} + \frac{\beta}{\lambda}, & \text{if } x < b \\ kx + D, & \text{if } x \geq b \end{cases}$$

We find the minimum cost by obtaining the first and second derivatives, as follows.

$$v'(x) = \begin{cases} B_{\gamma_2}x^{\gamma_2-1} + \alpha\zeta x^{2n-1} + \frac{\beta}{\lambda}, & \text{if } x < b \\ k, & \text{if } x \geq b \end{cases}$$

$$v''(x) = \begin{cases} B_{\gamma_2}(\gamma_2 - 1)x^{\gamma_2-2} + \alpha\zeta 2n(2n - 1)x^{2n-2} + \frac{\beta}{\lambda}, & \text{if } x < b \\ 0, & \text{if } x \geq b \end{cases}$$

From initial conditions $v'(b^+) = v'(b^-)$ and $v''(b^+) = v''(b^-)$, we obtain the constants b, B and D explicitly, as a function of the parameters $(k, n, \lambda, \mu, \sigma, \alpha, \beta)$.

We obtain,

$$k = B_{\gamma_2}b^{\gamma_2-1} + \alpha\zeta b^{2n-1}$$

$$0 = B_{\gamma_2}(\gamma_2 - 1)b^{\gamma_2-2} + \alpha\zeta 2n(2n - 1)b^{2n-2}$$

Solving for B we have:

$$B = -\frac{\alpha\zeta 2n(2n - 1)}{\gamma_2(\gamma_2 - 1)}b^{2n-\gamma_2} < 0 \quad (30)$$

Substituting for B in $k = B_{\gamma_2}b^{\gamma_2-1} + \alpha\zeta b^{2n-1}$

We obtain

$$k = -\frac{\alpha\zeta 2n(2n - 1)}{\gamma_2(\gamma_2 - 1)}b^{2n-\gamma_2}b^{\gamma_2-1} + \alpha\zeta 2nb^{2n-1}$$

Making b the subject of the formula and rearranging we have:

$$b = \left(\frac{k(\gamma_2 - 1)}{-\alpha\varsigma 2n(2n - 1) + \alpha\varsigma 2n(\gamma_2 - 1)} \right)^{\frac{1}{2n-1}}$$

$$b = \left(\frac{1}{\alpha\varsigma} \frac{k(\gamma_2 - 1)}{(\gamma_2 - 2n)2n} \right)^{\frac{1}{2n-1}} \quad (31)$$

Using b and k , and Equation(31) we obtain

$$D = Bb^{\gamma_2} + \alpha\varsigma b^{2n} + \frac{\beta}{\lambda} - kb \quad (32)$$

4.4 Parameters

The resulting solution for the optimal debt ceiling will vary according to the values of the subjective variables. Some variables are measurable and objective while others will be preference or subjective variables. We used the data to derive estimates of the mean and volatility of the state process. One can hence determine to what extent the results are changed when one selects different parameter estimates or preference variables.

Definition of Parameters

1. k

k represent the marginal cost of reduction of debt. That is, the government incurs the cost $k > 0$, for each unit of debt reduction. k is normalized by setting $k = 1$.

2. n

n is a subjective parameter. It captures debt aversion. It can also be a measure of debt intolerance. Precedence shows that for countries which have not defaulted on public debt, the value of $n = 2$ has been used. We will replicate this value in Kenyan case since there has been no public debt default.

3. λ

This is the government's discount rate. In the Kenyan case we used the Central Bank Rate as the risk-free rate which is 0.095. This rate is given autonomously by the Central Bank of Kenya.

