Statistical Theory of Integer Partitions

Dimbinaina Ralaivaosaona

African Institute for Mathematical Sciences
South Africa

International Mathematics Research Meeting,
July 24, 2012

Introduction

Definition

A partition of n is a non-decreasing sequence of positive integers $c_1, c_2, \cdots, c_t$ such that

$$n = c_1 + c_2 + \cdots + c_t.$$

We usually denote a partition of n as $(c_1, c_2, \cdots, c_t)$.

Example

Partitions of 5:

$$\begin{aligned} 5 &= 1+1+1+1+1 \\ &= 1+1+1+2 \\ &= 1+1+3 \\ &= 1+2+2 \\ &= 1+4 \\ &= 2+3 \\ &= 5, \end{aligned}$$

so there are 7 partitions of 5.

Restricted partitions

- Partitions which has no parts with multiplicity more than one are called restricted partitions,

Restricted partitions

- Partitions which has no parts with multiplicity more than one are called restricted partitions,
- partitions whose members are elements of a given sequence of positive integers λ are called λ-partitions.

Number of partitions

Given a positive integer n, how many partitions of n there are?

Asymptotic results

Theorem (Hardy-Ramanujan, 1918)

The total number of partitions of n admits the asymptotic formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$$

as $n \to \infty$.

Asymptotic results

Rademacher

We have

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k\geq 1} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(x-\frac{1}{24})}\right)}{\sqrt{(x-\frac{1}{24})}} \right]_{x=n}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \omega_{h,k} e^{-2\pi inh/k}$$

and $\omega_{h,k}$ is a 24kth root of unity.

Number of λ-partitions

Under certain technical conditions on λ, now called the Meinardus scheme, we have the the following theorem:

Theorem (Meinardus, 1954)

The number of λ-partitions satisfies the asymptotic formula:

$$p_\lambda(n) \sim \kappa_1 n^{\kappa_2} \exp\left(\kappa_3 n^{\alpha/(\alpha+1)}\right) \tag{1}$$

as $n \to \infty$, where the κ_i's are constants that can be determined in terms of α and A, and α and A are constants associated to the sequence λ.

Number of summands in partitions

Study of partitions the probabilistic approach.

Number of summands in partitions

Assume all partitions of n are equally likely. Let ϖ_n be the number of summands in a random partition of n, μ_n and σ_n its mean and standard deviation respectively.

Number of summands in partitions

Theorem (Erdős-Lehner, 1941)

For any real number x

$$\mathbb{P}\Big(\varpi_n \leq \mu_n + x\sigma_n\Big) \sim e^{-e^{-x}}$$

as $n \to \infty$. Furthermore, we have the following asymptotic estimates:

$$\mu_n = \frac{\sqrt{6n}}{2\pi}\Big(\log n + 2\gamma - \log(\pi^2/6)\Big) + \mathcal{O}(\log n),$$

and

$$\sigma_n^2 = n + \mathcal{O}(\sqrt{n}\log^2 n).$$

Number of summands in partitions

Theorem (Erdős-Lehner, 1941)

for any real number x we have

$$\mathbb{P}\left(\varpi_n^* \leq \mu_n^* + x\sigma_n^*\right) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \qquad (2)$$

where μ_n^* and σ_n^* are the mean and the standard deviation of ϖ_n^* respectively. Furthermore,

$$\mu_n^* = \frac{2\log 2}{\pi}\sqrt{3n} + \frac{3\log 2}{\pi^2} - \frac{1}{4} + \mathcal{O}(n^{-1/2})$$

and

$$\sigma_n^{*2} = \left(\frac{\sqrt{3}}{\pi} - \frac{12\sqrt{3}\log^2 2}{\pi^3}\right)\sqrt{n} + \mathcal{O}(1)$$

as $n \to \infty$.

Number of summands in λ-partitions

For the unrestricted λ-partitions, central limit theorems have been given by Haselgrove and Temperley (1954), under certain some technical conditions on λ. The results were later improved by Richmond and Lee.

