On unconditional Banach space Ideal Property

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Abstract

Let Lw' denote the assignment which associates with each pair of Banach spaces X, Y the vector space $Lw'(X, Y)$ and $K(X, Y)$ be the space of all compact linear operators from X to Y. Let $T \in Lw'(X, Y)$ and suppose $(Tn) \subset K(X, Y)$ converges in the dual weak operator topology (w') of T. Denote by $Ku((Tn))$ the finite number given by

$$Ku\big((Tn)\big) := sup_{n\in N}\big\{max\{\| Tn \|, \| T - 2Tn \|\}\big\}.$$

The u-norm on $Lw'(X, Y)$ is then given by

$$\| T \| u := inf\{Ku\big((Tn)\big): T = w' - lim_n Tn, \quad Tn \in K(X, Y)\}.$$

It has been shown that $(Lw'(X, Y) \|.\| u)$ is a banach operator ideal. We find conditions for $K(X, Y)$ to be an unconditional ideal in $(Lw'(X, Y) \|.\| u)$.

Key words: *Ideal projection, separable reflexive space, unconditional compact approximation property (UKAP), u-ideal.*

1. Introduction

In section 8 of paper [2], the authors established necessary conditions on a Banach space X such that the space $K(X)$ of compact operators is a u-ideal in the space $L(X)$ of bounded linear operators, showing that this is the case if X is separable and has (UKAP) (unconditional compact approximation property, i.e. if there exists a sequence (K_n) in $K(X)$ such that $\lim K_n x = x$ for all $x \in X$ and $\lim_n \| id_X - 2K_n \| = 1$).

Johnson proved in [5] that if Y is a Banach space having the bounded approximation property then the annihilator $K(X, Y)^{\perp}$ in the (continuous) dual space $L(X, Y)^*$ is the kernel of a projection on $L(X, Y)^*$. The range space of the projection is isomorphic to the dual space $K(X, Y)^*$. K. John showed in [3] that Johnson's result is also true in case of any separable Pisier space $X = P$ and its dual $Y = P^*$, both being spaces which do not have the approximation property. This motivated his more general results in a later paper,(cf [4]).

In the paper [1] an alternative (operator ideal) approach is followed to prove similar (and more general) versions of John's results. Having proved that $(L^{w'}, \|.\| u)$ is a Banach operator ideal (cf [6]), we shall build on the results in [1] to obtain conditions for the space $K(X,Y)$ of compact operators to be a u-ideal in a suitable subspace $(L^{w'}(X,Y), \|.\| u)$ of $L(X,Y)$. If $L^{w'}(X,Y) = L(X,Y)$, our results states conditions on $L(X,Y)$ so that $K(X,Y)$ is a u-ideal in $L(X,Y)$.

Before we investigate the u-ideal property of $K(X,Y)$ in $\left(L^{w'}(X,Y), \|.\| u\right)$, we recall from [2] the ideal property of $K(X,Y)$ in $L^{w'}(X,Y)$ with respect to the $|\ \|.\|\ |$-norm.

Theorem 1.1: (cf. [1], Theorem 2.5) *There exists a continuous bilinear form*
$J: L_{w'}(X,Y)^* \times L_{w'}(X,Y) \rightarrow \boldsymbol{K}$ *such that*
(a) $J(\phi,T) = \phi(T)$ *for all* $(\phi,T) \in L_{w'}(X,Y)^* \times L_{w'}(X,Y)$
(b) $|J(\phi,T)| \leq \|\phi\|\,|\,\|(T)\|\,|$ *for all* $T \in L_{w'}(X,Y)$ *and* $\phi \in L_{w'}(X,Y)^*$.
(c) $J(\phi,T) = lim_n\,\phi(T_n)$, *where* (T_n) *is any sequence of compact operators* $T_n \in K(X,Y)$ *tending to T in w'-topology.*

In particular,
Corollary 3.1.2: *Let X, Y be Banach spaces. There is a projection*

$$P: (L_{w'}((X,Y),|\ \|.\|\ |)^*) \rightarrow (L_{w'}((X,Y),|\ \|.\|\ |)^*)$$

such that

$$Ker(P) = K(X,Y)^{\perp} = \{\phi \in L_{w'}(X,Y)^* : \phi\, K(X,Y) = 0\}, \|P\| \leq 1$$

and the range of P is isomorphic to $K(X,Y)^*$. *The projection P is given by* $P\phi(T) = \lim_n \phi(T_n) = J(\phi,T)$. *Thus* $K(K,Y)$ *is an ideal in* $(L_{w'}(X,Y),|\ \|.\|\ |)$.

We shall make use of the properties of a Banach ideal operator defined in terms of compact approximation properties and separable reflexive spaces.

2. Unconditional Ideal property.

The authors in [2] call a sequence (K_n) of compact operators from X into X *a compact approximation sequence* if $lim_n k_n x = x$ for all $x \in X$. In [2] it is also agreed to say that X has (*UKAP*) if there is a compact approximation sequence
$K_n: X \rightarrow X$ such that $lim_{n\rightarrow\infty} \|I - 2k_n\| = 1$.

It is also proved in [2] that if X is a separable Banach space, then X has (*UKAP*) if and only if for every $\epsilon > 0$ there is a sequence (A_n) of compact operators such that for every $x \in X$ and every n and every $\theta_j = \pm 1,\ 1 \leq j \leq n$ we have $\sum_{i=1}^{\infty} A_n x = x$ and $\|\sum_{i=1}^{\infty} \theta_j A_j x\| \leq (1+\epsilon)\|x\|$. In particular, this means that if we let $K_n = \sum_{i=1}^{n} A_i$, then $K_n x \rightarrow x,\ \forall x \in X$ and

$$\|K_n x\| \leq (1+\varepsilon)\|x\|, \quad \forall x \in X, \quad \forall n \in N.$$

Moreover, also $\|I - 2K_n\| \leq 1 + \varepsilon$.

