On The Applicability of the Geometric Brownian Motion for Frontier and Developed Markets: A Comparative Approach.

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Submitted in partial fulfillment of the requirements for the Degree of Bachelor of Science in Financial Economics at Strathmore University

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August 2024

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DECLARATION

I declare that this work has not been previously submitted and approved for the award of a degree by this or any other University. To the best of my knowledge and belief, the Research Project contains no material previously published or written by another person except where due reference is made in the Research Project itself.

..…………….. [*Name of Candidate*]

……………... [*Signature*]

……………... [*Date*]

This Research Project has been submitted for examination with my approval as the Supervisor.

..…DR. MARIAN CHATORO.............................. [*Name of Supervisor*]

……… .... [*Signature*]

……24/01/2025.. [*Date*]

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Abstract

The goal of every investor participating in the stock market is to accurately predict stock prices and gain positive return. This necessity has led to the creation of technical analysis models which attempt to carry out this objective, one of which is the Geometric Brownian Motion (GBM). However, its applicability across different market conditions, ranging from stable developed markets to dynamic frontier markets, has been a subject of ongoing research. This study seeks to investigate the applicability of the Geometric Brownian Motion (GBM) model in predicting stock prices across both developed and frontier markets, using the American S&P 500 Index and the Kenyan NSE 20 Share Index as the main representative cases. The analysis revealed that while the GBM demonstrated strong predictive accuracy in developed markets with stable volatility and normality, its performance in frontier markets was mixed, with periods of market stability yielding better results than volatile, non-normal conditions. These findings emphasize the role of market stability and normality in shaping GBM's predictive accuracy, while also identifying the challenges posed by non-normal returns and volatility clustering. This study contributes to the discussion on adapting stochastic models for enhanced applicability in frontier markets, providing a foundation for future research and practical applications in financial forecasting.

Key Words: Geometric Brownian Motion (GBM), S&P500, NSE 20, Mean Absolute Percentage Error (MAPE), Developed Markets, Frontier Markets, Volatility, Predictive accuracy.

1. INTRODUCTION:

1.1 Background of the study

The stock market serves as a marketplace for buyers and sellers to trade equity shares issued by companies. These shares represent a part of the ownership of a company. Numerous factors such as the company's performance, industry conditions, macroeconomic factors, and investor sentiment cause share prices to fluctuate over time (Ahtesham & Verma, 2024). A common goal for every investor who participates in the stock market is to accurately predict prices of certain stocks and gain a positive return. However, due to the several factors that affect stock prices, it is not easy to do so. This motivation in seeking to know the future value of a stock's price has led to creation of different forecasting methods to predict stock prices (Mallikarjuna & Prabhakara Rao, 2019).

The two common approaches to forecasting stock prices are fundamental and technical analysis (Fama, 1965). Fundamental analysis refers to the evaluation of the intrinsic value of a security by seeking price-related information of a company such as its financial statements, management quality, competitive advantage, and industry trends so as to determine the stock's future performance. Technical analysis analyzes historical price movements and volume of a security over time and forecasts its future price movement. In our case we seek to use a technical analysis approach to simulate stock prices, using the Geometric Brownian motion (hereafter GBM).

The GBM is a stochastic process used to model various processes that change over time with uncertainty such as stock prices. It has gained wide acceptance as a valid model in simulating the return of assets as it takes into account the randomness as observed by stock price movements by incorporating the Brownian motion (Ermogenous, 2006). However, recent studies of the GBM's capabilities in modelling stock prices in developing countries have suggested that the stochastic process is unable to accurately predict prices in instances where the stock price distribution is heavily tailed, long-range correlations and does not account for periods of constant values (Marathe & Ryan, 2005; Sengupta, 2004).

1.2 Problem statement:

Accurately predicting stock prices is a common goal for all investors seeking to gain a positive return in the stock market. However, the inherent volatility and unpredictable nature of stock prices poses a challenge to this. This issue is further heightened by the fact that different types of markets exhibit different behaviors and characteristics unique to them. Traditional models such as the GBM have been used in simulating stock prices with great success. However, recent studies have highlighted limitations in its predictive accuracy, particularly in frontier markets, which exhibit characteristics such as volatility jumps, market illiquidity, and periods of constant values.

Several advanced models have been created to address the shortcomings of the traditional GBM in such market contexts such as jump-diffusion models, stochastic volatility models, local volatility models and regime-switching models. Despite these models providing significant improvements, they still use the GBM as an underlying process and rely on its foundational framework. Using the traditional GBM in this study allows for a more straightforward implementation and analysis. Moreover, it can serve as a standard benchmark against which more complex models can be suggested and compared. Understanding the GBM's performance and limitations within frontier markets will allow us to provide a baseline for future research in such market contexts.

1.3 Research objectives:

1. Evaluate the performance of GBM in predicting stock prices in both developed and frontier markets.
2. Analyze the factors that may impact the effectiveness of GBM in underdeveloped markets.
3. Compare the simulated stock prices generated by GBM with actual stock prices in the Kenyan NSE 20 Share Index and the American S&P 500 Index.

1.4 Research questions:

1. How effective is the traditional Geometric Brownian Motion (GBM) model in predicting stock prices in both developed and frontier markets?
2. What factors impact the predictive accuracy of the GBM model in frontier markets?
3. How do the simulated stock prices generated by the GBM model compare with the actual stock prices in developed and frontier markets?
4. How can the traditional GBM model be modified or extended to better account for the unique characteristics of frontier markets?
5. What are the implications of using the traditional GBM model for investment decision-making in frontier markets?

1.5 Significance of the study:

The main aim of this study is to evaluate the applicability of the GBM in forecasting prices across both developed and frontier markets. We seek to ascertain whether the GBM serves as a valid predictor of stock prices in under-developed markets, and if not, identify factors that may contribute to its inadequacy in such contexts such as market illiquidity, constant prices values and high volatility etc. Insights from this research will aid in proposing more effective modifications to the GBM to aid in stock price modelling in the unique environments of frontier markets and as well as provide investors wishing to invest in frontier markets more robust models in stock price prediction.

To achieve this, we will contrast the data obtained from simulated stock prices of stocks in the Kenyan Nairobi Securities Exchange 20 (NSE 20) Share index and the American Standard and Poor's 500 (S&P500) index.

2. LITERATURE REVIEW:

2.1 Theoretical review:

2.1.1 Overview of the Geometric Brownian Motion:

The stock market is a popular investment due to the potential of investors to gain high returns. However, with its potential to gain high returns, there is also the risk of going at a loss due to the stock market's unpredictable behavior. This inherent uncertainty has driven the development of mathematical models to forecast stock prices and address issues such as volatility, seasonality and time dependence. One such model is the Geometric Brownian Motion (GBM), which has become popular in financial modeling.

2.1.2 Historical Development of the GBM:

The Brownian motion was first discovered by Robert Brown in 1827 as he was investigating the random movement of pollen grains on water. Albert Einstein provided its theoretical backing in 1905, linking it to the kinetic theory of heat and demonstrating that the random movement was caused by collisions of water molecules with the pollen grains (Einstein, 1905). Nobert Wiener later provided rigorous mathematical framework for Brownian motion in 1923, leading to the alternative term "Wiener process" (Wiener, 1923). This development allowed several applications of this fundamental concept across various fields, including physics, finance, and engineering.

Later, Louis Bachelier theorized that price changes within the Paris stock exchange exhibited similar movements to the Brownian motion and published it in his PhD thesis, "Théorie de la spéculation". Bachelier's work also proves that future changes in stock prices are independent of their current value and are normally distributed. This model laid the groundwork for future developments of stochastic processes to financial modelling (Bachelier, 1900).

Building on this foundation, Paul Samuelson further refined the model by introducing the Geometric Brownian Motion. Samuelson's study was significant as it ensured that stock prices remain positive by modelling the stock price S_t as an exponential function of a Brownian motion. Bachelier's arithmetic Brownian motion allowed the possibility of stock prices to fall below zero. Through this refinement, Samuelson found that Bachelier's theory could be used to price options, formerly known as warrants (Samuelson, 1965).

The application of the GBM to financial markets was solidified with the creation of the Black Scholes model, which provides a closed-form solution for pricing European call and put options. Robert Merton further expanded their work, incorporating additional theoretical insights, which together led to the comprehensive Black-Scholes-Merton model. Black-Scholes-Merton model uses the GBM to describe the dynamics of underlying stock prices and has gained worldwide acceptance as the standard formula in pricing European options (Black & Scholes, 1973; Merton, 1973).

2.1.3 Components of the GBM:

The Geometric Brownian Motion is made up of two components, the drift and the stochastic term. A stochastic process S_t is said to follow a GBM if it follows the following stochastic differential equation (SDE):

$$dS_t = \mu S_t d_t + \sigma S_t dW_t$$

Whereby:

- μ is the drift coefficient, representing the expected return of the stock.
- σ is the volatility coefficient, representing the stock's volatility.
- S_t is the stock price at time t.
- dW_t is the Wiener process term, introducing randomness into the model.

The analytic solution of the equation is given as:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

For an initial value of S_0.

From the above equation, and the uncertain component of the stock price is described as the product of the stock's volatility and the Wiener (Brownian) process which includes an element of random volatility (Brewer, Feng and Kwan, 2012).

2.1.4 Importance of the GBM:

GBM is favored in financial modeling because it accounts for the randomness observed in stock price movements. Its ability to incorporate the stochastic nature of prices makes it suitable for simulating asset returns, despite certain limitations. Recent studies have highlighted that while GBM performs well in developed markets, it may struggle with accurate predictions in markets characterized by heavy-tailed distributions, long-range correlations, and periods of constant values (Marathe & Ryan, 2005; Sengupta, 2004).

2.2 Empirical review:

The application of Geometric Brownian Motion (GBM) in financial modeling has been extensively studied, primarily focusing on developed and emerging markets across various economic conditions. GBM has consistently demonstrated its effectiveness in modeling stock prices under different economic scenarios

Brătian et al. (2022) conducted a forecast of stock prices within several stock market indexes such as the DAX (Deutscher Aktien Index) of Germany, the S&P 500 of America, and the SHANGHAI Composite of China to prove the effectiveness of the GBM, accounting for different economic conditions like non-crisis and financial crisis scenarios. Their study enhances the drift and diffusion(stochastic) components of the model based on the probability of the economic conditions. Through their research, they found that the GBM, when modelled to capture the conditions of the economic environment, is effective in forecasting stock market index values and supports the informational efficiency of these markets.

Similarly, a study conducted by Si and Bishi (2020) applied the GBM to the S&P BSE index in the Bombay Stock Exchange to assess the model's short-term accuracy in predicting future stock prices. They conducted rigorous statistical tests such as the Kolmogorov-Smirnov test and Q–Q plots to confirm normality in the stock price data. The stock prices were simulated for 23 days using the log volatility equation and upon evaluation of the accuracy in prediction, they obtained a Mean Absolute Percentage Error (MAPE) of 5.41%. Despite the study's focus on a single market and a short time frame, the positive correlation coefficient in simulated and actual stock prices shows the GBM's reliability in capturing short term market movements. However, there is a need for further research, especially in the context of frontier financial markets.

For the GBM to accurately model stock prices, several assumptions must be made. These include: the log returns of the stock prices must follow a log-normal distribution, log returns are independent of each other, meaning that past returns do not influence future returns, statistical properties of the distribution such as the mean and variance remain constant over time and stock prices follow a random walk hence unpredictable over time.

Given the assumptions stated above, Marathe and Ryan (2005) conducted a study to evaluate how the GBM assumptions hold across various contexts such as stock prices, natural resource prices, and demand for products or services. They conducted multiple tests to confirm their datasets follow the GBM assumptions such as normality tested using Q-Q plots, independence using the Chi-squared tests, stationarity and random walk using the augmented Dickey-Fuller test. Through their modelling they found that while some datasets conformed to the GBM assumptions after deseasonalization, others did not. Hence this outlines the importance of validating these assumptions before applying the GBM model.

Despite the success of the GBM as a predictive model, it has several limitations that lie within its assumptions that make it unsuitable in modelling stock prices in certain contexts. A study by Stojkoski et al (2020) focuses on the log-normal assumption of returns, which can lead to inaccuracies when the actual distribution is not log-normal and exhibits skewness and heavy tails which is common in financial markets. They introduced a modification to the GBM called the Entropy Corrected Geometric Brownian Motion (EC-GBM) which addresses the weakness of the log-normal assumption. The model improved the predictive accuracy by incorporating entropy corrections, making it a robust alternative for the traditional GBM.

Similarly, the Heston model developed by Steven L. Heston addresses the constant volatility assumption of the Black Scholes model and the GBM by introducing a component of stochastic volatility in the options pricing framework. The model assumes that volatility is stochastic and follows its own random process essentially incorporating ‘volatility’ in the volatility of the stock. The volatility follows the mean-reverting Ornstein-Uhlenbeck process which assumes that volatility tends to revert to a long-term average over time, which is a realistic feature observed in financial markets. This model solves the multiple limitations of the GBM model such as failure to account for continuous price paths without jumps, asymmetric distribution of returns and clustering of large stock price movements (Heston, 1993).