Number of summands in λ-partitions

For the restricted λ-partitions, the result by Hwang deserved to be mentioned, in which convergence to the Gaussian distribution is proved for λ satisfying the Meinardus scheme.

Meinardus scheme

Let λ be a non-decreasing and unbounded sequence of positive integers, and consider the Dirichlet series:

$$D(s) = \sum_{\lambda} \frac{1}{\lambda^s}.$$

We say that a sequence λ satisfies the Meinardus scheme if the following three conditions are satisfied:

Meinardus scheme

(M1) The Dirichlet series $D(s)$ converges in the half-plane $\mathrm{Re}(s) > \alpha > 0$, and can be analytically continued into the half-plane $\mathrm{Re}(s) \geq -\alpha_0$ for some $\alpha_0 > 0$. For $\mathrm{Re}(s) \geq -\alpha_0$, $D(s)$ is analytic except for a simple pole at $s = \alpha$ with residue A.

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(M2) There is a constant $c > 0$ such that $D(s) \ll |t|^c$ uniformly for $\mathrm{Re}(s) \geq -\alpha_0$ as $|\mathrm{Im}(s)| = |t| \to \infty$.

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(M3) Let $\chi(\tau) = \sum_\lambda e^{-\lambda\tau}$, where $\tau = r + iy$ with $r > 0$. Then

$$\chi(r) - \mathrm{Re}(\chi(\tau)) \gg \left(\log \frac{1}{r}\right)^2$$

uniformly for $r^{1+\frac{\alpha}{2}} \leq |y| \leq \pi$ as $r \to 0$.

Prime partitions

The sequence of prime number does not satisfy the Meinardus scheme!

Prime partitions

Theorem

The number of summands in a random restricted partition of n into distinct primes is asymptotically normally distributed with mean and variance satisfying the following asymptotic formulas:

$$\mu_n = \frac{2\log 2}{\pi}\sqrt{\frac{6n}{\log n}}\left(1 - \frac{\log\log n}{2\log n} + \mathcal{O}\left(\frac{1}{\log n}\right)\right)$$

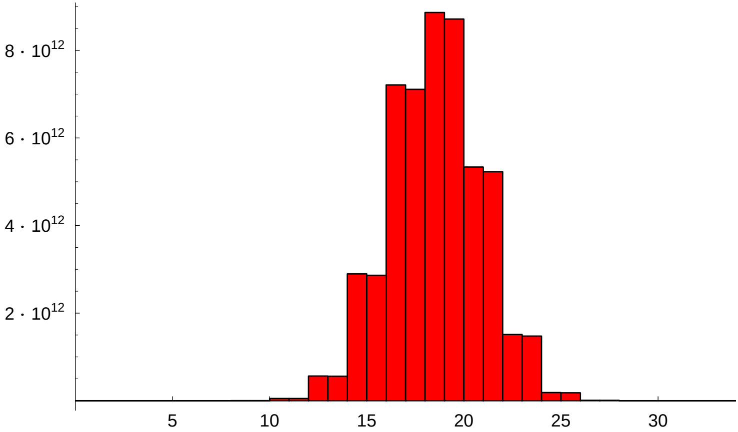
and

$$\sigma_n^2 = \frac{\sqrt{6}}{\pi}\left(1 - \frac{12\log^2 2}{\pi^2}\right)\sqrt{\frac{n}{\log n}}\left(1 - \frac{\log\log n}{2\log n} + \mathcal{O}\left(\frac{1}{\log n}\right)\right)$$

respectively as $n \to \infty$.

Example

For n=2012 we have the following histogram:



Generating function

Consider the bivariate generating function

$$Q(u,z) = \prod_{p}(1 + uz^{p}) = 1 + \sum_{n>1} Q_n(u)z^n$$

where

$$Q_n(u) = \sum_{k=0}^{n} q_n(k)u^k$$

and $q_n(k)$ is the number of partition of n into k distinct primes.