4. μ

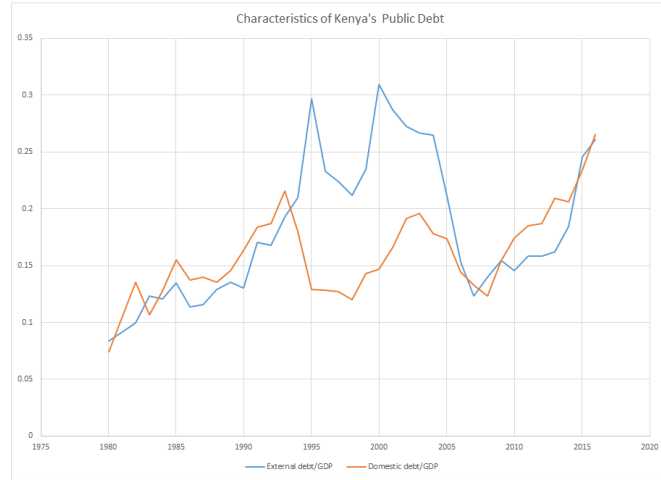
This is the mean of the Debt Ratio which follows a Brownian Motion. The parameter has been estimated from Kenyan Data for the period 1980-2016.

5. σ

This is the volatility of the debt ratio. The parameter was estimated from the Kenyan historical data for the period 1980-2016.

6. α

The parameter represents nature of debt itself. For instance, a country with a higher proportion of domestic debt than foreign debt will have a small α relative to k . This is the case since Domestic debt owners allow a higher debt ratio than foreign debt owners. In 2016, Kenya's ratio of domestic debt to foreign debt was 0.50 : 0.50. Cumulatively, from the year 1980 to 2016 the ratio of domestic debt to foreign debt was 0.49 : 0.51. Hence, Kenyan debt is half dominated by foreign debt. We will therefore consider a α of 0.51 in the Kenyan case.



7. β

β is a scale parameter. A scale parameter is related to dispersion parameter that defines the spread of a distribution.

Parameter estimation of Geometric Brownian Motion.

The dynamics of the Debt Ratio are assumed to follow Geometric Brownian Motion (GBM).

The SDE for GBM is given as:

$$dX(t) = \mu X(t)dt + \sigma X dW(t) \quad (33)$$

Using ito's lemma to solve the SDE (33) we let, $Y = \ln X(t)$

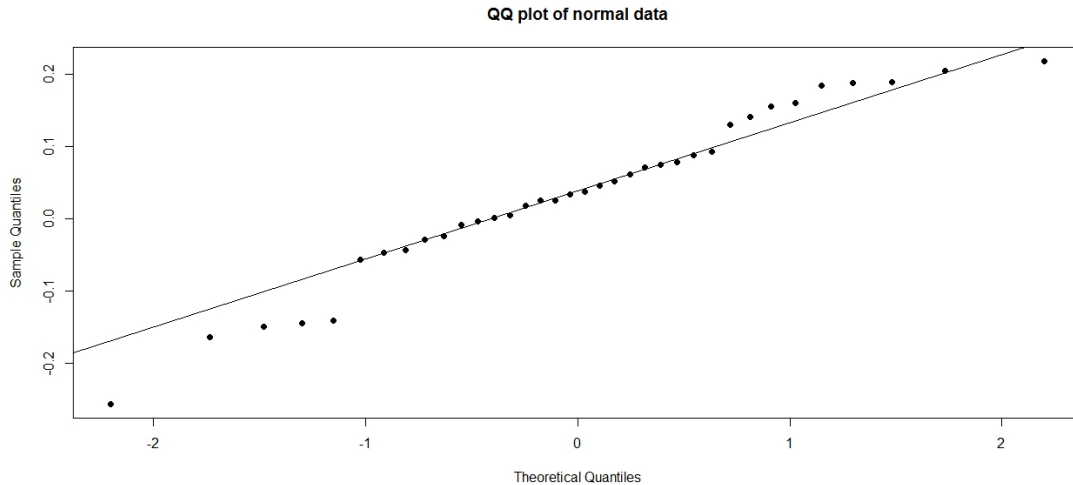
$$\begin{aligned}
dY(t) &= \left[\frac{\partial Y(t, x)}{\partial t} + \mu x \frac{\partial Y(t, x)}{\partial dX} + \frac{\sigma^2 X^2}{2} \frac{\partial^2 Y(t, x)}{\partial X^2} \right] dt + \sigma X \frac{\partial Y(t, x)}{\partial X} dw(t) \\
&= 0 + \mu X \frac{1}{X} - \frac{1}{2} \sigma^2 X^2 \frac{1}{X^2} + \sigma X dW(t) \\
&= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma X dW(t) \\
Y(t) &= Y(0) + \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma (W(t) - W(0)) \\
\ln X(t) &= \ln X(0) + \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma (W(t) - W(0)) \\
X(t) &= X(0) \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma (W(t) - W(0)) \right\}
\end{aligned}$$