Given the GBM's limitations mentioned above as well as the model modifications created to enhance the accuracy in predicting future stock prices, there is a need to assess comparatively the factors that differentiate the differences in stock price modelling in developed versus frontier markets by the traditional GBM. Moreover, several factors as seen in frontier markets such as volatility jumps or clustering, periods of constant values, illiquidity and lack of incorporating regime switching (Reddy and Clinton, 2016) highlights the need to further investigate how these factors impact the accuracy of GBM modelling.

3. METHODOLOGY

3.1 Research Design

The research will follow an empirical research design to evaluate the performance of the GBM, answer the research questions and fulfill the research objectives. This design is chosen as it allows collection and analysis of numerical stock price data necessary for modelling and comparison. Moreover, this will also allow us to test hypothesis and draw conclusions on the effectiveness of the GBM in different market environments.

3.2 Population and Sampling

The study will focus on historical stock price data of the American Standard and Poor's 500 (S&P 500) Index, which would represent the developed market and the Nairobi Stock Exchange 20 (NSE 20) Share Index, which would represent the frontier market.

The S&P 500 Index represents the top 500 companies listed on stock exchanges in the US, while the NSE 20 share index represents the top 20 companies listed on the Nairobi Securities Exchange. The data will use similar time horizons and frequency of stock price data to ensure the forecasted stock price data is adequate for comparison.

3.3 Data collection

The study utilizes secondary data comprising of historical stock prices. The S&P 500 data is retrieved from the Yahoo Finance website while the NSE 20 data is retrieved from the Nairobi Securities Exchange website. The data is downloaded from the relevant websites and will span a period of 4 years, from 1st January 2021 to 1st January 2024. These financial databases are adequate for downloading large volumes of stock data.

The selected time period captures the most recent market trends and behaviors in both developed and frontier markets. This time period allows for sufficient data collection to conduct comprehensive statistical analyses and balances the need to capture short-term fluctuations and long-term trends within the market. It also encompasses the post-COVID-19 recovery phase which had significant impacts on global financial markets. Moreover, starting the time period in 2021 helps reduce the impact of seasonality caused by the pandemic in 2020, hence ensures the data reflects more typical market conditions.

3.4 Data Analysis:

3.4.1 Derivation of the Stochastic Differential Equation (SDE) of the Geometric Brownian Motion (GBM)

Before going into the data analysis, we first derive the stochastic differential equation of the GBM which we will use in modelling the stock prices. The GBM is characterized by the following SDE:

$$dS_t = \mu S_t d_t + \sigma S_t dW_t$$

Applying the functional form:

$$f = f(t, S_t) = In(S_t)$$

For a function $f(t, S_{t,}) = In(S_{t,})$, Ito's lemma, after applying rules of box calculus, states:

$$df = \frac{\partial f}{\partial t} dt + \partial St \frac{\partial f}{\partial St} dSt + \frac{\partial^2 f}{\partial 2St^2} dSt^2$$

Using this to find the partial derivatives of f:

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial St} = \frac{1}{St}$$

$$\frac{\partial^2 f}{\partial S^2 t} = -\frac{1}{St^2}$$

Plugging back to the original Ito's equation:

$$df = 0 + \frac{1}{St} dSt - \frac{1}{St^2} dSt^2$$

And since $dS_t = \mu S_t d_t + \sigma S_t dW_t$, the equation is given as follows:

$$df = 0 + \frac{1}{St}(\mu S_t d_t + \sigma S_t dW_t) - \frac{1}{St^2} (\mu S_t d_t + \sigma S_t dW_t)^2$$

Which simplifies to:

$$df = (\mu - \frac{1}{2}\sigma)dt - \sigma dW_t$$

Our goal is to derive the solution of the stock price S_t. Recall the functional form:

$$f = f(t, S_t) = In(S_t)$$

We therefore replace f in our equation with $In(S_t)$:

$$d(In(S_t)) = (\mu - \frac{1}{2}\sigma)dt - \sigma dW_t$$

Integrating the equation from time t to time 0:

$$\int_0^t d(In(S_u)) = \int_0^t (\mu - \frac{1}{2}\sigma)du - \sigma dW_u$$

Removing the constants out of the integral:

$$\int_0^t d(In(S_u)) = (\mu - \frac{1}{2}\sigma)\int_0^t du - \sigma\int_0^t dW_u$$

Solving the integral gives us:

$$\ln S_t + \ln S_0 = \left(\mu - \frac{1}{2}\sigma\right)(t) + \sigma W_t$$

Making $\ln S_t$ the subject of the equation:

$$\ln S_t = \ln S_0 + \left(\mu - \frac{1}{2}\sigma\right)(t) + \sigma W_t$$

Exponentiating both sides of the equation to return to the original variable S_t:

$$S_t = S_0 . e^{\left(\mu - \frac{1}{2}\sigma\right)(t) + \sigma W_t}$$

This is the analytical or explicit solution of the GBM, which will be used in the modelling of the stock prices.

Note that integrating from a time T to time t would give us a solution of:

$$S_T = S_t . e^{\left(\mu - \frac{1}{2}\sigma\right)(T-t) + \sigma(W_T - W_t)}$$

3.4.2 Data Analytics:

The stock price data will be analyzed and modelled using the statistical software R and Python. The GBM assumes that stock prices follow a log-normal distribution, therefore testing for normality of stock prices before modelling will be crucial. Techniques such as Q-Q plots and the Shapiro Wilk test will be used to check for normality.

The drift component μ will be estimated as the average of logarithmic returns:

$$\mu = \frac{1}{T}\sum_{t=1}^{T} \ln(\frac{S_t}{S_{t-1}})$$

The volatility σ will be estimated as the standard deviation of logarithmic returns:

$$\sigma = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(\ln\left(\frac{S_t}{S_{t-1}}\right) - \mu)^2}$$

3.4.3 Methods for comparing Simulated and Actual Stock Prices:

a) Correlation coefficient:

The first method will involve calculating the correlation coefficient of the simulated and actual stock prices. This will measure the strength and direction of the relationship of the two prices whereby 1 indicates a perfect linear relationship and -1 indicates a negative linear relationship between the two prices. The formula is given as:

$$\rho = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{[(n(\sum x^2) - (\sum x)^2)(n(\sum y^2) - (\sum y)^2)]^2}}$$

Where x represent the simulated stock prices and y represent the actual stock prices.

b) Mean Absolute Percentage Error:

The second method will be Mean Absolute Percentage Error (MAPE) which measures the prediction accuracy of the model. According to Abidin and Jaffar, a MAPE value of less than 10 percent is considered to show highly accurate forecasts and as the value increases, the model signifies lower predictive accuracy, whereby 51% percent and above suggests highly inaccurate forecasts (Abidin and Jaffar, 2012).

$$MAPE = \frac{1}{n}\sum_{t=1}^{n} \left| \frac{S_{act,t} - S_{sim,t}}{S_{act,t}} \right|$$

3.5 Model Validation:

Validation of the GBM model to be used in forecasting stock prices is essential to assess its reliability when applied to different datasets and market conditions. To ensure the generalizability of the GBM, this study will also test the GBM on additional developed markets such as the German DAX and Australian ASX, as well as frontier markets, including the Bangladesh DSE index and the Nigerian NSE All Share Index. The techniques that will be used in validating the model are included below:

a. Out-of-Sample testing:

Out-of-Sample testing involves evaluating the performance of the model by applying it to a different dataset. The main goal of this method is to ensure that the model not only performs well on the data it was trained (in-sample data) on but also on new data. The procedure entails dividing the dataset into two parts: in-sample period for training and out-of-sample periods for testing. For instance, in this study we will apply the GBM on data from 2017 to 2020 (pre-covid period) to train the model and then test its performance from 2021 onwards. This would ensure that the model would make good predictions in a real-life trading scenario.

b. Cross-Validation:

Cross-validation ensures that the model stays consistent across different subsets of the data. The process involves dividing the dataset into a number of subsets, for instance, k-subsets. The model is then trained on k-1 subsets and tested on the remaining subset. This process is repeated several times, and the descriptive statistics are then averaged to gauge the model's overall performance. This method of validation also prevents overfitting of the model on the training data and ensures it also performs on unseen data.

c. Back testing

This involves assessing the model's predictive accuracy by training the model on the in-sample data and comparing the predicted values with actual values. The performance metrics mentioned above such as the correlation coefficient and the MAPE will be computed to provide insights on the model's performance.

3.6 Overall Results Interpretation:

The procedure to conducting the overall results interpretation from the GBM model is outlined as follows; Firstly, the results of the normality tests such as the Shapiro-Wilk test and Q-Q plots will be analyzed to ensure that the log-normal distribution assumption of the GBM holds. Secondly, the assessment of the key performance metrics, the correlation coefficient and MAPE will provide insights on the model's performance in the developed and frontier markets. A low MAPE (20% and below) and a high correlation coefficient (a value approaching positive one) after back testing will indicate that the GBM model was highly accurate and is effective for stock price prediction in that particular market environment. However, a high MAPE (above 20%) and a low correlation coefficient (a value approaching negative one) will suggest that the GBM model found it difficult to capture the complexities of that particular market environment. This will allow us to investigate the specific factors that would affect the model's performance such as high volatility or illiquidity and provide insights to any necessary model refinements.

4. RESULTS AND ANALYSIS

4.1 Introduction:

This section will cover the analysis of the GBM's accuracy in capturing the market dynamics of all the markets applied to the GBM.

4.2 Normality analysis:

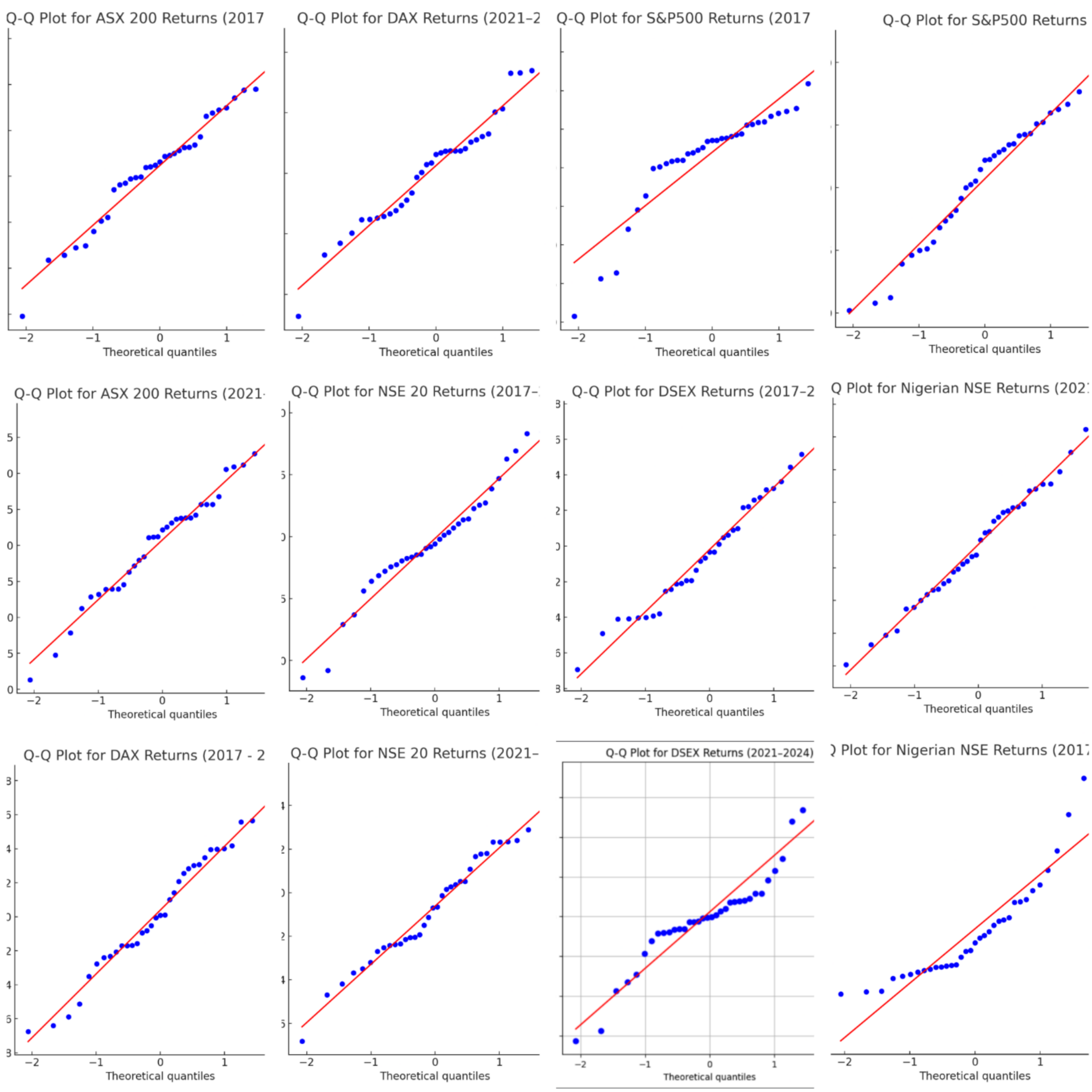
Before conducting the GBM modelling of stock prices, we first assess the normality of our data. This was conducted using the Shapiro-Wilk test and Q-Q plots to adequately capture the dynamics of the stock market data of each specific market. For a particular dataset's returns to be deemed normal, it's p-value must be greater than 0.05 in order to reject the null hypothesis, suggesting that the data aligns with the GBM's normality assumption. The Q-Q plots also give a visual representation of the normality of the data, indicated by the dataset's points aligning with the theoretical quantile line. While confirming normality is important, it does not guarantee that the model will capture the full range of market dynamics. Financial markets are inherently complex, with underlying price dynamics influenced by factors like sudden market corrections, shifts in financial sentiment, and periodic increases in volatility. These factors can introduce irregularities in market behavior that, while subtle, may challenge the GBM's predictive performance. The table below shows a summary of the Shapiro-Wilk tests results for each market:

- Table 1: Normality analysis

Market	Index	Period	Shapiro-Wilk W	p-value	Normality Interpretation
Developed Market	S&P 500	2017–2020	0.886	0.0017	Non-normal; high volatility
	S&P 500	2021–2024	0.951	0.1914	Approx. normal; stabilized post-COVID
	DAX	2017–2020	0.969	0.4221	Approx. normal; regular market conditions
	DAX	2021–2024	0.963	0.2936	Approx. normal; increased volatility
	ASX 200	2017–2020	0.963	0.639	Approx. normal; low volatility
	ASX 200	2021–2024	0.957	0.5858	Approx. normal; moderate volatility
Frontier Market	NSE 20	2017–2020	0.921	0.2807	Approx. normal; higher stability
	NSE 20	2021–2024	0.971	0.8145	Approx. normal; higher stability
	DSEX	2017–2020	0.942	0.7282	Approx. normal; low data variance
	DSEX	2021–2024	0.939	0.0184	Non-normal; higher volatility

	NSE All (Nigeria)	2017–2020	0.931	0.0004	Non-normal; erratic market behavior
	NSE All (Nigeria)	2021–2024	0.957	0.9567	Approx. normal; improved stability

➢ Figure 1: Q-Q plots



4.2.1 Market by market justification and interpretation:

1. Developed Markets:

a) S&P 500 index (USA):

The pre-covid (2017-2020) p-value is low (0.0017) suggesting that the data deviates from a normal distribution. This can be attributed to the high market volatility and sporadic price jumps within the US market at the time, likely due to the US-China trade war (Bown, 2020; Irwin, 2019) and the Federal Reserve's rate hikes and cuts that may have brought about market uncertainty (Federal Reserve, 2018; Smialek, 2019). However, in the post-covid period (2021-2024) the p-value increased to 0.1914 indicating a closer approximation to normality as markets stabilized after the COVID-19 pandemic. This change supports GBM's applicability in capturing trends under more stable market conditions in mature economies.

b) DAX index (Germany):

In this index, normality was generally consistent for both periods, with p-values of 0.4221 (2017-2020) and 0.2936 (2021-2024). These values suggest that the DAX returns were approximately normally distributed, aligning with the GBM model's assumptions.

c) ASX 200 (Australia):

Similarly, the ASX 200 index in Australia displayed consistent normality in returns in both periods, (0.639 in 2017 to 2020 and 0.5858 from 2021 to 2024). This suggests that the GBM model's assumptions may hold well under stable market conditions in developed economies.

2. Frontier markets:

a) Nairobi Stock Exchange (NSE 20):

For the 2017 to 2020 period for the Nairobi NSE 20, a p-value of 0.2807 is observed, which points to normality in the return distribution. This was unforeseen for a frontier market which as compared to developed market, is more susceptible to factors such as economic shocks, limited investor participation and low liquidity which may cause rapid shifts in normality. Moreover, the 2021 to 2024 period reveals a surprising shift, with a p-value of 0.8145 indicating strong normality. This unusual result for a frontier market may suggest that, during this period, the Kenyan stock market experienced relatively stable movements, potentially driven by specific economic or regulatory conditions fostering market stability. Nevertheless, it's crucial to recognize that the Shapiro-Wilk test may not fully capture periods of abrupt volatility shifts, as it assesses the overall distribution rather than detecting intermittent spikes or structural breaks.

b) Nigerian NSE All Share Index:

Similarly, the Nigerian normality test for the 2017–2020 period indicates pronounced non-normality, with a p-value of 0.0004, reflecting the typical characteristics of a frontier market. In contrast, the 2021–2024 period reveals an unexpected shift, with a p-value of 0.9567 suggesting strong normality—an unusual outcome for a frontier market. This shift could highlight the inherent unpredictability of frontier markets, where significant swings in normality can occur between periods, influenced by changing economic conditions, regulatory impacts, or shifts in investor behavior (Salvatore, 2020). The drastic difference between these periods underscores the volatile and often unpredictable nature of frontier markets, capable of exhibiting distribution patterns that even surpass those seen in developed markets.

c) Bangladesh DSEX Index:

For the DSEX Index, we observe the opposite pattern compared to the Kenyan and Nigerian indices. The 2017 period shows strong normality, with a p-value of 0.7282, suggesting that market conditions at that time may have been relatively stable, fostering a return distribution close to normal. However, the 2021 period reflects non-normality, with a p-value of 0.0184. This shift could indicate increased volatility or structural changes within the market, possibly driven by post-pandemic economic adjustments or shifts in investor sentiment, which may disrupt normal distribution patterns in frontier markets.

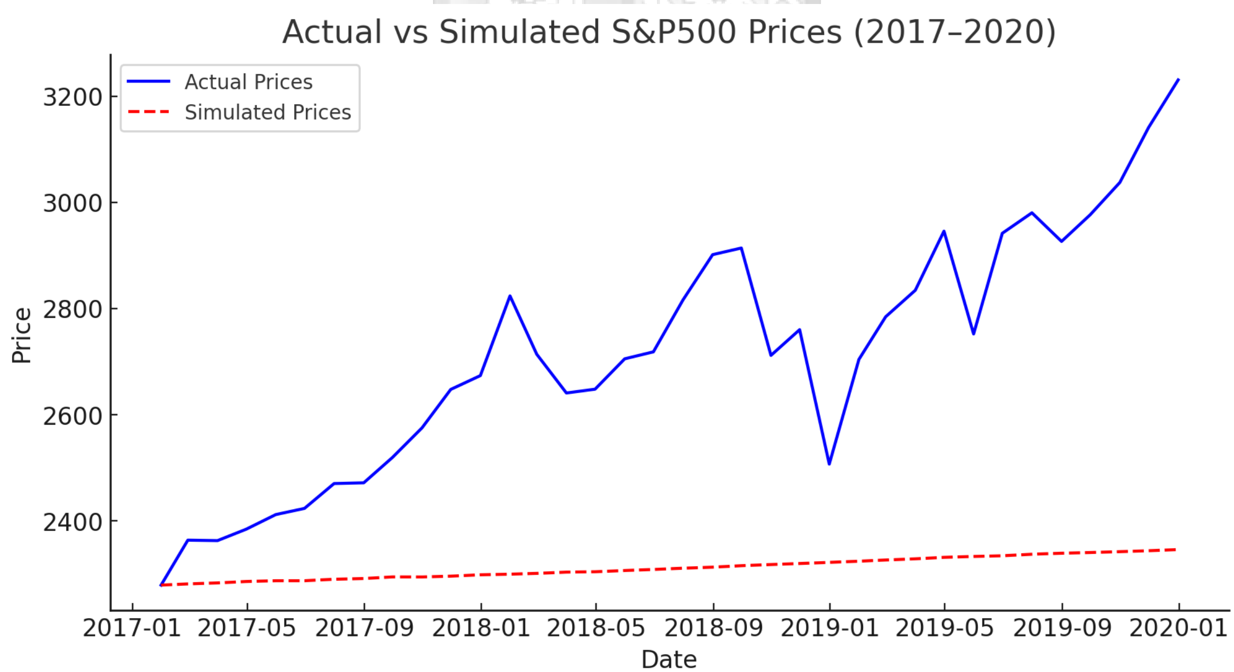
4.3 Comparison of Actual vs Simulated Prices (Back testing):

4.3.1 Developed Markets

a) S&P 500:

i. 2017 to 2020 period:

➢ Figure 2: S&P 500 Actual vs Simulated graph (2017 – 2020)

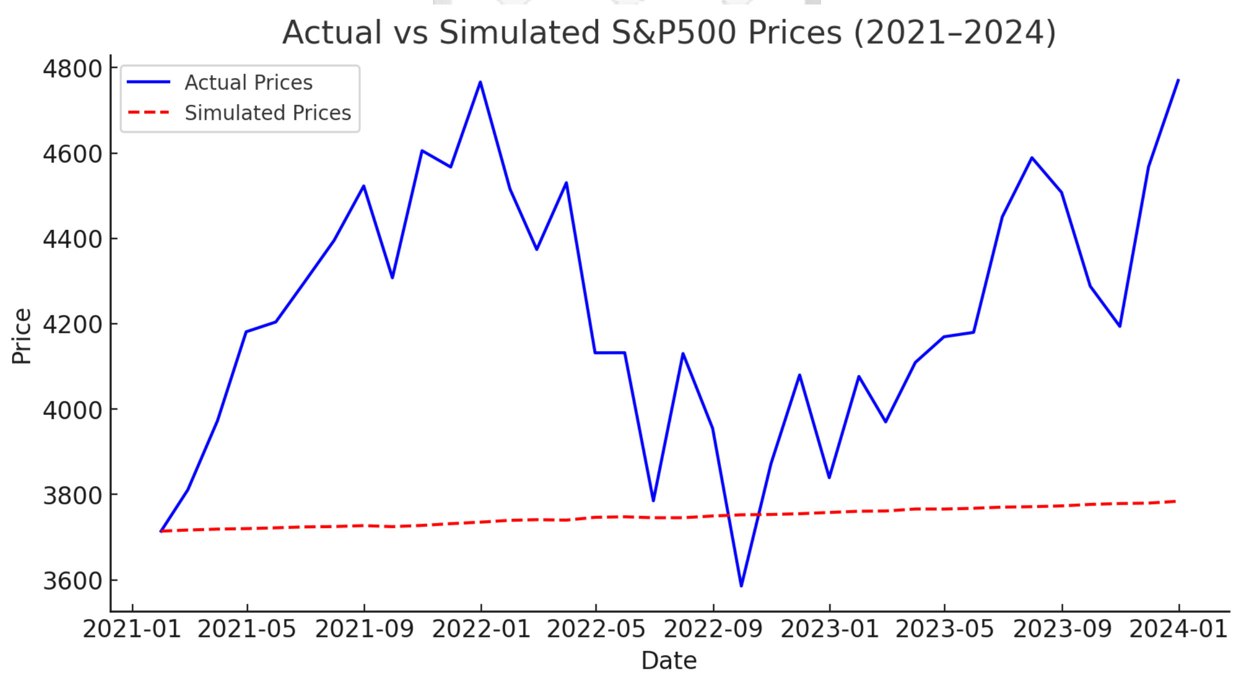


From both the table and the graph above, it is clear that the GBM struggled to adequately model the stock prices of the S&P500 during the 2017 to 2020 period. This can be largely attributed to the non-normality of the data, evidenced by its low p-value of 0.0017 from the Shapiro-Wilk test. Moreover, despite having a low annualized volatility of 0.0358 over the 3-year period, the GBM still encountered significant challenges in modelling due to the fact that the data failed to satisfy its assumption that the data should be normal. This highlights the limitations of the GBM as a predictive model in market environments that exhibit non-normality.