Generating function

Then, we have the probability generating function for ϖ_n (number of summands in a random restricted prime partition):

$$\frac{Q_n(u)}{Q_n(1)} = \mathbb{E}(u^{\varpi_n}).$$

We also have the mean and variance:

$$\mu_n = \frac{\partial}{\partial u}\mathbb{E}(u^{\varpi_n})\Big|_{u=1},$$

and

$$\sigma_n^2 = \frac{\partial^2}{\partial^2 u}\mathbb{E}(u^{\varpi_n})\Big|_{u=1} + \mu_n - \mu_n^2.$$

Saddle point method

We have

$$Q_n(u) = \frac{1}{2\pi i} \oint_{|z|=e^{-r}} Q(u,z) \frac{dz}{z^{n+1}},$$

which is the same as

$$Q_n(u) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp(int + f(u, r+it))dt,$$

for any $r > 0$, where

$$f(u,\tau) = \log Q(u, e^{-\tau}).$$

Saddle point method

The following asymptotic formula holds for the coefficient of z^n in $Q(u,z)$:

$$Q_n(u) = \frac{1}{\sqrt{2\pi f''(u,r)}} e^{nr+f(u,r)} \left(1 + \mathcal{O}\left(n^{-1/7}\right)\right).$$

uniformly in u, as $n \to \infty$. Here, $r = r(u,n)$ is now the unique positive solution of the equation

$$n = \sum_p \frac{pue^{-pr}}{1 + ue^{-pr}}.$$

Moment Generating Function

Then we have,

$$\frac{Q_n(u)}{Q_n(1)} = \exp\left(n(r - r_0) + f(u,r) - f(1,r_0)\right)\left(1 + \mathcal{O}\left(n^{-1/7}\right)\right).$$

where r_0 satisfies the equation

$$n = \sum_p \frac{pe^{-pr_0}}{1 + e^{-pr_0}}.$$

Moment Generating Function

The moment generating function of the random variable ϖ_n can be written as a function of $Q_n(u)$ as follow

$$M_n(t) = \mathbb{E}(e^{(\varpi_n - \mu_n)t/\sigma_n})$$
$$= \exp\left(\frac{-\mu_n t}{\sigma_n}\right) \frac{Q_n(e^{t/\sigma_n})}{Q_n(1)}.$$

We need to expand this around $t = 0$.

Moment Generating Function

Finally, we have the expression of the moment generating function

$$M_n(t) = \exp\left(\frac{t^2}{2} + O(n^{-2/7})\right)$$

for a bounded t.

Phase transition

We consider partitions of a positive integer n such that the multiplicity of each summand is less than a given number d.

Known results

Theorem (Mutafchiev, 2005)

If $d \sim \alpha\sqrt{n}$ then among all partitions of n the set of partitions with no parts of multiplicity greater than d has a positive density asymptotically equal to

$$\prod_{\lambda}(1 - e^{-\alpha\lambda})^{-1}.$$

Phase transition

Theorem

Let $S_{d,n}$ be the set of partitions of an integer n with no parts of multiplicity greater than d (d may be a function of n) and assume that all partitions in $S_{d,n}$ are equally likely. Then we have the following behaviour for the limit distribution of the number of summands in a random partition:

Phase transition

Theorem

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- if $d = o(\sqrt{n})$ then it is asymptotically Gaussian,

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Theorem

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- if $d = o(\sqrt{n})$ then it is asymptotically Gaussian,
- if $dn^{-1/2}$ is unbounded then the distribution is asymptotically Gumbel,

Phase transition

Theorem

If $d \sim b\sqrt{n}$ where b is a positive constant, then when normalized, the distribution of the number of summands converges to a distribution with moment generating function given by:

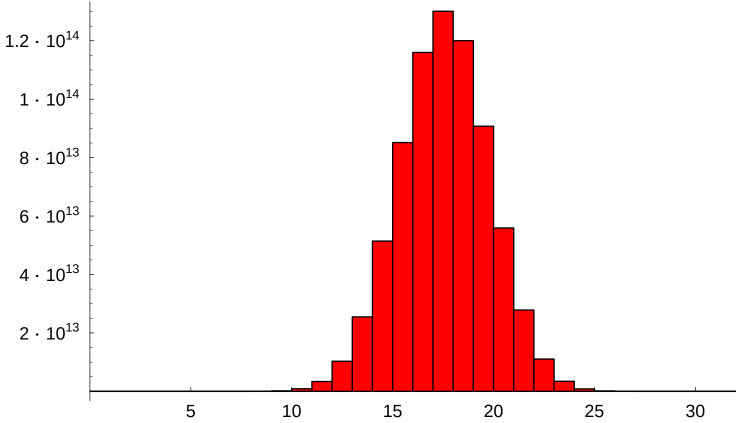
$$M(x) = \prod_{\lambda} \frac{e^{-\frac{a}{\lambda}}}{1 - \frac{a}{\lambda}} \prod_{\lambda} \left(\frac{1 - e^{-(\lambda - a)\vartheta}}{1 - e^{-\lambda\vartheta}} \right) e^{a\vartheta/(e^{\lambda\vartheta} - 1)}$$

where the product is taken over the set of positive integers, and

$$a = \frac{x}{\sqrt{\frac{\pi^2}{6} - \kappa}}, \quad \vartheta = \frac{\pi}{\sqrt{6}} b, \quad \text{and} \quad \kappa = \sum_{\lambda} \frac{\vartheta^2 e^{-\lambda\vartheta}}{(1 - e^{-\lambda\vartheta})^2}.$$