Hence, $X(t) \sim LN(\ln X(0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$

This means that the log of Debt Ratio is normally distributed. Using historical data on debt Ratio, we transformed the data to log and tested normality on the transformed data using QQ-plot and shapiro test since shapiro test is more sensitive to small sample size.

Normality Test

The QQ-plot for the data is given below:



The shapiro test for normality is given as:

H_0 : The data is normally distributed.

Shapiro-Wilk normality test	
data	Log(Debt Ratio)
W = 0.96489	P-value = 0.3027

Since P-value is greater than 0.05, the null hypothesis is not rejected hence normality is assumed.

We estimate the parameters from the transformed historical data. Given that,

$$\frac{dX}{X} = \mu\Delta t + \sigma\mathcal{E}\sqrt{\Delta t}$$

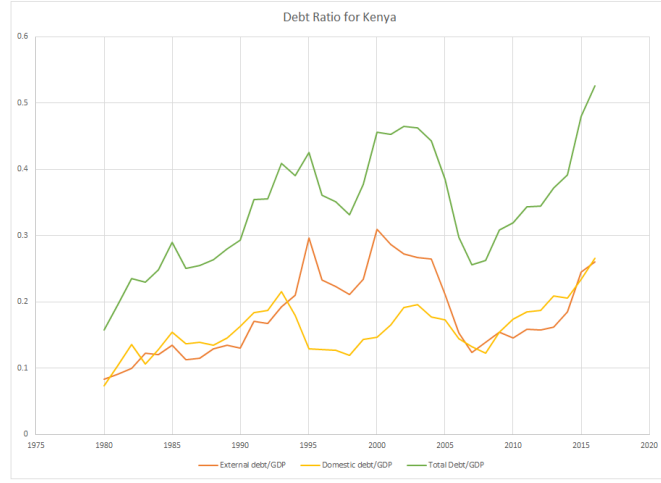
This means that

$$\frac{dX}{X} \sim N(\mu\Delta t, \sigma\sqrt{\Delta t})$$

We therefore estimate the parameters μ and σ from historical data. This is done by setting the time interval, Δt in years. From the data, we obtain the innovations given by the the sample mean, $\hat{\mu}$ and the sample variance, $\hat{\sigma}$. $\hat{\mu}$ is an estimate of $\mu\Delta t$ and $\hat{\sigma}$ is an estimate of $\sigma\sqrt{\Delta t}$.

From the data, the parameter μ is 0.039 while the parameter σ is 0.1134. The R code for the estimates is in the appendices.

The calibration of the historical data of Debt Ratio from 1980-2016 is below.



The parameter values that were used in the analysis are given in the table below:

k	μ	σ	n	α	λ	β	r	g
1	0.04	0.01134	2	0.51	0.095	0	0.08	0.04

In the table above, we have indicated the values of the parameter values that we used in our analysis. The meaning of these parameters is: $\mu = (r - g)$ is the difference between real interest rate and the economic growth rate, σ is debt ratio volatility, k is the marginal cost of debt intervention, n is debt aversion, α represents debt characteristics, scale parameter β and risk free rate, λ .

To apply the optimal debt ceiling explicit formula, we assumed that the macroeconomic variables, economic growth and interest rates are constant.

σ represents volatility in the debt dynamics which means that the debt ratio changes are attributed to deficits or excesses beyond the government control.

Using parameters set in the table above we calculate the value of the optimal debt ceiling, b and the corresponding value function as shown below. The R code is given in the appendix.