When a separable Banach space X has *UKAP*, it is easily seen that for each $T \in L(X)$, $TK_n \to T$ $(as\ n \to \infty)$ is the weak operator topology. If X is also reflexive, then $TK_n \to T$ $(as\ n \to \infty)$ in the w'-topology and it follows that

$$K_u((TK_n)) \leq (1 + \epsilon) \| T \|.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\| T \|_u \leq \| T \|, i.e\ \| T \| = \| T \|_u$ in this case.

Putting $T_n := TK_n$, it follows that $T_n \overset{w'}{\to} T$ and $\| T - 2T_n \| \leq (1 + \epsilon) \| T \|\ for\ all\ n \in \mathbf{N}$. Consequently, it follows that

$$\| Id_{(Lw')^*} - 2P \| = sup_{\|\phi\| \leq 1} \| \phi - 2P\phi \|$$

$$= sup_{\|\phi\| \leq 1} sup_{\|\phi\| \leq 1} |\phi(T) - 2P\phi(T)|$$

$$= sup_{\|\phi\| \leq 1} sup_{\|\phi\| \leq 1} | lim_n\, \phi(T - 2T_n)|$$

$$\leq sup_{\|T\| \leq 1} sup_n \| T - 2T_n \| \leq 1 + \epsilon$$

This being so for all $\epsilon > 0$, it is clear that:

Proposition 2.1: *(Special case of [2], Proposition 8.2).* Let X be a separable reflexive Banach space. If X has (*UKAP*), then $K(X)$ is a *u*-ideal in $L(X)$.

If X satisfies the conditions in Proposition 2.1 and Y is any Banach space, then for each $T \in L(X, Y)\ and\ each\ \epsilon > 0$, we may choose the sequence $(K_n) \subset K(X)$ to satisfy the properties in the above proof of Proposition 2.1.

Again, put $Tn = TKn\ for\ all\ n$. Then, as before, $T_n \overset{w'}{\to} T$ and we still have the inequalities

$$\| T - 2T_n \| \leq (1 + \epsilon) \| T \|\ and\ K_u((T_n)) \leq (1 + \epsilon) \| T \|.$$

Hence $\| T \|_u \leq (1 + \epsilon) \| T \|\ for\ all\ \epsilon > 0$. The existence of a contractive projection $P : L(X, Y)^* \to L(X, Y)^*\ with\ Ker(P) = K(X, Y)^{\perp}$ follows from the Theorem 1.1 and Corollary 1.2, since in this case we have $(L(X, Y), \|.\|) = (Lw'(X, Y), \|.\|\ u)$.

Therefore, $K(X, Y)\ is\ an\ ideal\ in\ L(X, Y)$.

The argument in the proof of Proposition 2.1 then shows that:

Proposition 2.2: *Let X be a separable reflexive Banach space and Y any Banach space. If X has* (*UKAP*), $then\ K(X, Y)\ is\ a\ u - ideal\ in\ L(X, Y)$.

In the discussion of the proof of Proposition 2.1 it is important to realise that for each $T \in L(X, Y)\ and\ each\ \epsilon > 0$ the sequence $(T_n) \subset K(X, Y)$ can be chosen to satisfy $T_n \overset{w'}{\to} T$ and $\| T - 2T_n \| \leq (1 + \epsilon) \| T \|\ and\ \| T_n \| \leq (1 + \epsilon) \| T \|$.

With the conditions on the Banach space X in Proposition 2.1, the norms $\|.\|, |\|.\||\ and$ $\|.\|_u$ coincide on $L(X, Y)$, exactly because we can choose the sequence (T_n) as such. Therefore, it is natural to formulate the following definition:

Definition 2.3: *Let X and Y be Banach spaces. We say an operator $T \in L(X,Y)$ has $(w' - UKAP)$ (i.e. it has the "w'-uniform compact approximation property" if each $\epsilon > 0$ there exists a sequence $(T_n) \subset K(X,Y)$ such that* $T = w' - lim_n T_n, \| T - 2T_n \| \leq (1 + \epsilon) \| T \|$ *and* $\| T_n \| \leq (1 + \epsilon) \| T \|$ *for all* n.
It follows from the above discussion that:

Proposition 2.4: *Suppose each $T \in (L_{w'}(X,Y), \|.\|_u)$ (respectively, each $T \in L(X,Y)$) has $(w' - UKAP)$.Then $K(X,Y)$ is a u-ideal in $(L_{w'}(X,Y), \|.\|_u)$ (respectively, each $T \in L(X,Y)$).*

We follow the prove of proposition 2.1 which was already discussed in [2] that a separable reflexive Banach space has $(UKAP)$ if and only if $K(X)$ is a u-ideal in $L(X)$.
As a motivation for the condition $(w' - UKAP)$ in Proposition 2.4, we introduce another property on Banach spaces:
A sequence (K_n) of compact operators from X into X is called a *w'-compact approximating sequence* if $w' - lim_n K_n = I$.

If X is reflexive, then clearly each compact approximating sequence is w'-compact approximating. We say X *has* $(w' - UKAP)$ *if for each* $\epsilon > 0$ there is a w'-compact approximating sequence $K_n : X \to X$ such that $\| K_n x \| \leq (1 + \epsilon) \| x \| \quad \forall x \in X, \forall n \in \boldsymbol{N}$ and $\| I - 2K_n \| \leq 1 + \epsilon$ *for all* n.
Finally from proposition 2.4 we formulate the following corollary:

Corollary 2.5: *If X has $(w' - UKAP)$, then $K(X)$ is a $u - ideal$ in $L(X)$.*

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