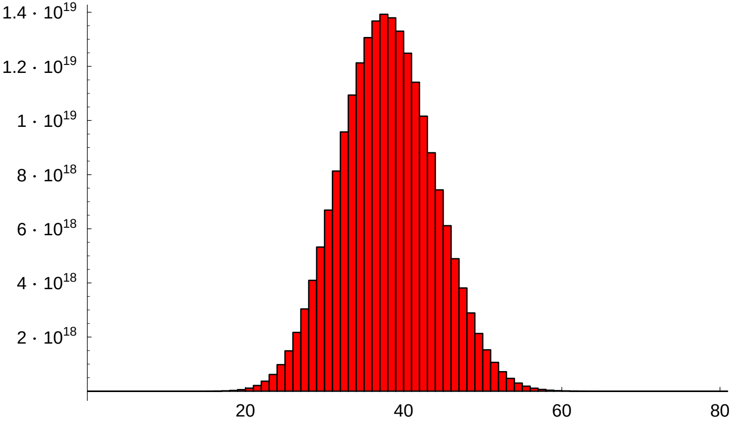
Examples

For $n = 500$ and $d = 2$



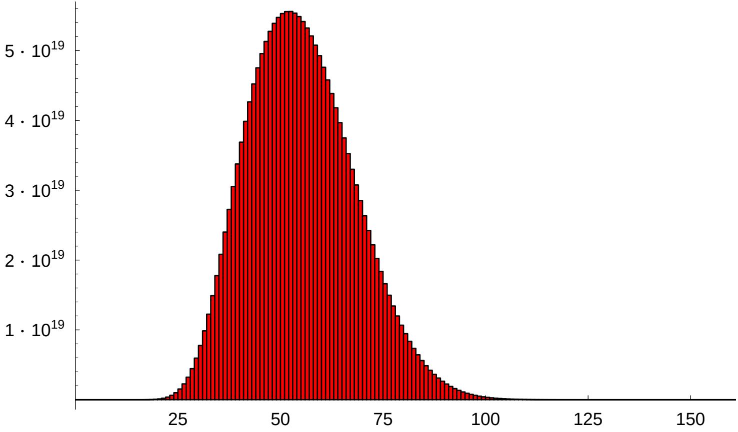
Examples

For $n = 500$ and $d = 9$



Example

For $n = 500$ and $d = 33$



Example

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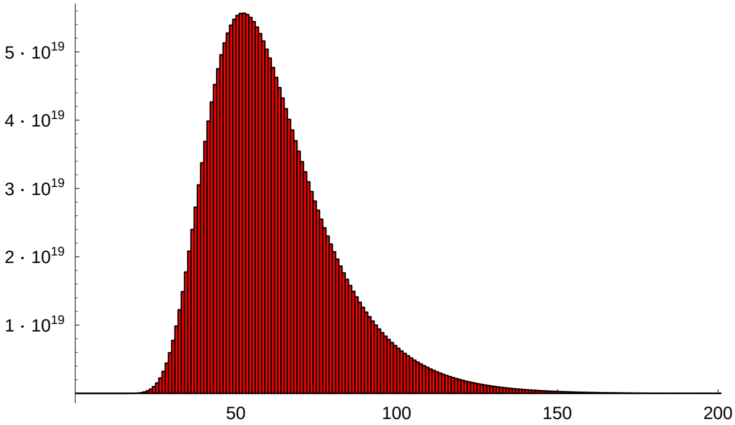


The number of parts of given multiplicity

Multiplicities in partitions were studied, amongst others, by Corteel et al, who showed that a randomly selected part of a random partition has multiplicity d with probability tending to $\frac{1}{d(d+1)}$.

The number of parts of given multiplicity

Theorem (Brennan, Knopfmacher, Wagner)

Let d be a positive integer, then the limit distribution of the number of parts having multiplicity d or more in a random partition of n is Gaussian.

The number of parts of given multiplicity

Theorem

The limit distribution of the number of parts of multiplicity d is:

The number of parts of given multiplicity

Theorem

The limit distribution of the number of parts of multiplicity d is:

- Gaussian with mean and variance asymptotically equal to $\frac{\sqrt{6n}}{\pi d(d+1)}$ for $d = o(n^{1/4})$,

The number of parts of given multiplicity

Theorem

The limit distribution of the number of parts of multiplicity d is:

- Gaussian with mean and variance asymptotically equal to $\frac{\sqrt{6n}}{\pi d(d+1)}$ for $d = o(n^{1/4})$,
- Poisson with parameter $\frac{\sqrt{6}}{\pi\alpha^2}$ for $d \sim \alpha n^{1/4}$,

The number of parts of given multiplicity

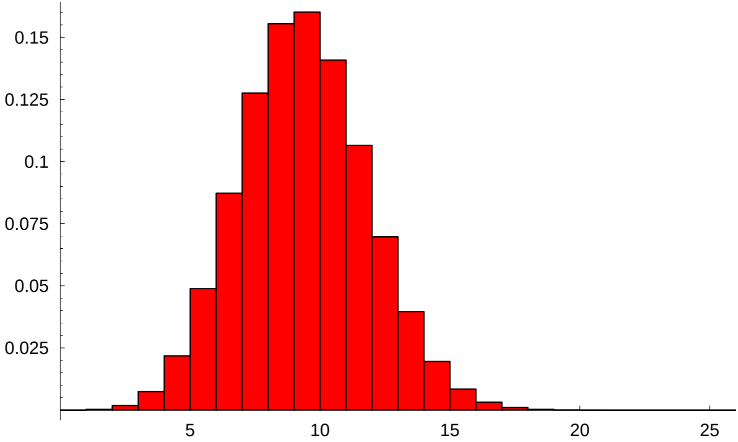
Theorem

The limit distribution of the number of parts of multiplicity d is:

- Gaussian with mean and variance asymptotically equal to $\frac{\sqrt{6n}}{\pi d(d+1)}$ for $d = o(n^{1/4})$,
- Poisson with parameter $\frac{\sqrt{6}}{\pi\alpha^2}$ for $d \sim \alpha n^{1/4}$,
- degenerate at zero for $dn^{-1/4} \to \infty$.