The optimal debt ceiling is given by:

$$b = \left(\frac{k(\gamma_2 - 1)}{-\alpha_5 2n(2n - 1) + \alpha_5 2n(\gamma_2 - 1)} \right)^{\frac{1}{2n-1}}$$

$$b = 0.2782979$$

Hence, using the specific parameters for Kenya, the optimal debt ceiling is 27.82979%. This means that when the debt ratio is at 27.8% and crosses the optimal debt ceiling, the government should put fiscal adjustments measures in place to control the debt ratio from crossing b . If the initial debt ratio x is greater than the debt ceiling, b , the control process Z , jumps from $Z_0 = 0$ to $Z_0 = x - b$. In the same manner, the debt ratio transits from $X_0 = x$ to $X_0 = b$. Therefore, if $x > b$, the fiscal adjustments measures should be taken to increase the primary surplus by $(x - b)$.

Therefore at any period when the debt ratio of Kenya is below 27.82979%, no fiscal interventions are needed; if the debt ratio equals 27.82979%, then control should be put in place to prevent it from crossing b ; if 27.82979% , then the government should immediately put control measures that aim at lowering the debt ratio to the optimal level.

Using the values given in the table above, the value function corresponding to the optimal government debt ceiling 27.82979% is

$$V(x) = \begin{cases} 2.875095x^{1.834206} - \frac{0.51}{8.759838}x^4 & \text{if } x < 0.2782979 \\ x - 0.02981958 & \text{if } x \geq 0.2782979 \end{cases}$$

The value function is gives the smallest expected total cost. By total cost we mean, the sum of the cost of having the debt and the intervention cost. The value function gives the minimum cost that can be obtained when the initial debt ratio is x and all acceptable controls are considered.

For instance in 2016, the debt ratio for Kenya was 0.5263393. The value function associated with the debt ratio, x is $kx + D$ since $x > 0.2782979$. This implies that

the minimum expected total cost the government incurred is 0.497 i.e. (0.5263393-0.02981958).

On the other hand, we consider a case in which $x < 0.2782979$. In 2008 the debt ratio for Kenya was 0.2628896 hence below the optimal debt ceiling. The value function associated with this ratio is given by $2.875095x^{1.834206} - \frac{0.51}{8.759838}x^4$ which evaluates to a minimum expected cost of 0.247691.

This implies that the government incurs a higher cost when the debt ratio \geq optimal debt ceiling than when the the debt ratio $<$ optimal debt ceiling.

5 Conclusion and Recommendation

5.1 Conclusion

The study aimed at establishing the optimal debt ceiling for Kenya and the corresponding value function. To achieve this we used an explicit formula for optimal debt ceiling. We chose to use the formula because it is a function of macroeconomic variables such as the economic growth, interest rate, debt volatility, debt aversion, risk free rate that are unique for each country. This implies that using a particular country's data it is possible to use the formula to determine the optimal debt ceiling for that country.

From the study, we established that the volatility of debt dynamics in Kenya is 0.01134. This means that the Kenyan debt ratio can increase or decrease by 0.01134 due to factors beyond the control of government.

We found out that the optimal debt ceiling for Kenya is 27.82979%. This means that at any point when the debt ratio is above 27.82979%, the government incurs an intervention cost in addition to the running cost of having a debt. This was illustrated in the study where we found out that the value function was higher in cases when the debt ratio, x , was greater than the optimal debt ceiling, b , and vice-versa. This additional cost can have negative effects in the economy such as tax distortion due to increased taxation and less growth of capital stock.

Further, a debt ratio above the optimal debt ceiling would imply that debt is growing faster than the Gross Domestic Product (GDP). This would increase the riskiness of a country which may lead to downgrading of a government's credit rating by rating agencies. For instance, Kenya was recently downgraded by Moody's from B2 to B1 due to a rise in debt levels and deterioration in debt affordability (Moody's, 2018). At the time of downgrade Kenya's debt ratio was at 54%. This downgrade is consistent with the findings of this study that indicate that the debt ratio in Kenya is above the optimal debt ceiling. Hence, the government should

take fiscal adjustments measures aimed at maintaining the debt ratio at or below the optimal debt ceiling.

