On the distribution of multiplicities in integer partitions

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Abstract. We study the distribution of the number of parts of given multiplicity (or equivalently ascents of given size) in integer partitions. In this paper we give methods to compute asymptotic formulas for the expected value and variance of the number of parts of multiplicity d (d is a positive integer) in a random partition of a large integer n and also prove that the limiting distribution is asymptotically normal for fixed d. However, if we let d increase with n, we get a phase transition for d around $n^{1/4}$. Our methods can also be applied to so called λ-partitions where the parts are members of a sequence of integers λ.

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1. Introduction

Let d be a positive integer. An ascent of size d in a partition $(c_1, c_2, \cdots, c_t)$ of an integer n is a succession of two parts c_i, c_{i+1} such that $c_{i+1} - c_i = d$. If $c_1 = d$ then we assume that the partition has already one ascent of size d. Then the number of ascents of size d in a given partition is exactly the number of parts having multiplicity d in its conjugate partition.

Multiplicities in partitions were studied, amongst others, by Corteel et al [3], who showed that a randomly selected part of a random partition has multiplicity d with probability tending to $\frac{1}{d(d+1)}$. As a main step in their proof, they provide an asymptotic formula for the average number of parts of multiplicity d. A similar result was found by Knopfmacher and Munagi [9] for the number of ascents (successions) of size d. Here we improve on these results by proving a central limit theorem which can be stated as follows:

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Theorem 1. *The limit distribution of the number of parts having multiplicity d (or ascents of size d) in a random partition of n is Gaussian with mean and variance given by the asymptotic formulas:*

$$\mu_n = \frac{\sqrt{6n}}{\pi d(d+1)} + \frac{3}{\pi^2 d(d+1)} + o(1) \tag{1}$$

and

$$\sigma_n^2 \sim \left(\frac{1}{\pi d(d+1)} - \frac{1}{2\pi d(d+1)(2d+1)} - \frac{3}{\pi^3 d^2(d+1)^2}\right)\sqrt{6n} \tag{2}$$

respectively as $n \to \infty$.

A similar limit theorem was shown by Brennan, Knopfmacher and Wagner [2] for ascents of size d or more (equivalently, parts of multiplicity d or more). Later in this paper we give a generalisation of Theorem 1 and the results in [2] to λ-partitions satisfying the so called Meinardus scheme. For a complex variable s, the Dirichlet series associated to a sequence of positive integers λ is

$$D(s) = \sum_{\lambda} \lambda^{-s}, \tag{3}$$

and λ is said to satisfy the Meinardus scheme if the following three conditions hold:

- (M1) The Dirichlet series $D(s)$ converges in the half-plane $\mathrm{Re}(s) > \alpha > 0$, and can be analytically continued into the half-plane $\mathrm{Re}(s) \geq \alpha_0$ for some $\alpha_0 > 0$. In $\mathrm{Re}(s) \geq \alpha_0$, $D(s)$ is analytic except for a simple pole at $s = \alpha$ with residue A.
- (M2) There is a constant $c > 0$ such that $D(\sigma + it) \ll |t|^c$ uniformly for $\sigma \geq \alpha_0$ as $|t| \to \infty$.
- (M3) Let $\chi(\tau) = \sum_{\lambda} e^{-\lambda\tau}$, where $\tau = r + iy$ with $r > 0$ then
$$\chi(r) - \mathrm{Re}(\chi(\tau)) \gg \left(\log \frac{1}{r}\right)^2$$
uniformly for $r^{1+\frac{\alpha}{2}} \leq |y| \leq \pi$ as $r \to 0$.

There are many sequences of positive integers satisfying the Meinardus scheme including the sequence $\lambda = \mathbb{Z}^+$. We shall see more examples in Section 4.

In the above results, d was considered fixed. But when we let d increase with n and $d \to \infty$ as $n \to \infty$ then the following phase transition can be observed:

Theorem 2. *The limit distribution of the number of parts of multiplicity d is:*

- *Gaussian with mean and variance asymptotically equal to $\frac{\sqrt{6n}}{\pi d(d+1)}$ for $d = o(n^{1/4})$,*
- *Poisson with parameter $\frac{\sqrt{6}}{\pi\alpha^2}$ for $d \sim \alpha n^{1/4}$,*
- *degenerate at zero for $dn^{-1/4} \to \infty$.*

Other examples of Gaussian central limit theorems in the context of partitions include those by Goh and Schmutz [7] for the number of distinct parts and by Hwang [8] for the number of parts in a restricted partition (all multiplicities are less or equal to 1), generalising a result of Erdős and Lehner [5].