Contrary to our expectations, the actual and simulated values give a MAPE score of 13.91%, a moderate score despite the fact that the two lines in the graph were distant and did not overlap. This, however, can be attributed to the small scale of deviation that is exhibited by the two sets of values. MAPE measures the relative difference between two sets of values and therefore due to the small percentage deviations, the MAPE score is moderate. Moreover, the values also give us a correlation coefficient value (rho) of 0.9048, which was also unexpected. Rho measures the strength of the linear relationship between two sets of values, not the absolute difference. It captures the general directional trend for the actual and simulated values and therefore, despite their divergence in magnitude in the two values, the high correlation value suggests that the GBM was reasonably aligned with the movement of the actual index values.

ii. 2021 to 2024 period:

- Figure 3: S&P 500 Actual vs Simulated graph (2021 – 2024)



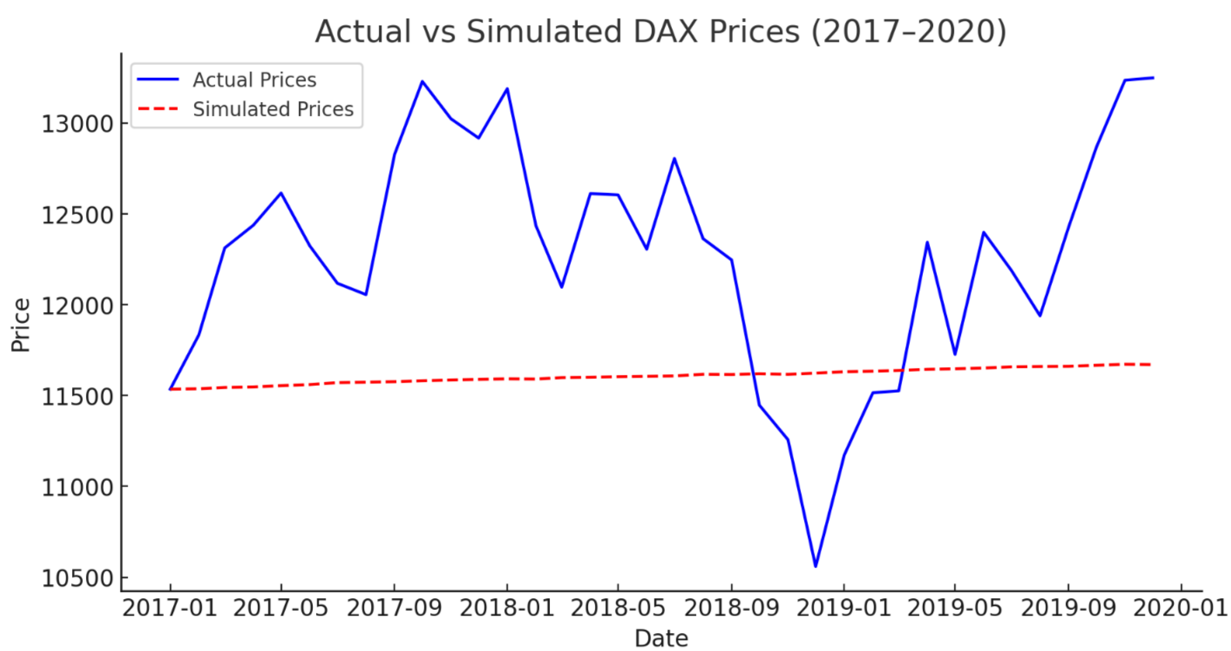
The GBM demonstrated an improved ability to model the S&P500 stock prices during the 2021 to 2024 period. This is due to the fact that the data aligns more closely to a normal distribution as indicated by its p-value of 0.1914 hence satisfies the GBM's core assumption. Additionally, global markets during this period began to stabilize after the COVID-19 pandemic, as evidenced by the market's low annualized volatility of 0.0513352 over the 3-year period. Despite the improvements, we can observe from the graph that the GBM struggled to track the market accurately particularly during the 2021 to mid-2022 period. This may suggest that shortening the forecast period may enhance the model's predictive accuracy.

The period exhibited a MAPE score of 11.45% which is an improvement from the previous period. This shows that the GBM did better due to the more stable post-pandemic market conditions. Additionally, the period also gave a correlation coefficient value of 0.9527 which indicates an even stronger alignment of direction between the movements of the actual and simulated prices. Therefore, this suggest that the GBM performed better in general in the post-pandemic period.

b) German DAX:

i. 2017 to 2020 period:

➢ Figure 4: German DAX Actual vs Simulated graph (2017 – 2020)

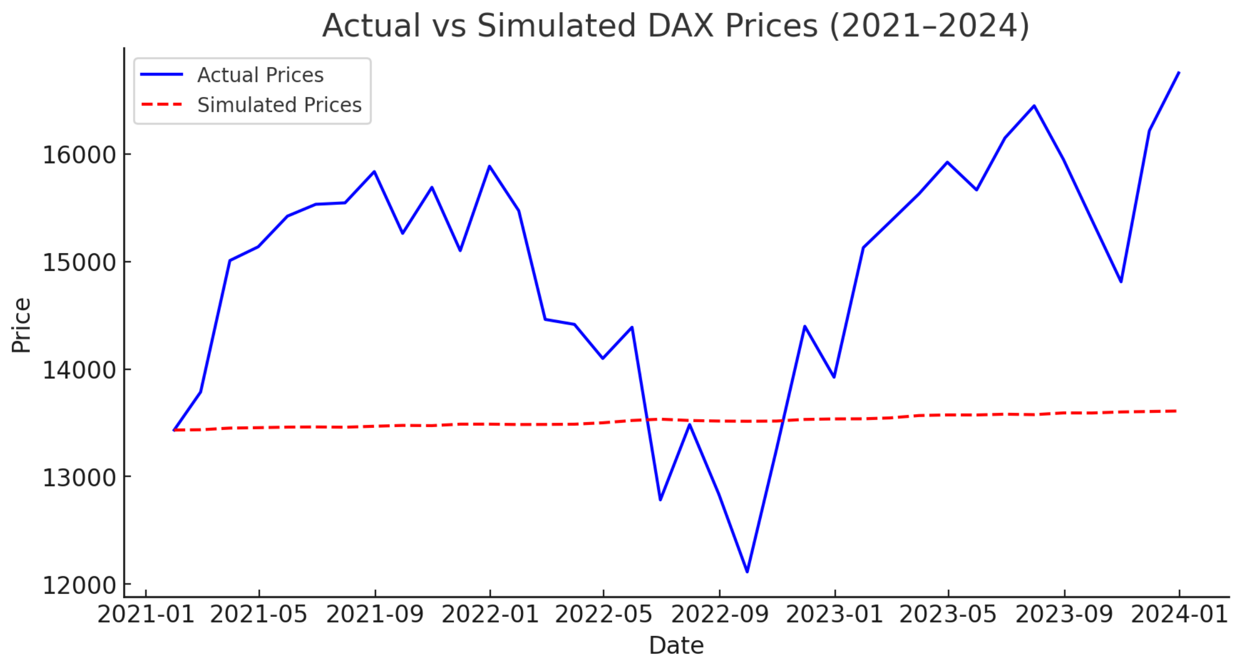


The GBM performed relatively well in simulating stock prices for the German DAX during the pre-covid period. The simulated prices align very closely with the actual prices throughout the period, intercepting multiple times as shown in the graph. This can be attributed by the high Shapiro-Wilk test p-value of 0.4229 which indicated that the dataset strongly aligns to a normal distribution. Moreover, a low annualized 3-year volatility of 3.66% suggests that there was little variability in the stock prices which favored the GBM's predictive ability.

Additionally, a low MAPE score of 6.2% further highlights the model's effectiveness, indicating that there were very small percentage deviations between the actual and simulate values. However, the dataset gives a correlation coefficient value of -0.06, which suggest that there was a disconnect in the directional trends of the actual and simulated prices. While this may seem like a minor issue since the model still fairly aligned with the actual prices, it is important to consider that the negative correlation score may be caused by volatility clustering or external market shocks that may not be well captured by the GBM. Despite this, the dataset illustrates that under conditions of low volatility and strong normality, the GBM can serve as a reliable predictive model.

ii. 2021 to 2024 period:

➢ Figure 5: German DAX Actual vs Simulated graph (2021 – 2024)



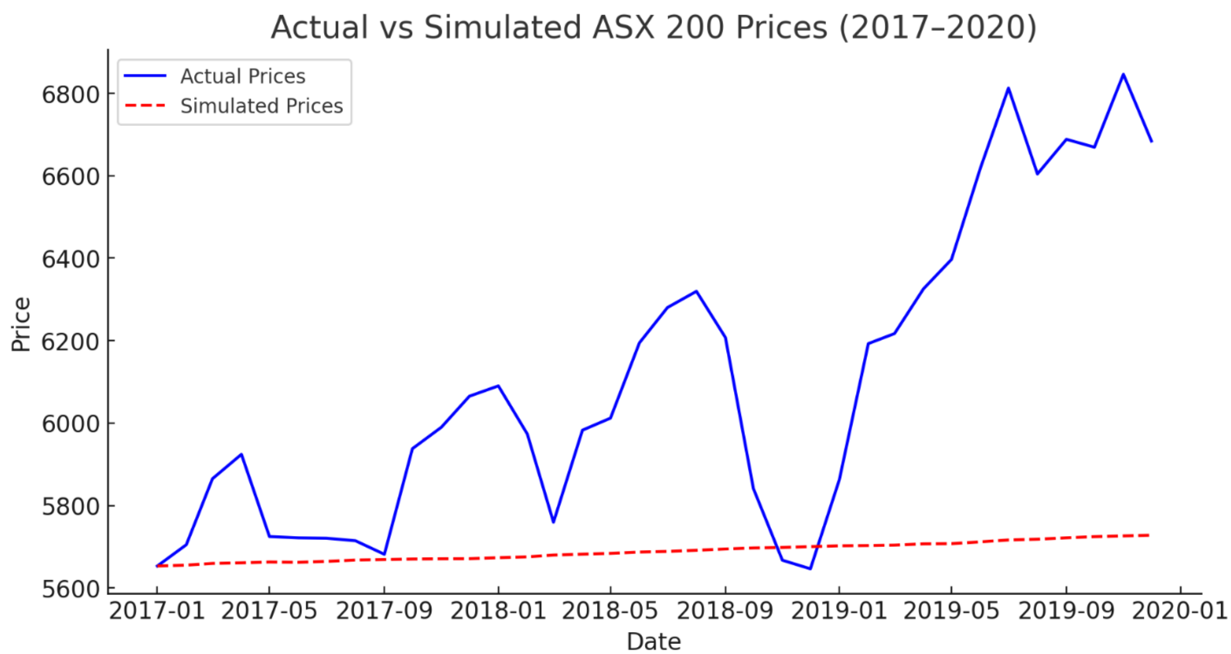
The GBM also exhibited a moderate performance during the 2021 period. The Shapiro-Wilk test p-value of 0.2936 shows that the data aligns with a normal distribution and a low annualized volatility of 4.87% reflects a stable market environment, both being conducive for the GBM's application.

A MAPE of 9.84% suggests that the percentage deviations between the actual and simulated values were small, supporting the model's predictive accuracy. A correlation coefficient of 0.1824 is an improvement from the previous period but is still however low. It indicates a weak positive relationship in replicating the directional patterns of the actual and simulated prices, showing that the GBM struggled to match the movements of the simulated prices to the actual prices. Overall, the GBM still performed relatively well within this period.

c) Australian ASX index:

i) 2017 to 2020 period:

➢ Figure 6: Australian ASX Actual vs Simulated graph (2017 – 2020)

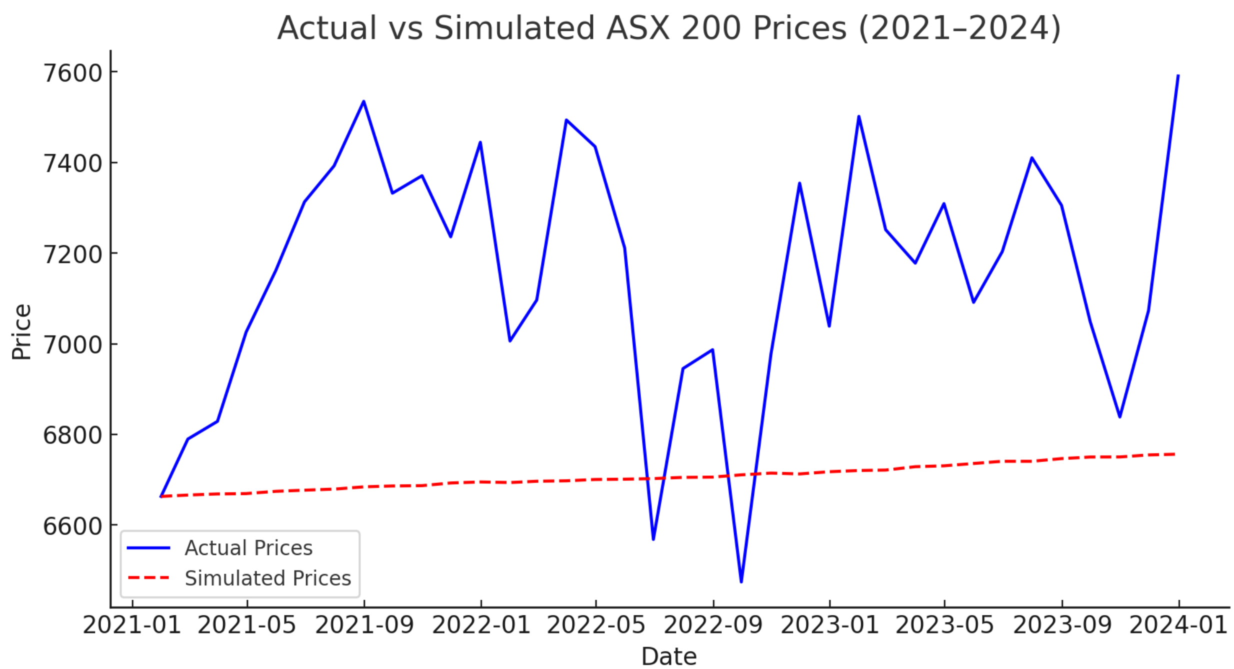


The GBM did exceptionally well in the Australian AXS during the 2017 to 2020 period. An annualized volatility of 2.55% reflects that the market was relatively stable, favoring the GBM's application. Moreover, a p-value of 0.639 indicates that the data strongly aligns with a normal distribution, a key assumption of the GBM.