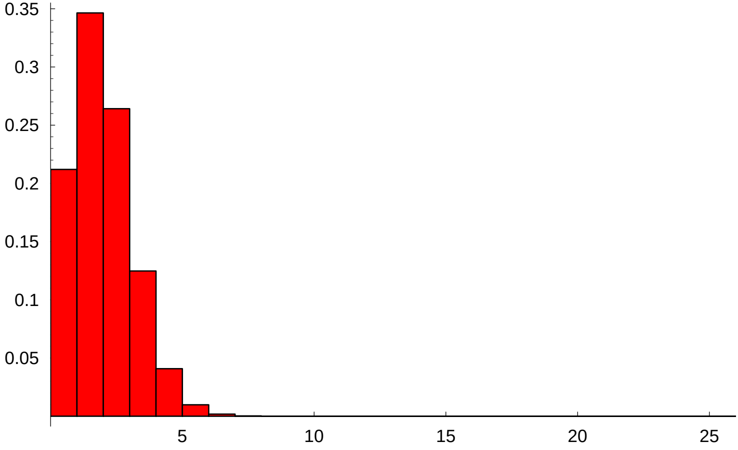
Example

number of parts of multiplicity 1 in partitions of 500:



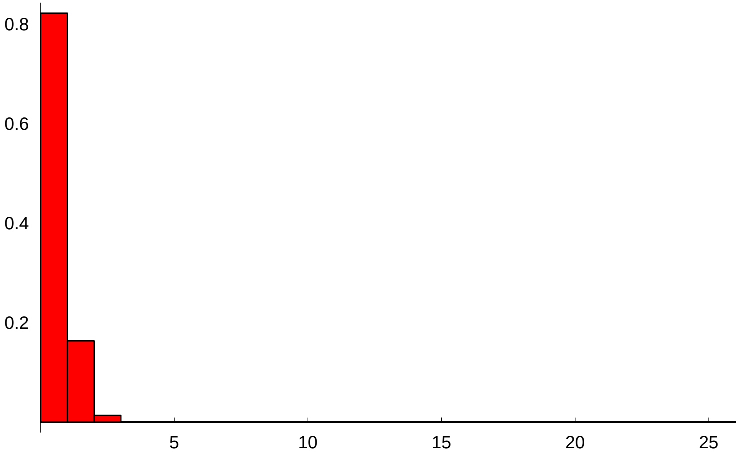
Example

number of parts of multiplicity 3 in partitions of 500:



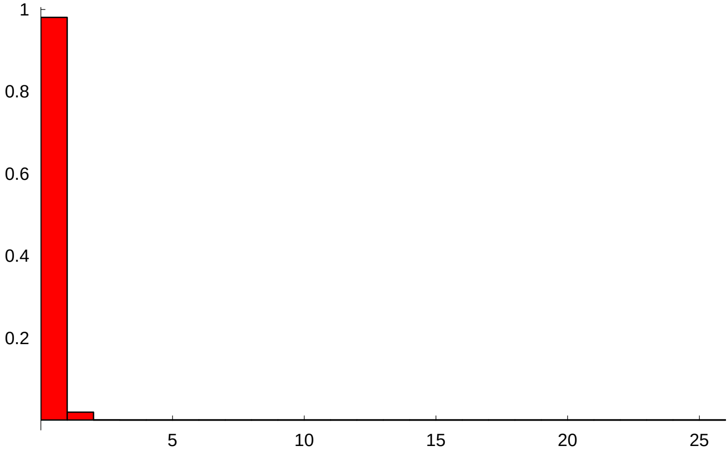
Example

number of parts of multiplicity 9 in partitions of 500:



Example

number of parts of multiplicity 27 in partitions of 500:



Thanks

Thank you!