We present our results in the following way: We shall give a detailed proof of Theorem 1 in Section 2, and in Section 3 we discuss how the proof of Theorem 1 can be adapted to prove Theorem 2. In these proofs we use methods that can be generalised to the case of λ-partitions. Then Section 4 gives a generalisation of Theorem 2 to the case of λ-partitions and finally in Section 5 we discuss the generalisation of the results in [2].

2. Proof of Theorem 1

Throughout this section, d is a fixed positive integer. For a large positive integer n, we assign a uniform probability measure to the set of all partitions of n. Then the random variable ϖ_n is the number of parts of multiplicity d (which is the same as the number of ascents of size d in this case) in a random partition, its mean and standard deviation will be denoted by μ_n and σ_n respectively. We shall use $\prod_\lambda$ and $\sum_\lambda$ as abbreviations for the product and sum over all positive integers respectively. The reader should take note of the change in the other sections as we shall use the same notation but with different meaning.

The following is the generating function for the distribution of the number of parts of multiplicity d:

$$Q(u,z) = \prod_\lambda \left(\frac{1}{1-z^\lambda} + (u-1)z^{\lambda d} \right), \tag{4}$$

that is

$$\frac{Q_n(u)}{Q_n(1)} = \mathbb{E}(u^{\varpi_n}), \tag{5}$$

where $Q_n(u)$ is the coefficient of z^n in $Q(u,z)$. Note that $Q_n(1)$ is the total number of partitions of n. We also introduce the following functions:

$$f(\tau) := -\sum_\lambda \log(1-e^{-\lambda\tau}) \tag{6}$$

and

$$\phi(v,\tau) := \sum_\lambda \log(1+ve^{-d\lambda\tau}(1-e^{-\lambda\tau})). \tag{7}$$

Then we have

$$\log(Q(u,e^{-\tau})) = f(\tau) + \phi(u-1,\tau). \tag{8}$$

For simplicity we shall use the following abbreviation: if $F(\tau)$ is a function of a complex variable τ, and if it is analytic in some domain containing an element τ_0 in its interior, then we write $F_k(\tau_0)$ for

$$\frac{\partial^k}{\partial^k \tau} F(\tau)\Big|_{\tau=\tau_0}.$$

Let us first recall the asymptotic formula for $Q_n(1)$.

Lemma 3. *The number of partitions of n is given by the following asymptotic formula*

$$Q_n(1) = \frac{e^{nr}}{\sqrt{2\pi f_2(r)}} Q(1, e^{-r})(1 + O(n^{-1/7})) \tag{9}$$

as $n \to \infty$, where r is the positive solution of the equation

$$n = \sum_{\lambda} \frac{\lambda}{e^{\lambda r} - 1}. \tag{10}$$

Note that the above asymptotic formula implies the well known Hardy-Ramanujan formula:

$$Q_n(1) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}),$$

but one does not require it explicitly to prove our results. The proof of Lemma 3 is based on the use of the saddle point method that we shall not present here, see for instance [1] for a detailed presentation. However we use a similar approach to obtain the asymptotic formulas for the mean and variance given in Theorem 1. As mentioned earlier, results on the mean and variance can already be found in the literature (see [3, 9]), but the method we apply here is easier to generalise to λ-partitions.

2.1. Mean and Variance

By definition, the mean of the random variable ϖ_n is

$$\begin{aligned}\mu_n &= \frac{\partial}{\partial u} \mathbb{E}(u^{\varpi_n}) \Big|_{u=1} \\ &= \frac{1}{Q_n(1)} \frac{\partial}{\partial u} Q_n(u) \Big|_{u=1}.\end{aligned}$$