A MAPE value of 6.27% indicates that the percentage deviations between the actual and simulated prices were minimal. Moreover, a correlation coefficient value of 0.7972 indicates that the simulated prices strongly mimicked the movement of the actual prices. However, as seen from the graph, the simulated prices consistently underestimate the actual prices, this being attributed to the GBM's assumption of constant drift and volatility. This may suggest incorporating varying volatility models such as the Heston model or drift components adjustments could further improve the model's performance. Overall, the GBM still performed relatively well during this period.

ii) 2021 to 2024 period:

- Figure 7: Australian ASX Actual vs Simulated graph (2021 – 2024)



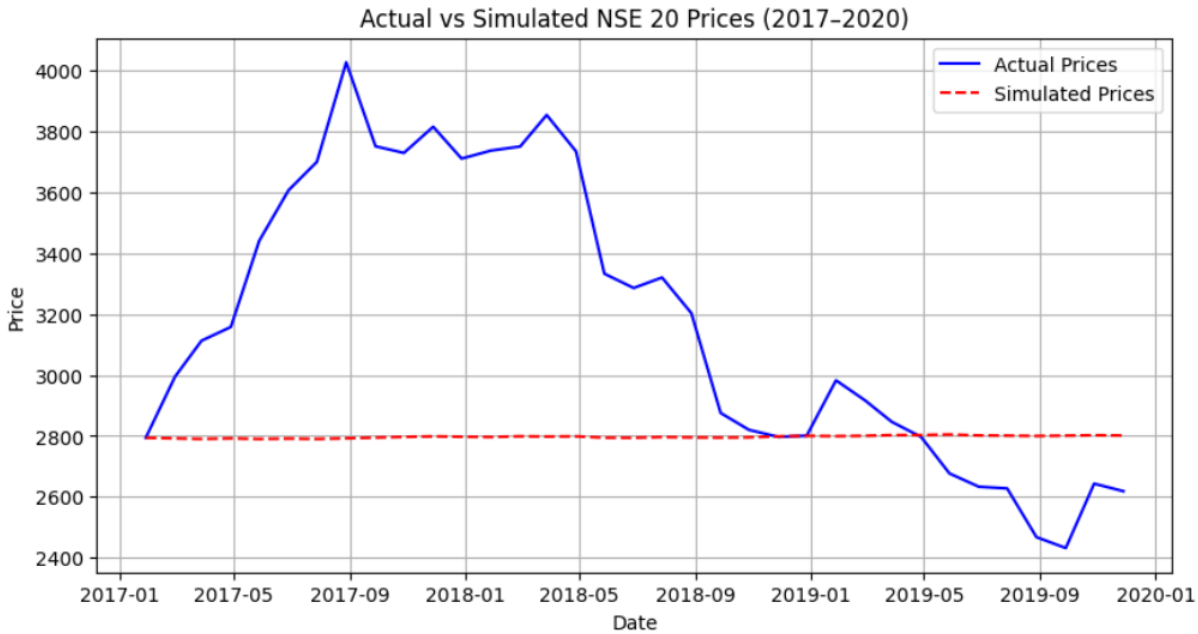
The GBM also demonstrated moderate performance within the 2021 to 2024 period. A p-value of 0.5858 supports the GBM's assumption of normality and a low annualized volatility of 4.05% indicates that the market was stable. The MAPE of 6.43% highlights low deviations between actual and simulated prices; however, the weak correlation coefficient ($\rho = 0.0877$) suggests the model struggled to replicate directional trends accurately. Its limitation of constant drift and volatility components still arise as seen in the graph with its constant underestimation of the actual prices. Overall, the GBM provided reasonable predictive accuracy.

4.3.2 Frontier Markets:

a) Kenyan NSE 20:

i. 2017 to 2020 period:

➢ Figure 8: Kenyan NSE 20 Actual vs Simulated graph (2017 – 2020)

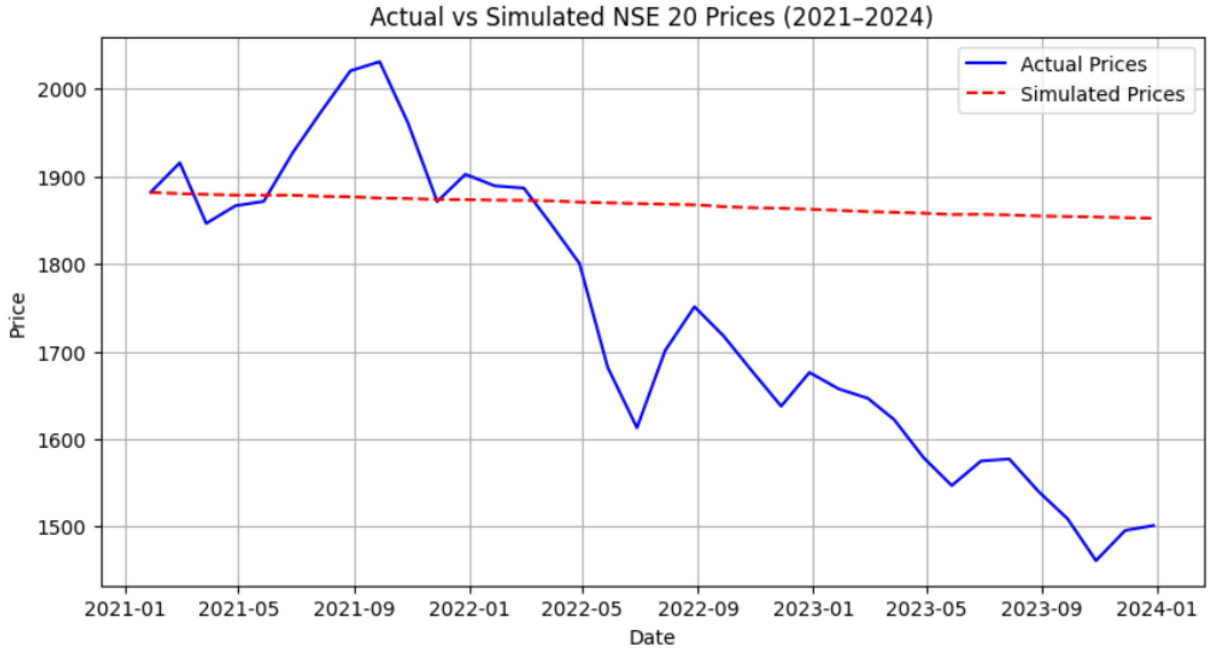


The GBM performed moderately well in the 2017 period for the NSE 20, given the favorable market conditions. The p-value of 0.2807 indicates that the data approximately aligns with a normal distribution, being one of the key assumptions of the GBM hence explains the model's performance. Moreover, the annualized volatility of 4.78% reflects minor market fluctuations during the period. The market's stability can be attributed to Kenya's steady economic growth throughout the period and peaceful 2017 elections which bolstered investor confidence.

The MAPE of 13.28% shows that there were moderate percentage deviations between the actual and simulated prices, with the simulated prices underestimating the actual price levels as seen from the graph and table. The correlation coefficient of 0.7591 also demonstrates a positive relationship between the movement of the actual and simulated prices. Overall, the GBM provided reasonable predictive insights given the usual characteristics of a frontier market.

ii. 2021 to 2024 period:

- Figure 9: Kenyan NSE 20 Actual vs Simulated graph (2021 – 2024)



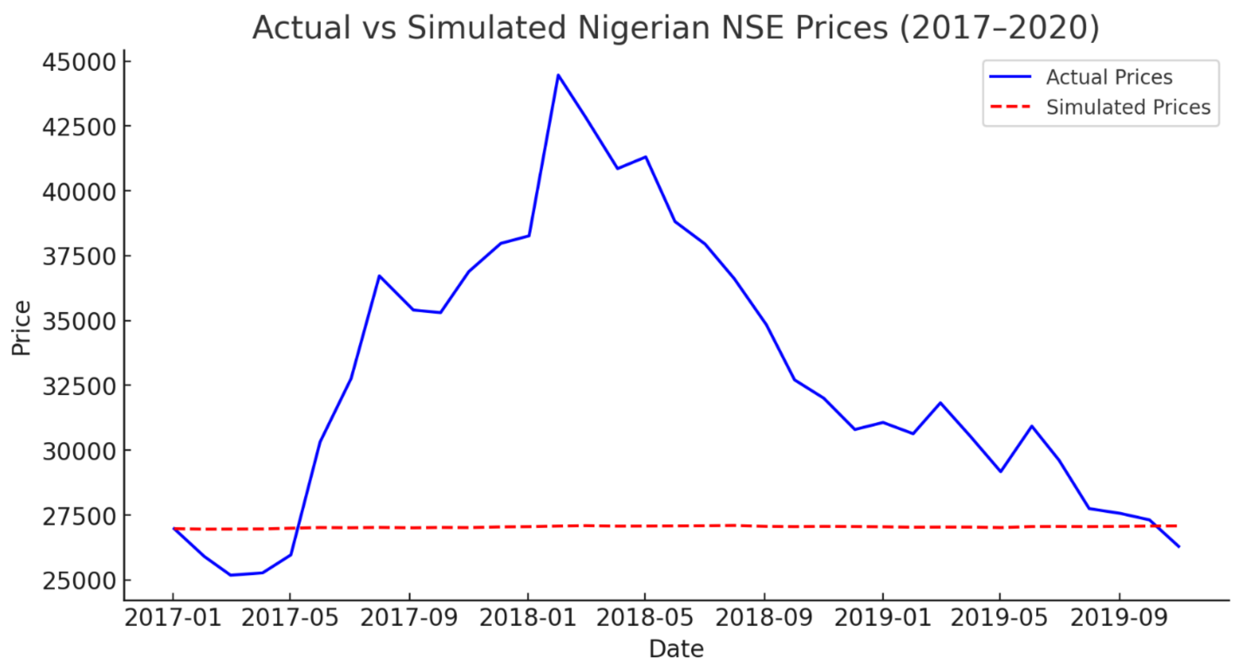
The GBM performed exceptionally well in the 2021 period for the NSE 20. Given the favorable conditions which attributes to its performance such as the high p-value of 0.8145 which signifies that the data closely aligns to a normal distribution and the low annualized volatility of 2.59% signifying that there were minimal market fluctuations within the period. This stability can be attributed to the rebound in economic activity from the disruptions caused by the pandemic and stable exchange rates. It is important to note this period of stability ceased and was followed by a large stock market downturn caused by reduced investor sentiment due to political tensions from protests and large decrease in the market capitalization.

A MAPE score of 9.24% indicates that the percentage deviations between actual and simulated prices were minimal and the correlation coefficient value of 91.48% shows that the GBM's simulated prices aligned closely with the movement of the actual prices hence further proving the GBM’s good performance during the period. This performance was unforeseen for a frontier market like Kenya. However, the results emphasize that given the right parameters, the GBM can perform exceptionally well even in markets that are typically considered more unpredictable.

b) Nigerian NSE All Share Index:

i. 2017 to 2020 period:

- Figure 10: Nigerian NSE All Share Index Actual vs Simulated graph (2017 – 2020)

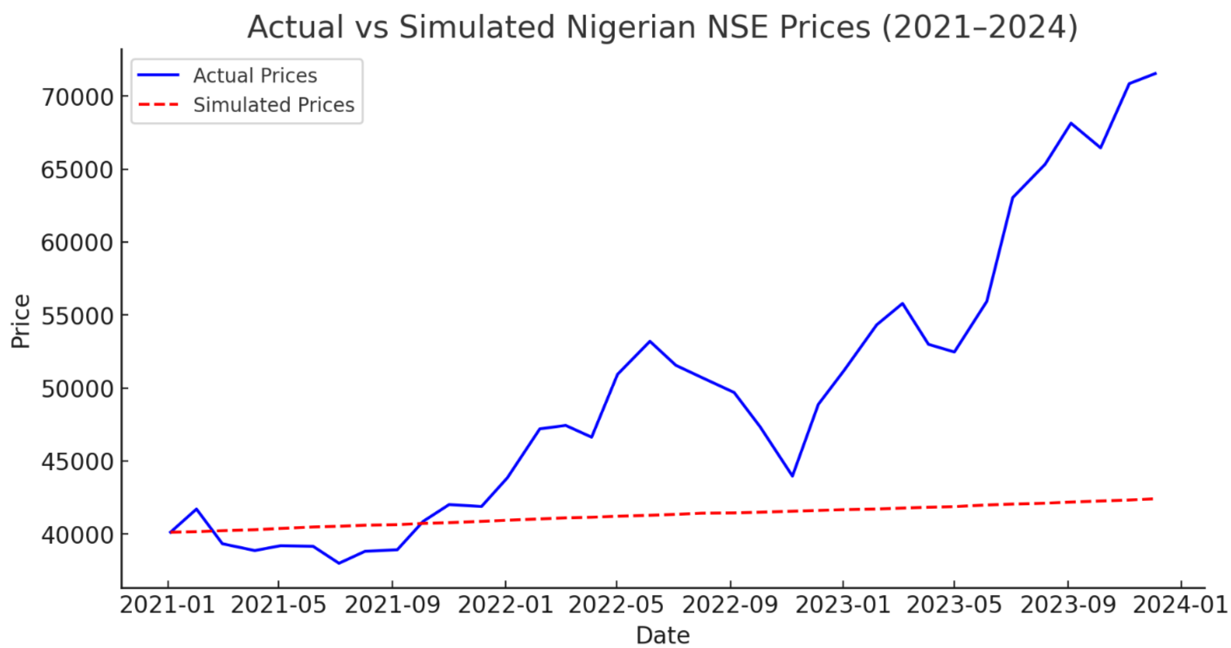


The model showed mixed performance when simulating the Nigerian All Share Index during the 2017 period. The Shapiro-Wilk test returned a p-value of 0.0004683 hence suggesting that the returns deviate from normality, explaining the model's weak performance. The annualized 3-year volatility of 5.64% suggests that there were moderate market fluctuations during the period.