The novel aspects covered in this study are:

1. In calculating the value of the subjective parameter α which represents the characteristics of a country's Public debt, we obtained the ratio of cumulative domestic debt and foreign debt to total public debt. Cumulatively, the ratio of cumulative domestic debt and foreign debt to total public debt was 0.49: 0.51. We hence set the value of α to 0.51 to reflect the dominance of total public debt by foreign debt.
2. Further, we used the Central Bank risk-free Rate rather the Discount rate due to the fact that the variable is usually determined by an autonomous Central Bank, whose action is not modelled in the model.

5.2 Recommendation for Further study

This study has used one dimensional stochastic optimal control model for study of optimal debt ceiling. A more comprehensive multidimensional approach could be used to determine the optimal debt ceiling. For the multidimensional approach, the debt dynamics would depend on macroeconomic factors such as exchange rate, inflation that would be modelled exogenously.

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List of Appendices

Appendix A: Data and R Code

```
> library(MASS)
> data1=read.csv("data1.csv")
> data1
  Year Real.interest.rate. GDP.growth..annual... mu..r.g.. Debt.ratio.X
1  2016           7.90           5.8486654      2.05133464      0.5263393
2  2015           5.90           5.7133829      0.18661708      0.4800054
3  2014           7.82           5.3518399      2.46816014      0.3912295
4  2013          11.55           5.8797639      5.67023613      0.3717170
5  2012           9.46           4.5632002      4.89679983      0.3450157
6  2011           3.84           6.1116135     -2.27161346      0.3434337
7  2010          12.03           8.4022771      3.62772294      0.3201139
8  2009           2.84           3.3069398     -0.46693981      0.3086104
9  2008          -0.98           0.2322827     -1.21228275      0.2628896
10 2007           4.82           6.8507298     -2.03072977      0.2564523
11 2006          -8.01           6.4724943    -14.48249430      0.2977767
12 2005           7.61           5.9066661      1.70333392      0.3849126
13 2004           5.05           5.1042998     -0.05429978      0.4431448
14 2003           9.77           2.9324755      6.83752445      0.4629141
15 2002          17.36           0.5468595     16.81314047      0.4644360
16 2001          17.81           3.7799065     14.03009350      0.4528388
17 2000          15.33           0.5996954     14.73030461      0.4565314
18 1999          17.45           2.3053886     15.14461140      0.3777534
19 1998          21.10           3.2902137     17.80978628      0.3315987
20 1997          16.88           0.4749019     16.40509808      0.3508920
21 1996          -5.78           4.1468393     -9.92683927      0.3612668
22 1995          15.80           4.4062165     11.39378347      0.4258026
23 1994          16.43           2.6327845     13.79721548      0.3901414
24 1993           3.41           0.3531973      3.05680274      0.4088029
25 1992           1.83          -0.7994940      2.62949396      0.3551427
26 1991           5.75           1.4383468      4.31165321      0.3546678
27 1990           7.33           4.1920510      3.13794903      0.2939164
28 1989           6.82           4.6903488      2.12965123      0.2807307
29 1988           8.03           6.2031838      1.82681618      0.2641773
30 1987           8.16           5.9371074      2.22289255      0.2553798
31 1986           4.86           7.1775554     -2.31755539      0.2509517
32 1985           5.26           4.3005618      0.95943818      0.2899927
33 1984           3.84           1.7552170      2.08478302      0.2484237
34 1983           3.57           1.3090502      2.26094976      0.2298940
35 1982           2.61           1.5064783      1.10352175      0.2355210
36 1981           1.41           3.7735442     -2.36354420      0.1959680
37 1980           0.94           5.5919762     -4.65197621      0.1576095
> attach(data1)
> class(data1)
[1] "data.frame"
```

```

> mean(mu..r.g..)
[1] 3.608417
> mean(mu..r.g..)/100
[1] 0.03608417
> var(mu..r.g..)/100
[1] 0.5280112
> sd(mu..r.g..)/100
[1] 0.07266438
> mean(Real.interest.rate.)/100 ### interest rate, r
[1] 0.07454054
> mean(GDP.growth..annual...)/100### Economic growth, g
[1] 0.03845637
> {mean(Real.interest.rate.)-mean(GDP.growth..annual...)} /100
[1] 0.03608417

> library(MASS)
> DebtRatio=read.csv("DebtRatio.csv")
> DebtRatio
  YEAR LN..DebtRatio.
1  1981    0.217831112
2  1982    0.183848795
3  1983   -0.024181712
4  1984    0.077517473
5  1985    0.154720006
6  1986   -0.144595380
7  1987    0.017491464
8  1988    0.033868353
9  1989    0.060775672
10 1990    0.045899293
11 1991    0.187886215
12 1992    0.001338146
13 1993    0.140713397
14 1994   -0.046723763
15 1995    0.087466676
16 1996   -0.164359123
17 1997   -0.029138337
18 1998   -0.056552960
19 1999    0.130316081
20 2000    0.189415756
21 2001   -0.008121142
22 2002    0.025287550
23 2003   -0.003282343
24 2004   -0.043644944
25 2005   -0.140880327
26 2006   -0.256672210
27 2007   -0.149401272
28 2008    0.024791432
29 2009    0.160345566
30 2010    0.036597141
31 2011    0.070317378