In order to find an asymptotic formula for μ_n we shall consider the following instead

$$Q_n(1)(\mu_n - g(r)) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp(int + f(r + it))(g(r + it) - g(r))dt \tag{11}$$

for any $r > 0$, where

$$g(\tau) = \sum_{\lambda} e^{-d\lambda\tau}(1 - e^{-\lambda\tau}).$$

We approximate this integral by means of the saddle point method where r is chosen to be the same as defined in Lemma 3, that is the solution of the equation

$$n = \sum_{\lambda} \frac{\lambda}{e^{\lambda r} - 1}.$$

The series on the right hand side is a monotone decreasing function of r therefore the solution exists and it tends to zero as $n \to \infty$. Now we claim that the integral (11) can be approximated by

$$\frac{e^{nr}}{2\pi} \int_{-r^{1+\beta}}^{r^{1+\beta}} \exp(int + f(r+it))(g(r+it) - g(r))dt \tag{12}$$

when $1/3 < \beta < 1/2$. This is not surprising since we know that without the term $g(r+it) - g(r)$ the estimate holds from the fact that for $|t| > r^{1+\beta}$

$$\begin{aligned} \frac{|Q(1, e^{-(r+it)})|}{Q(1, e^{-r})} &= \exp\Big(- \sum_{k \geq 1} \frac{1}{k} \sum_{\lambda} e^{-\lambda k r}(1 - \cos(\lambda k t))\Big) \\ &\leq \exp\Big(- \sum_{\lambda} e^{-\lambda r}(1 - \cos(\lambda t))\Big) \\ &\ll \exp\Big(- c|\log r|^2\Big) \end{aligned}$$

is smaller than any power of r^{-1} as $r \to 0$ (this is in fact the case for any sequence satisfying the condition (M3) in the Meinardus scheme). But note that for any t

$$|g(r+it) - g(r)| \ll \sum_{\lambda} e^{-\lambda r} \ll r^{-1}$$

as $r \to 0$. The claim follows from these two observations. Now for $|t| \leq r^{1+\beta}$ we have

$$f(r+it) = f(r) + if_1(r)t - f_2(r)\frac{t^2}{2!} - if_3(r)\frac{t^3}{3!} + \tag{13}$$

$$f_4(r)\frac{t^4}{4!} + if_5(r)\frac{t^5}{5!} + O\Big(|t|^6 \sup_{|\eta| \leq |t|} |f_6(r+i\eta)|\Big). \tag{14}$$

In order to obtain asymptotic estimates for $f_k(r)$ (and later also other quantities) we apply the method of Mellin transforms, in particular the following lemma from [6]:

Lemma 4. *Let $F(x)$ be a continuous function in $(0, \infty)$ with Mellin transform $F^*(s)$ having a non empty fundamental strip $\langle \alpha, \beta \rangle$. Assume that $F^*(s)$ admits a meromorphic continuation to the strip $\langle \gamma, \beta \rangle$ for $\gamma < \alpha$ with a finite number of poles there, and is analytic on $\mathrm{Re}(s) = \gamma$. Assume also that there exists a real number $\eta \in (\alpha, \beta)$ such that*

$$F^*(s) = O(|s|^{-c})$$

with $c > 1$, when $|s| \to \infty$ in $\gamma \leq \mathrm{Re}(s) \leq \eta$. If $F^*(s)$ admits the singular expansion for $s \in \langle \gamma, \alpha \rangle$

$$F^*(s) \asymp \sum_{(\xi,k) \in A} \frac{d_{\xi,k}}{(s-\xi)^k},$$

then an asymptotic expansion of $F(x)$ at 0 is

$$F(x) = \sum_{(\xi,k)\in A} \frac{(-1)^{k-1} d_{\xi,k}}{(k-1)!} x^{-\xi} (\log x)^{k-1} + O(x^{-\gamma}).$$

Specifically, the Mellin transform of $f_k(r)$ is

$$(-1)^k \zeta(s-k+1)\Gamma(s)\zeta(s-k).$$

Therefore, Lemma 4 gives

$$f_k(r) \sim (-1)^k k! \frac{\pi^2}{6r^{k+1}} \tag{15}$$

as $r \to 0$, in particular $n \sim \frac{\pi^2}{6} r^{-2}$. Furthermore, to estimate the error term in Equation (13) we have

$$|f_6(r+i\eta)| \ll \sum_{\lambda} \frac{\lambda^6 e^{-\lambda r}}{|1 - e^{-\lambda(r+i\eta)}|^6}.$$