A MAPE of 17.2% indicates that there were significant percentage deviations between the actual and simulated prices, showing that the GBM struggled to capture the price movements. Moreover, a Pearson correlation coefficient value of 0.3454 indicates a weak posiive relationship in the movement of actual and simulated prices. From the graph, we can see that the GBM consistently underestimated the actual prices and failed to capture the upward spike in prices, common in volatile frontier market environments.

ii. 2021 to 2024 period:

- Figure 11: Nigerian NSE All Share Index Actual vs Simulated graph (2021 – 2024)



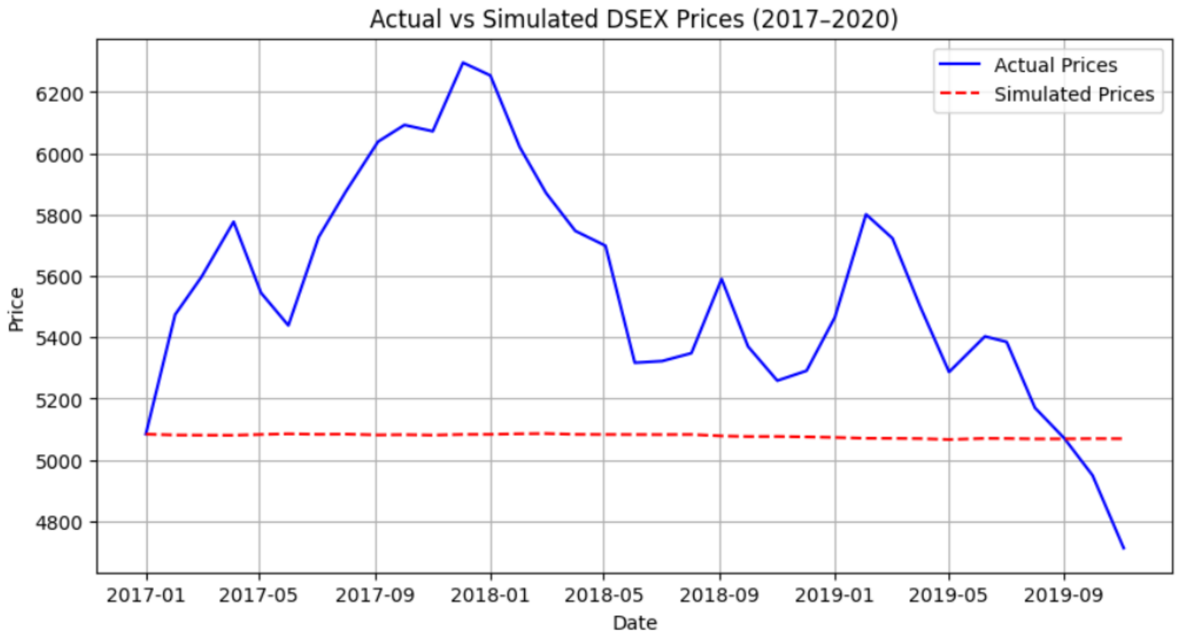
The GBM performed moderately well in the 2021 period for the Nigerian NSE All Share Index. The high p-value of 0.9567 suggests that the data aligns closely with a normal distribution, supporting the GBM's assumptions. Additionally, the annualized volatility of 4.65% indicates relatively low market fluctuations during the period, pointing to a stable market environment.

The MAPE score of 15.85% reflects relatively small deviations between the actual and simulated prices, showing that the model captured the price trend fairly accurately. The Pearson correlation coefficient of 0.9197 further confirms this, indicating a very strong positive relationship between the actual and simulated prices. However, despite these promising metrics, the simulated prices consistently underestimated the actual price levels, as seen in the graph hence showing that the GBM failed to capture the magnitude of price changes during the period.

c) Bangladesh DSE General Index:

i. 2017 to 2020 period:

➢ Figure 12: Bangladesh DSE Index Actual vs Simulated graph (2017 – 2020)

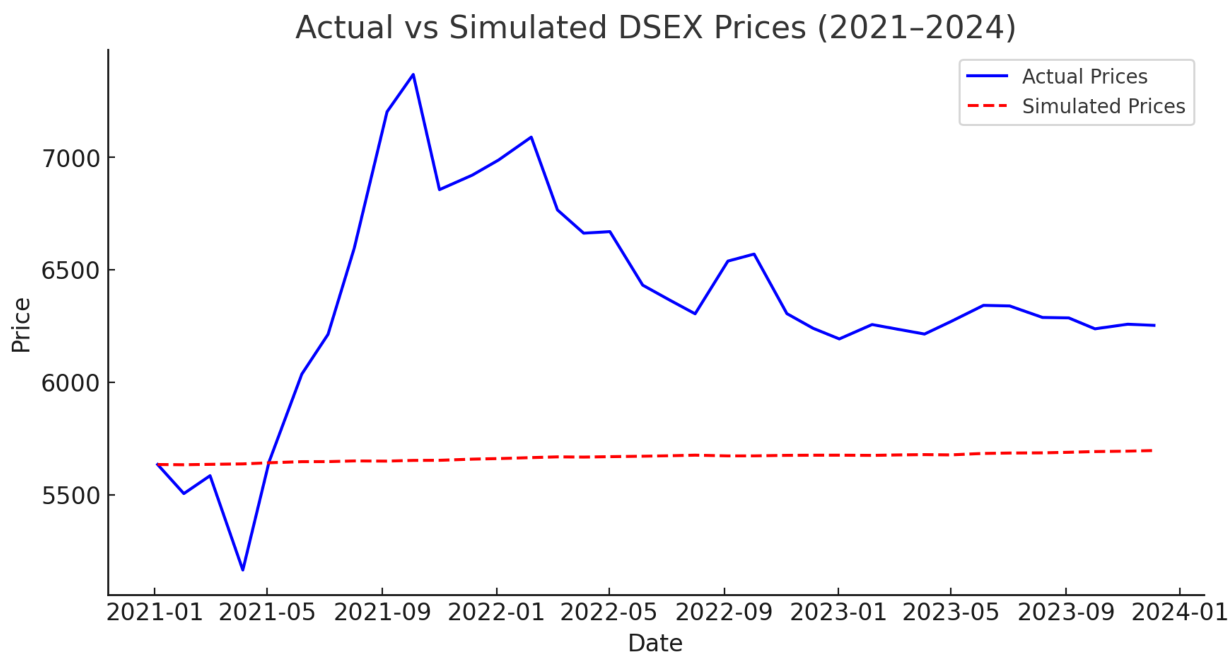


The GBM performed reasonably well in the 2021 period for the Bangladesh DSE General Index. The p-value of 0.7282 indicates that the data closely aligns with a normal distribution, supporting the GBM's assumption of normality. The annualized volatility of 3.41% suggests low market fluctuations during the period, which likely contributed to minimal price deviations and a smoother curve for the simulated prices.

The MAPE score of 8.99% reflects relatively small deviations between the actual and simulated prices, indicating that the GBM was able to capture the price movements fairly well. The Pearson correlation coefficient of 0.5509 shows a moderate positive relationship between the actual and simulated prices, meaning that while the GBM tracked the overall direction of the market, it did not always replicate the exact price levels. Overall, the low volatility during this period may have resulted in the simulated prices underestimating the actual price movements, but the relatively good MAPE and moderate correlation suggest that the GBM was a reasonably effective model for the market conditions.

ii. 2021 to 2024 period:

- Figure 13: Bangladesh DSE Index Actual vs Simulated graph (2021 – 2024)



The GBM showed mixed results for the Bangladesh DSE General Index. The low p-value of 0.0184 indicates that the data deviates from normality, suggesting that the GBM's assumption of normality was not met, which could explain some of the model’s limitations. The annualized volatility of 3.58% reflects low market fluctuations, which likely contributed to the low-price deviations seen in the simulated data and could in turn, may have helped achieve a relatively low MAPE of 11.27%, indicating that the GBM was able to predict the overall price direction with somewhat reasonable accuracy.

However, the Pearson correlation coefficient of 0.2212 is relatively low, suggesting a weak positive relationship between the actual and simulated prices. This indicates that the GBM struggled to accurately replicate the actual price movements, likely due to its inability to account for non-normality or sudden market shifts. Overall, while the low volatility and MAPE score suggest that the GBM performed reasonably well in terms of trend capture, non-normality limited the model’s predictive potential, which is a limitation of the GBM.

4.4 Cross-validation:

In this section, we will divide the S&P 500 and NSE 20 datasets into one-year subsets to train the Geometric Brownian Motion model individually for each subset. This approach aims to maintain model consistency, capture year-specific market dynamics, and assess the model's applicability on unseen data. The 3-year period (2021–2024) for each dataset will be systematically split into three subsets, each representing a single year. For each subset, normality tests will be conducted to evaluate whether the data aligns with the assumptions of the GBM model.

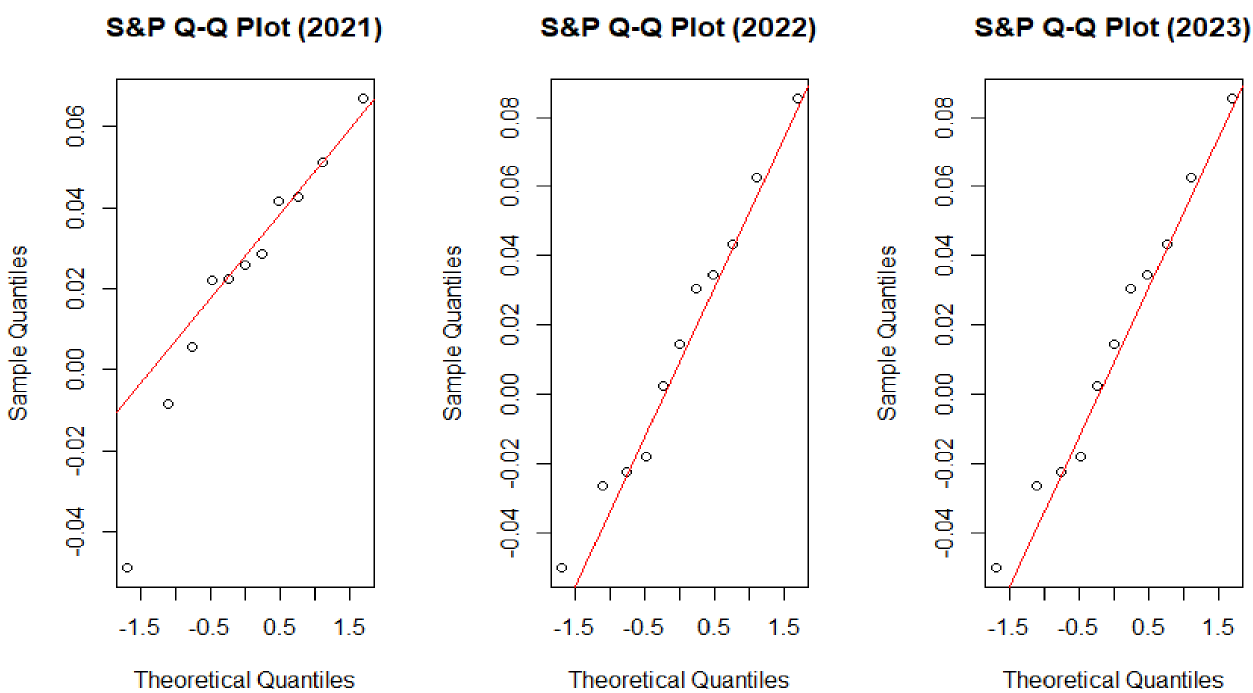
However, the Shapiro-Wilk test could not be applied due to the limited number of data points in the subsets. Instead, the Kolmogorov-Smirnov (KS) test was employed to assess normality more effectively in this context. Below is the analysis of the normality assessment of each subset.