```

```

32 2012    0.004595809
33 2013    0.074542743
34 2014    0.051161649
35 2015    0.204502890
36 2016    0.092148768

```

```

> class(DebtRatio)
[1] "data.frame"
> summary(LN..DebtRatio.)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
-0.25670 -0.02542  0.03523  0.03350  0.10170  0.21780
> mean(LN..DebtRatio.)
[1] 0.03349516
> sd(LN..DebtRatio.)
[1] 0.1134146
> qqnorm(LN..DebtRatio.,main="QQ plot of normal data",pch=19)
> qqline(LN..DebtRatio.)
>
> #####Hypothesis test for normality###
> shapiro.test(LN..DebtRatio.)

```

Shapiro-Wilk normality test

```

data: LN..DebtRatio.
W = 0.96489, p-value = 0.3027

```

```

>
> #####Parameter Estimation###
> dt=1
> mean=mean(LN..DebtRatio.)
> stdev=sd(LN..DebtRatio.)
> mean
[1] 0.03349516
> stdev
[1] 0.1134146
> sigma=stdev/sqrt(dt)
> sigma
[1] 0.1134146
> sigma
[1] 0.1134146
> mu=mean/dt+sigma^2/2
> mu
[1] 0.0399266
> #####Determination of optimal debt#####
> u=0.033
> sigma=0.1134
> v=sigma^2
> v

```

```

[1] 0.01285956
> mu= 0.04
> mu
[1] 0.04
> lambda=0.095
> n=2
> k=1
> alpha=0.51
> gamma1={(-mu-sqrt(mu^2+2*lambda*v))/(v)}
> gamma2={(-mu+sqrt(mu^2+2*lambda*v))/(v)}
> varsigma={1/(lambda-v*n*(2*n-1)-2*u*n)}
> gamma1
[1] -8.055258
> gamma2
[1] 1.834206
> varsigma
[1] -8.759838
> f=varsigma*alpha
> f
[1] -4.467517
> #####Optimal debt ceiling#####
> b=(k*(gamma2-1))/((-f*2*n*(2*n-1))+(f*2*n*(gamma2-1)))
> b
[1] 0.02155409
> b=b^{1/(2*n-1)}
> b
[1] 0.2782979
> #####Finding B #####
> s=0.3152285^{2*n-gamma2}
> s
[1] 0.08205893
> B=-{(alpha * varsigma *2* n *(2*n -1))/( gamma2 *( gamma2 -1))* s #####
> B
[1] 2.875095
> #####Finding D####
> D=B*(b^gamma2)+(alpha*varsigma)*b^(2*n)-(k*b)
> D
[1] -0.02981958

```