For $\lambda \geq r^{-1}$ we have

$$|1 - e^{-\lambda(r+i\eta)}| \geq 1 - e^{-1},$$

so

$$\sum_{\lambda \geq r^{-1}} \frac{\lambda^6 e^{-\lambda r}}{|1 - e^{-\lambda(r+i\eta)}|^6} \ll \sum_{\lambda} \lambda^6 e^{-\lambda r} \ll r^{-7}$$

and if $\lambda < r^{-1}$,then

$$|1 - e^{-\lambda(r+i\eta)}| \gg 1 - e^{-\lambda r} \gg \lambda r.$$

Therefore,

$$\sum_{\lambda < r^{-1}} \frac{\lambda^6 e^{-\lambda r}}{|1 - e^{-\lambda(r+i\eta)}|^6} \ll r^{-6} \sum_{\lambda < r^{-1}} e^{-\lambda r} \ll r^{-7}.$$

Hence for $|\eta| \leq r^{1+\beta}$

$$|f_6(r+i\eta)| \ll r^{-7},$$

and we have

$$e^{nit + f(r+it)} = e^{f_0(r) - f_2(r)\frac{t^2}{2}} \Big(1 - i f_3(r) \frac{t^3}{3!} + f_4(r)\frac{t^4}{4!} + i f_5(r) \frac{t^5}{5!} + O(r^{6\beta-2})\Big).$$

Similarly we have

$$g(r+it) - g_0(r) = i g_1(r) t - g_2(r) \frac{t^2}{2} + O(r^{3\beta-1}), \tag{16}$$

and we also have the following asymptotic formula:

$$g_k(r) = \frac{(-1)^k k!}{d(d+1)} \frac{1}{r^{k+1}} \Big(1 + O(r)\Big), \tag{17}$$

this can also be obtained elementarily since $g(\tau)$ is a difference of geometric series in this case. Therefore, one can approximate $Q_n(1)(\mu_n - g_0(r))$ by

$$\frac{e^{nr+f_0(r)}}{2\pi}\int_{-r^{1+\beta}}^{r^{1+\beta}} e^{-f_2(r)t^2/2}\left(-g_2(r)\frac{t^2}{2}+f_3(r)g_1(r)\frac{t^4}{3!}+O(r^{7\beta-3})\right)dt$$

with an exponentially small error term, since integrals involving an odd power of t are identically 0. We may now change the range of integration to $(-\infty,+\infty)$ with another exponentially small error term and then apply the formula for the Gaussian integral. Then we get the following expression for the mean in terms of r:

$$\mu_n = g_0(r) - \frac{g_2(r)}{2f_2(r)} + \frac{f_3(r)g_1(r)}{2f_2^2(r)} + O(r^{7\beta-3}), \tag{18}$$

which gives

$$\mu_n = \frac{1}{d(d+1)}r^{-1} + \frac{3}{2\pi^2 d(d+1)} + O(r^{7\beta-3}). \tag{19}$$

Now for the variance we have

$$\sigma_n^2 = \frac{\partial^2}{\partial^2 u}\mathbb{E}(u^{\varpi_n})|_{u=1} - \mu_n^2 + \mu_n. \tag{20}$$

So we need to find an approximation of the second factorial moment

$$\frac{\partial^2}{\partial^2 u}\mathbb{E}(u^{\varpi_n})|_{u=1} = \frac{e^{nr}}{2\pi Q_n(1)}\int_{-\pi}^{\pi}\exp(int+f(r+it))\psi(r+it)dt, \tag{21}$$

where $\psi(\tau) = g^2(\tau) - h(\tau)$ and

$$h(\tau) = \sum_{\lambda} e^{-2d\lambda\tau}(1-e^{-\lambda\tau})^2.$$

Now we use the same method as for the mean: we obtain

$$g^2(r+it) = g_0^2(r) + 2ig_0(r)g_1(r)t - (g_1^2(r) + g_0(r)g_2(r))t^2 + O(r^{3\beta-2})$$

and also

$$h(r+it) = h_0(r) + ih_1(r)t - h_2(r)\frac{t^2}{2} + O(r^{3\beta-1}).$$

Since $h_k(r)$ has the same order as $g_k(r)$, the contribution from $-h(r+it)$ in the integral is $-h_0(r)$ with an error of at most constant order. For $g^2(r+it)$ we proceed as we did for the mean, and the main term of the integral comes from