4.4.1 S&P500 Performance analysis:

- Table 2

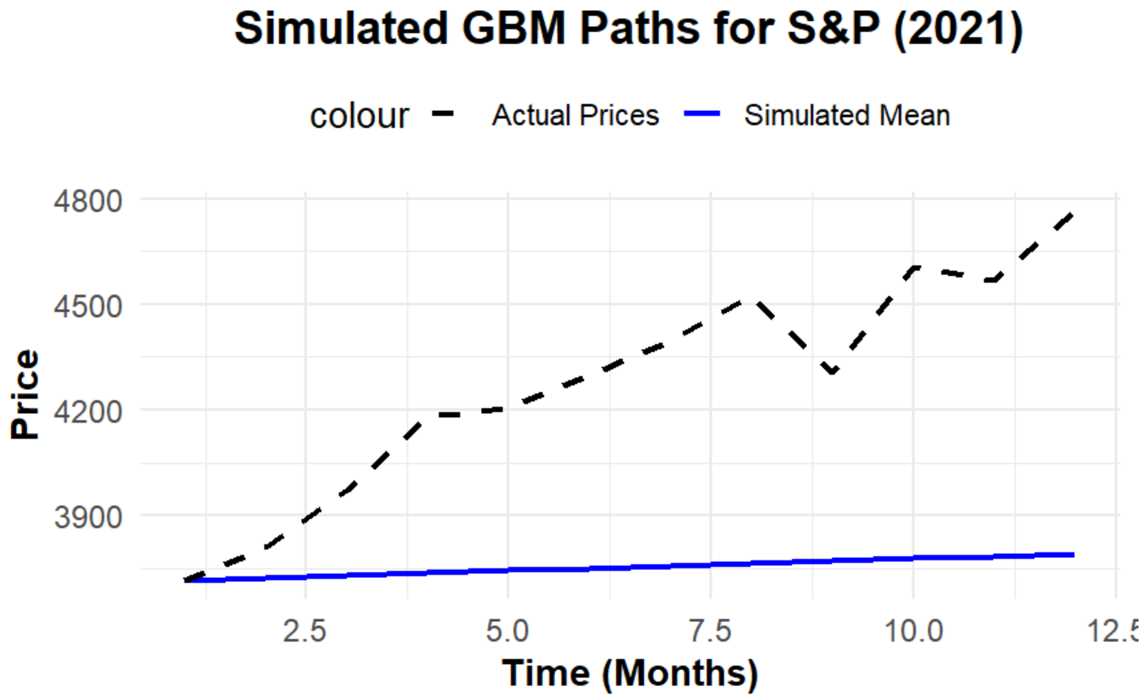
Year	MAPE (%)	Pearson Correlation (ρ)	Normality (KS Test)	Observations
2021	11.84	0.9584	D = 0.7469, $p < 0.0001$	Strong correlation, poor normality
2022	11.97	-0.8015	D = 0.6664, $p < 0.0001$	Moderate correlation, poor normality
2023	5.32	0.7593	D = 0.6664, $p < 0.0001$	Improved correlation, persistent non-normality

➢ Figure 14: S&P 500 Q-Q plots

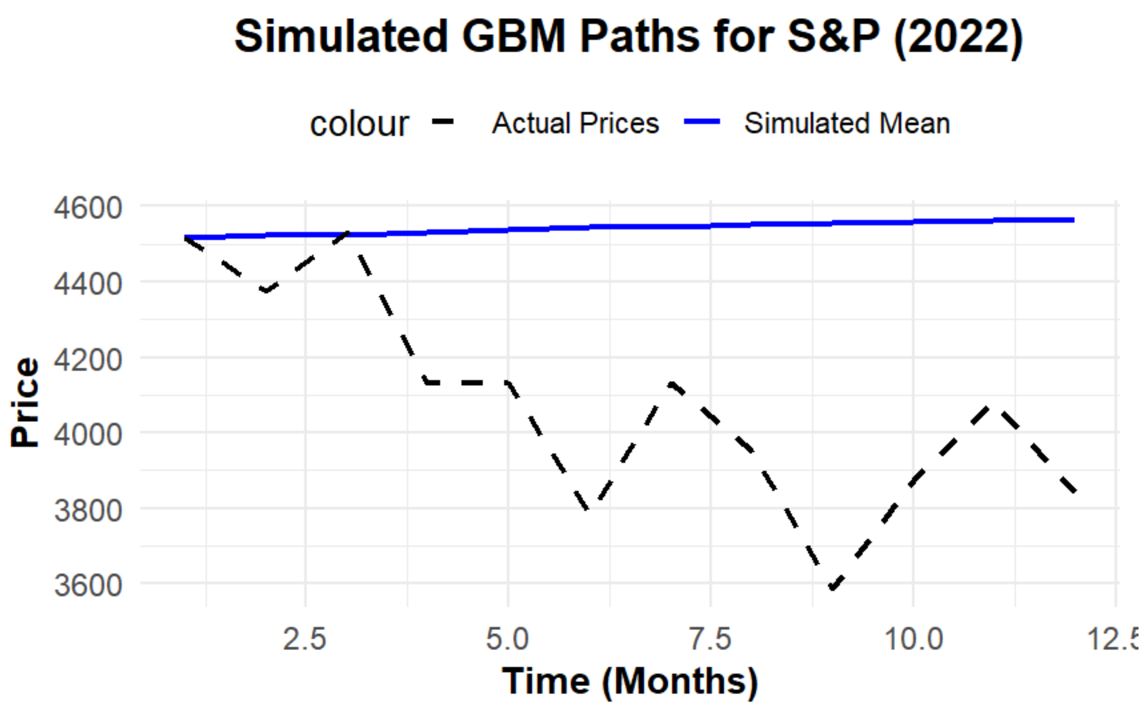


S&P 500 Actual vs Simulated price paths:

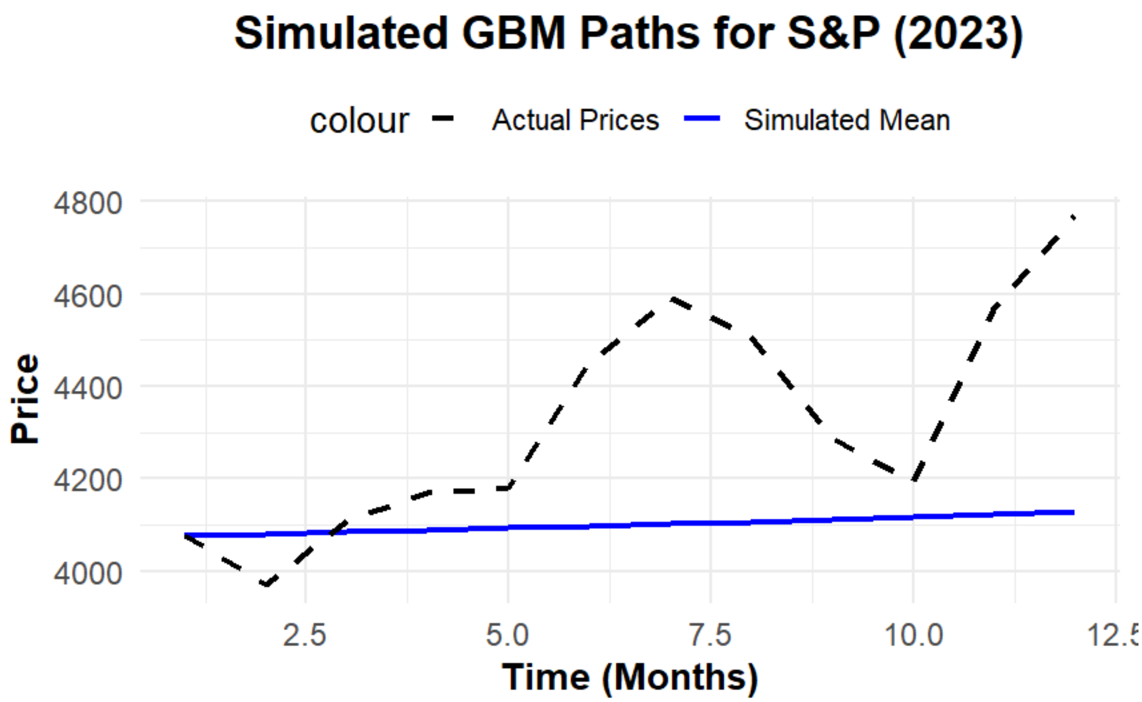
➢ Figure 15: S&P 500 Actual vs Simulated graphs (2021)



➢ Figure 16: S&P 500 Actual vs Simulated graphs (2022)



➢ Figure 17: S&P 500 Actual vs Simulated graphs (2023)



The GBM showed varying levels of performance when applied to the S&P500 subsets across the 3 years. Contrary to our expectations, the normality tests of the 3 subsets indicates that the data significantly deviated from normality as shown with a p-value of 0.0001, suggesting that we reject the null hypothesis and conclude that the data is non-normal. In the three diagrams, the simulated prices line can be seen to be almost straight. This can be attributed of the low mean and standard deviation of the actual prices within the subsets' period.

In 2021, the model had a high correlation coefficient of 0.9479, suggesting a strong positive relationship between the actual and simulated prices and a low MAPE score of 13.08% suggesting small percentage deviations between the actual and simulated prices. Despite the non-normality of the data, the model was still able to capture the overall trend of the actual prices.

In 2022, we observe a decline in the model's performance with a negative correlation coefficient of -0.8015, suggesting that the movement of the simulated prices did not align with that of the actual prices. A low MAPE score of 11.97% suggests minimal percentage deviations between the actual and simulated prices. This mismatch of the MAPE and correlation coefficient indicates that while the simulated prices followed the general magnitude of price changes, they failed to replicate the directional movement accurately. Overall, the GBM performed poorly in the 2022 period.

In 2023, we observe an improvement of the GBM's performance, with a positive correlation coefficient value of 0.7593 showing that the simulated prices fairly mimicked the movement of the actual prices. The lowest MAPE score of the 3 subsets of 5.32% indicated that the percentage deviations between the actual and simulated prices were very minimal. Despite non-normality, the model was able to perform fairly well during the 2023 period.

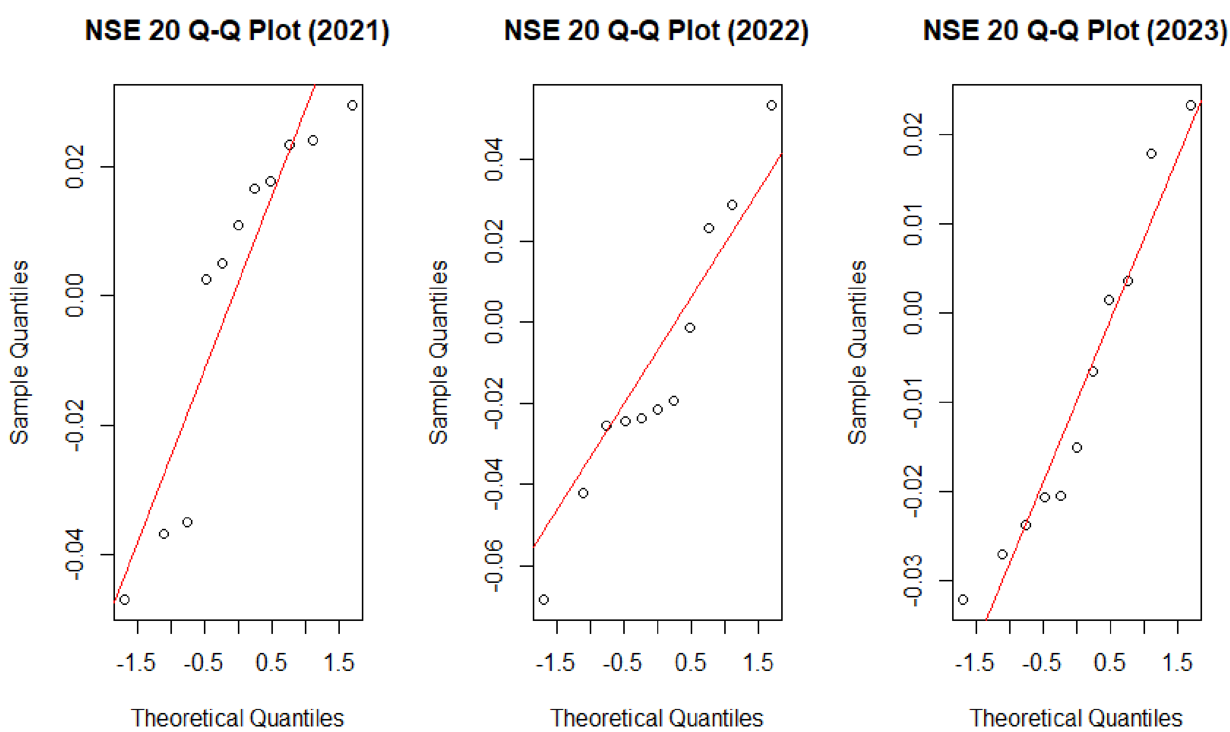
Overall, while GBM performed well in 2021 and 2023, its failure in 2022 highlights the limitations of relying on GBM for datasets with strong deviations from normality and structural volatility differences.

4.4.2 NSE 20 Performance Analysis:

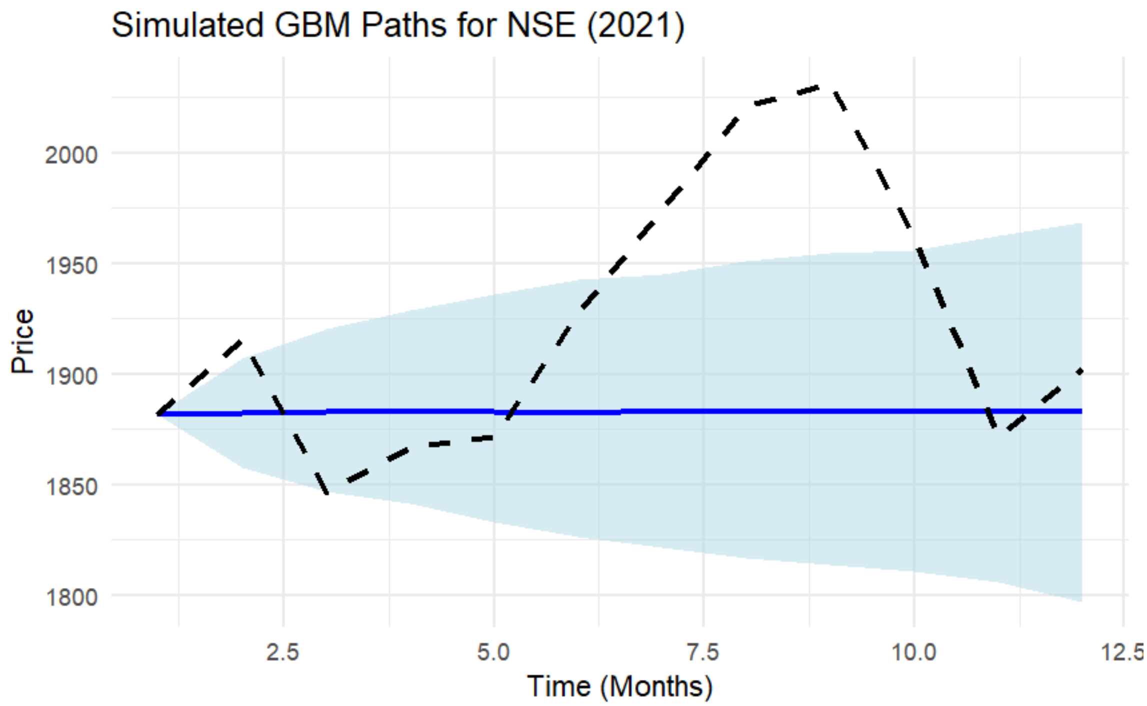
- Table 3:

Year	MAPE (%)	Pearson Correlation (ρ)	Normality (KS Test)	Observations
2021	2.65	0.2499	D = 0.2515, p = 0.42	Normality observed, weak correlation
2022	8.27	0.7457	D = 0.2328, p = 0.52	Normality observed, moderate correlation
2023	5.97	0.9284	D = 0.1874, p = 0.77	Strong correlation, normality maintained

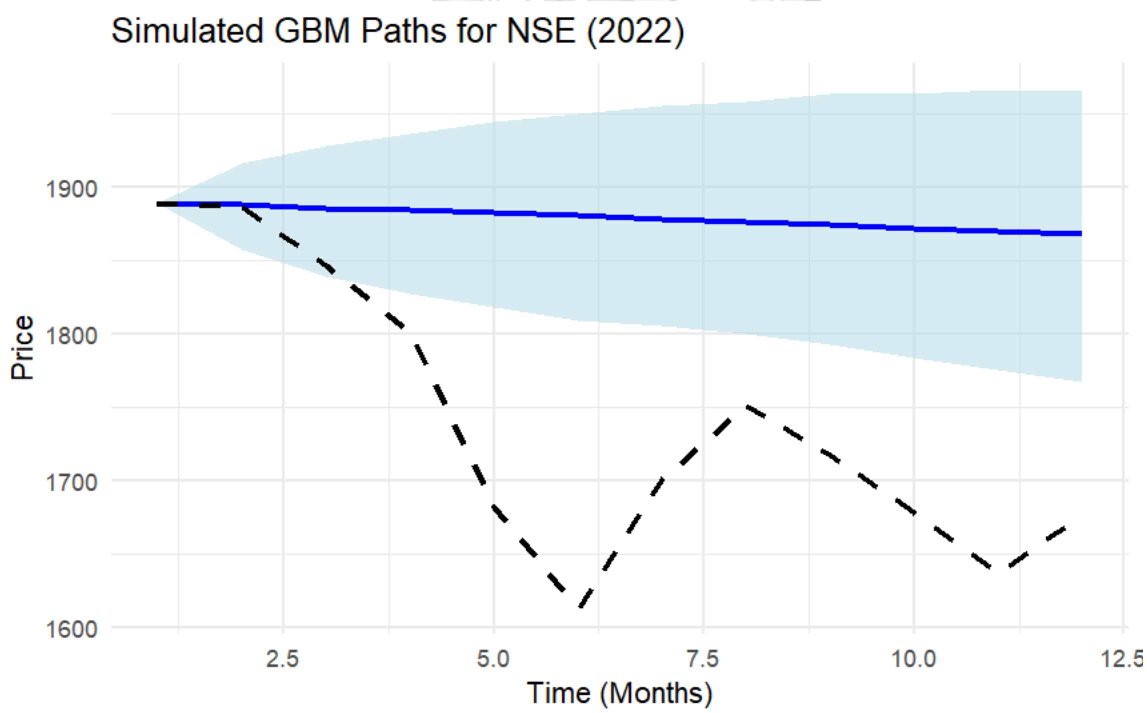
- Figure 18: NSE 20 Q-Q plots



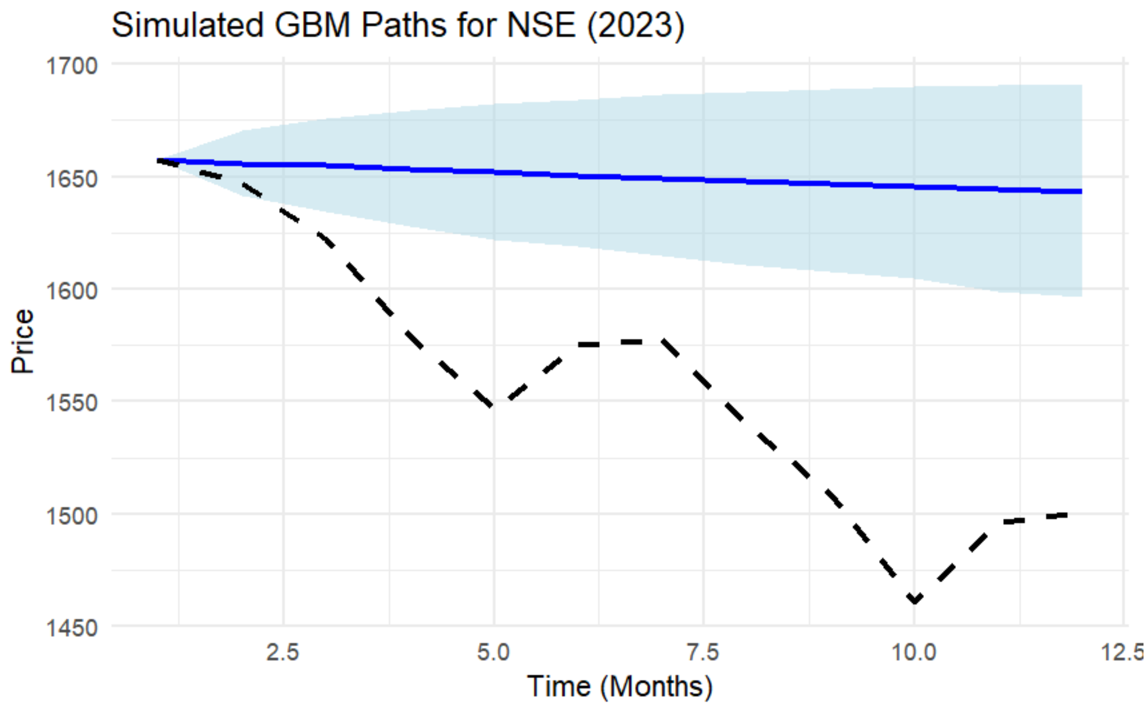
- Figure 19: Kenyan NSE 20 Actual vs Simulated price path (2021)



➢ Figure 20: Kenyan NSE 20 Actual vs Simulated price path (2022)



➢ Figure 21: Kenyan NSE 20 Actual vs Simulated price path (2023)



The GBM model performed consistently well across the 3 years in the NSE 20 dataset. Contrary to our expectations, the dataset adhered to normality assumptions across the 3 years with a p-value exceeding the 0.05 threshold, allowing us to fail to reject the null hypothesis and conclude that the data is indeed normal. This can explain the model's generally good performance across the 3 subsets.

In 2021, the model had a correlation coefficient of 0.2499 indicating a weak positive relationship between the actual and simulated prices. This implies that the model struggled to replicate the short-term price trends accurately. A low MAPE score of 2.65% suggests that the percentage deviations between the actual and simulated prices were minimal. As seen from the graph, low volatility and mean could explain the model's minimal movement across the period.

In 2022, the GBM model exhibited reasonable statistical performance with a correlation coefficient of 0.7457 and a MAPE score of 8.27%. However, a disparity arises from the diagram as the simulated prices consistently overestimates the actual prices, which exhibit a downward trend. This suggests that the model failed to adapt to the short-term directional trends, which were sudden and sharp, a common characteristic observed in frontier markets. This highlights the shortcomings of the GBM as a predictive model, particularly in highly volatile environments such as the Kenyan NSE stock market.

The same can be observed in 2023, whereby, while the statistical metrics indicate exceptional performance of the model, the visual representation shows otherwise. This can be attributed to the static parameters of the mean and volatility, which remain constant over time. Static parameters fail to adapt to structural changes or volatility clustering in financial markets which highlights a limitation of the GBM. This posits the need for other alternative predictive models which allow flexibility in the parameters such as regime-switching GBM or stochastic volatility models such as the Heston model.

4.5 Final Summary of the GBM model:

The analysis has demonstrated the model's varying performance across developed and frontier markets. Key findings indicate that the GBM generally performs better in developed markets where assumptions of normality and stable market conditions are prevalent. For instance, the German DAX and Australian ASX exhibited high predictive accuracy evidenced by the high correlation coefficient value, low Mean Absolute Percentage Error (MAPE) and strong alignment between the actual and simulated price trends.

In contrast, frontier markets like Kenya's NSE 20 and Nigeria's NSE All Share Index displayed varied results. In periods of strong normality, such as Kenya's post-pandemic recovery and Bangladesh's pre-pandemic period, the GBM performs exceptionally well, as shown by high correlation coefficients and low MAPE scores. However, periods characterized by high market volatility, market illiquidity and non-normality highlighted the model's limitations in capturing the market's erratic behavior. The GBM's inability to capture these market dynamics underscores its weaknesses in such environments.

Overall, the findings affirm that while the GBM is a robust tool for simulating stock prices under certain conditions, its predictive accuracy is highly dependent on the satisfaction of its core assumptions. Incorporations of other models which address these weaknesses such as the regime-switching volatility models (Hamilton, 1989) which addresses market states, the Heston model (Heston, 1993) which incorporates stochastic volatility and the Entropy-Corrected GBM (Stojkoski et al., 2020) model which adjust for deviation from normality can significantly enhance its application to different market dynamics and environments.

5. DISCUSSIONS AND CONCLUSION

This study set out to evaluate the Geometric Brownian Motion's applicability in simulating stock prices across developed and frontier markets, using the American S&P 500 and the Kenyan NSE 20 as our main representative cases. The results highlight the GBM's strengths and limitations in stock price prediction, providing insights into its role in financial modelling and areas of further research.

5.1 Key findings:

1. **Developed markets:** The GBM demonstrated strong predictive accuracy in developed markets, highlighted by high correlation coefficients values and low MAPE scores. This reflects the model's reliability under conditions of stable volatility and adherence to normality assumptions.
2. **Frontier markets:** The results for frontier markets were mixed. In instances of stable market volatility and normality, the GBM performed exceptionally well. However, in non-normal and volatile environments coupled with dramatic price jumps, the model struggled to capture the market's erratic behavior.
3. **Limitations:** The GBM's assumptions of log-normal returns, constant volatility and randomness limits its predictive accuracy in markets with non-normal environments, volatility clustering or structural shifts.

5.2 Implications of the study:

1. **For Investment:** The findings suggest that while the GBM can provide a solid baseline for forecasting stock prices, its use requires caution. It is valuable for scenarios where one would like to understand the range of potential future outcomes in order to simulate possible price trajectories for risk analysis or option pricing. Moreover, it is more applicable to be used where the market or the underlying asset adequately satisfies the GBM's assumptions such as normality and relatively stable market volatility. Investors may need to supplement the GBM with more advanced models or its model modifications to account for the dynamic characteristics of some markets.
2. **For model development:** Incorporating enhancements such as regime-switching volatility models, stochastic volatility models like the Heston model, or entropy-corrected approaches as discussed in the final summary of the GBM model can improve the robustness of the GBM in diverse market conditions and overcome its core limitations.

In conclusion, the study reinforces the GBM as a foundational tool for financial modelling while highlighting the need for tailored modifications to address the complexities of unpredictable market environments, such as those exhibited by frontier markets. This may allow future studies to contribute to more accurate and context-sensitive predictive models, which would enhance decision-making by both investors and policymakers.

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