$$(g^2(r+it) - g_0^2)\left(1 - if_3(r)\frac{t^3}{3!} + f_4(r)\frac{t^4}{4!} + if_5(r)\frac{t^5}{5!} + O(r^{6\beta-2})\right) =$$

$$2ig_0(r)g_1(r)t - (g_1^2(r) + g_0(r)g_2(r))t^2 + 2g_0(r)g_1(r)f_3(r)\frac{t^4}{3!} + O(r^{7\beta-4})$$

$$+ \text{ terms involving odd powers of } t.$$

When we apply the integral we get

$$\sigma_n^2 + \mu_n^2 - \mu_n - g_0^2(r) + h_0(r) = \frac{-g_1^2(r) - g_0(r)g_2(r)}{f_2(r)} + \frac{g_0(r)g_1(r)f_3(r)}{f_2^2(r)} + O(r^{7\beta-4}),$$

which implies that

$$\sigma_n^2 = \mu_n - h_0(r) - \frac{g_1^2(r)}{f_2(r)} + O(r^{7\beta-4}), \tag{22}$$

which gives

$$\sigma_n^2 = \left(\frac{1}{d(d+1)} - \frac{1}{2d(d+1)(2d+1)} - \frac{3}{\pi^2 d^2(d+1)^2}\right) r^{-1} + O(r^{7\beta-4}). \tag{23}$$

Having these formulas for the mean and variance we may use Lemma 4 to deduce asymptotic formulas as a function of n. First we need an asymptotic formula for $r = r(n)$. We have already mentioned an asymptotic dependence between n and r as a consequence of (15), and expanding further, using Lemma 4, we get

$$n = \frac{\pi^2}{6} r^{-2} - \frac{1}{2} r^{-1} + O(1),$$

which implies that

$$r^{-1} = \frac{\sqrt{6}}{\pi}\sqrt{n} + \frac{3}{2\pi^2} + O(n^{-1/2}). \tag{24}$$

So the equations (19) and (23) give the asymptotic formulas for the mean and variance

$$\mu_n = \frac{\sqrt{6}}{\pi d(d+1)}\sqrt{n} + \frac{3}{\pi^2 d(d+1)} + O(n^{-\epsilon}) \tag{25}$$

and

$$\sigma_n^2 \sim \left(\frac{1}{\pi d(d+1)} - \frac{1}{2\pi d(d+1)(2d+1)} - \frac{3}{\pi^3 d^2(d+1)^2}\right)\sqrt{6n} \tag{26}$$

as $n \to \infty$, where ϵ is a positive constant. These prove the asymptotic formulas in Theorem 1.

2.2. Moment Generating Function

We saw that the mean and variance are both tending to infinity, so in order to determine the limiting distribution we need to consider the normalised random variable

$$X_n := \frac{\varpi_n - \mu_n}{\sigma_n}. \tag{27}$$

The moment generating function of X_n is defined as

$$M_n(x) := \mathbb{E}(e^{xX_n}) \tag{28}$$

for a fixed real number x. To complete the proof of Theorem 1 we need to show that $M_n(x)$ converges pointwise to $e^{x^2/2}$ within a fixed interval containing 0. Note that $M_n(x)$ can also be written as follows:

$$M_n(x) = e^{-x\mu_n/\sigma_n} \frac{Q_n(e^{x/\sigma_n})}{Q_n(1)}. \tag{29}$$

Recall the formula for the coefficient

$$Q_n(u) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp(int + f(r+it) + \phi(u-1, r+it))dt. \tag{30}$$

We use the saddle point method again to find an asymptotic formula of the latter integral for u suitably close to 1, for now let us just say that $|u-1| \leq \delta$ for some fixed small $\delta > 0$. We shall be able to provide an asymptotic formula for $Q_n(u)$ by using a series of lemmas. We begin with the following which allows us to ignore the tails of the integral in (30).

Lemma 5. *There is a positive constant c_1, such that if $\pi > |t| > r^{1+c}$ where $1/3 < c < 1/2$ then*

$$\frac{|Q(u, e^{-(r+it)}|}{Q(u, e^{-r})} \ll e^{-c_1 |\log r|^2}$$

as $r \to 0^+$.

Proof. In fact this proof works for any sequence of positive integers λ satisfying the condition (M3) of the Meinardus scheme ($\frac{\alpha}{3} < c < \frac{\alpha}{2}$ for the general case), but also for arbitrary d. First we claim that for any complex number z such that $|z| \leq 2$ we have

$$\frac{|1+z|}{1+|z|} \leq e^{-\frac{1}{9}(|z| - \mathrm{Re}(z))}.$$

Indeed, for $|z| \leq 2$ we have

$$\begin{aligned}\frac{|1+z|^2}{(1+|z|)^2} &= 1 - 2\frac{|z| - \mathrm{Re}(z)}{(1+|z|)^2} \\ &\leq 1 - \frac{2}{9}(|z| - \mathrm{Re}(z)) \\ &\leq e^{-\frac{2}{9}(|z| - \mathrm{Re}(z))}.\end{aligned}$$

Now for any l that is a member of the sequence λ, and any z such that $|z| \leq 1$

$$|1 + z^l + z^{2l} + \cdots + z^{(d-1)l} + uz^{dl} + z^{(d+1)l} + \cdots| \tag{31}$$

$$\leq |1 + z^l| + |z^{2l} + z^{3l}| + \cdots . \tag{32}$$

Note that only one of the terms in (32) involves u, it is either $|uz^{dl} + z^{(d+1)l}|$ or $|z^{(d-1)l} + uz^{dl}|$ depending on the parity of d. We may assume that $1/2 \leq u \leq 2$. Using the inequality above, we find that for all positive real a and b such that $1/2 \leq b/a \leq 2$,

$$|az^{kl} + bz^{(k+1)l}| \leq e^{-\frac{1}{18}(|z|^l - \mathrm{Re}(z^l))} \Big(a|z|^{kl} + b|z|^{(k+1)l} \Big),$$

which implies that (31) is at most

$$e^{-\frac{1}{18}(|z^l|-\mathrm{Re}(z^l))}(1+|z|^l+|z|^{2l}+\cdots+|z|^{(d-1)l}+u|z|^{dl}+|z|^{(d+1)l}+\cdots).$$

Hence,

$$\frac{|Q(u,e^{-(r+it)}|}{Q(u,e^{-r})}\leq \exp\Big(-\frac{1}{18}\sum_{\lambda}e^{-\lambda r}(1-\cos(\lambda rt))\Big),$$

which proves the lemma if the sequence λ satisfies the condition (M3) of the Meinardus scheme. □

Note that the saddle point is chosen to be the solution of the equation

$$n=-f_1(r)-\phi_1(v,r). \tag{33}$$

So far nothing is known about the solution of Equation (33), we do not even know if a solution exists. For that we need the following lemma:

Lemma 6. *For any integer $j\geq 1$,*

$$\phi_j(v,r)\sim(-1)^j j! C(v,d) r^{-(j+1)}, \tag{34}$$

where

$$C(v,d)=\int_0^1\frac{\log(1+vx^d(1-x))}{x}dx, \tag{35}$$

these estimates are all uniform for $|v|\leq\delta$.

Proof. We write $\phi(v,\tau)$ as

$$\begin{aligned}\phi(v,\tau)&=\sum_{\lambda}\sum_{k\geq 1}(-1)^{k+1}e^{-kd\lambda\tau}(1-e^{-\lambda\tau})^k\frac{v^k}{k}\\&=\sum_{k\geq 1}(-1)^{k+1}\frac{v^k}{k}\sum_{\lambda}e^{-kd\lambda\tau}(1-e^{-\lambda\tau})^k.\end{aligned}$$

Then

$$\phi_j(v,r)=\sum_{k\geq 1}(-1)^{k+1}\frac{v^k}{k}\frac{\partial^j}{\partial^j\tau}\sum_{\lambda}e^{-kd\lambda\tau}(1-e^{-\lambda\tau})^k|_{\tau=r}$$

and we have the following Mellin transform:

$$\mathcal{M}\Big(\phi_j(v,r),s\Big)=(-1)^j\alpha(k,d,s-j)\Gamma(s)\zeta(s-j),$$

where

$$\alpha(k,d,s)=\sum_{k\geq 1}(-1)^{k+1}\frac{v^k}{k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\frac{1}{(kd+l)^s},$$

which is a Dirichlet series uniformly convergent in the right half-plane if $|v|<1/2$. Applying Lemma 4 to the function $\phi_j(v,r)$ for fixed v and j, this gives us the asymptotic formula in (34) with

$$C(v,d)=\alpha(k,d,1)=\int_0^1\frac{\log(1+vx^d(1-x))}{x}dx.$$

□

Lemma 6 along with the approximation of $f_k(r)$ in (15) imply the following

$$f_k(r) + \phi_k(v,r) \sim (-1)^k k! \Big(\frac{\pi^2}{6} - C(v,d)\Big) r^{-(k+1)} \tag{36}$$

for any $k \geq 1$. Furthermore, the constant $C(v,d)$ can be made arbitrarily small by making $v = u - 1$ small. From these observations, it follows that for fixed small v the function on the right hand side of (33) is a monotone decreasing function of r for $0 < r < \epsilon$ for some $\epsilon > 0$, and so there is a unique positive $r = r(u,n,d)$ satisfying Equation (33) provided that n is sufficiently large. One can already deduce an asymptotic relation

$$r^{-1} \sim \sqrt{\frac{6n}{\pi^2 - 6C(v,d)}} \tag{37}$$

as $n \to \infty$. We are now able to apply the saddle-point method.

Theorem 7. *The following asymptotic formula holds:*

$$Q_n(v+1) = \frac{1}{\sqrt{2\pi(f_2(r) + \phi_2(v,r))}} e^{nr + f(r) + \phi(v,r)} (1 + O(n^{1/7})) \tag{38}$$

as $n \to \infty$, uniformly for $|v| \leq \delta$.

Proof. Use Lemma 5 and Lemma 6 and apply the saddle point method in the same way as before. □

Now we go back to the formula for the moment generating function given in Equation (29). We shall adopt some new notations for the remaining part of this section so x will denote a fixed real number, $v = e^{x/\sigma_n} - 1$, $r = r(v)$ and $r_0 = r(0)$. From Theorem 7 it is not hard to show that

$$\frac{Q_n(v+1)}{Q_n(1)} \sim \exp\Big(nr + f(r) + \phi(v,r) - nr_0 - f(r_0)\Big). \tag{39}$$

It remains to estimate the exponent of (39). We recall the relation between n, v and r

$$n = -f_1(r) - \phi_1(v,r).$$

Then by means of implicit differentiation we get

$$\frac{\partial}{\partial v} r(v)\Big|_{v=\eta} = \frac{\frac{\partial}{\partial v}\phi_1(v, r(\eta))\Big|_{v=\eta}}{f_2(r(\eta)) + \phi_2(\eta, r(\eta))}. \tag{40}$$

If $|\eta| \leq e^{x/\sigma_n} - 1$, then r_0 and $r(\eta)$ are asymptotically equal. Therefore

$$f_2(r(\eta)) + \phi_2(\eta, r(\eta)) \gg r_0^{-3}.$$

Also by a similar technique as in the proof of Lemma 6 one may get

$$\frac{\partial}{\partial v}\phi_1(v, r(\eta))\Big|_{v=\eta} = O(r_0^{-2}).$$

Thus the difference $r - r_0$ is a $O(r_0|v|)$, that is of order $n^{-3/4}$ in terms of n. And so

$$f(r) - f(r_0) = f_1(r_0)(r - r_0) + f_2(r_0)\frac{(r-r_0)^2}{2} + O(r_0^{1/2}). \tag{41}$$

For $\phi(v, r)$ we use Taylor expansion with two variables

$$\begin{aligned}\phi(v,r) =& g(r_0)v + \phi_1(0,r_0)(r-r_0)\\ &+ h(r_0)\frac{v^2}{2} + g_1(r_0)(r-r_0)v + \phi_2(0,r_0)\frac{(r-r_0)^2}{2} + O(r_0^{1/2}).\end{aligned}$$

Adding up everything, we remain with

$$\frac{Q_n(v+1)}{Q_n(1)} \sim \exp\Big(g(r_0)v + \Big(-h(r_0) - \frac{g_1^2(r_0)}{f_2(r_0)}\Big)\frac{v^2}{2}\Big). \tag{42}$$

Moreover,

$$v = \frac{x}{\sigma_n} + \frac{x^2}{2\sigma_n^2} + O\Big(\frac{x^3}{\sigma_n^3}\Big). \tag{43}$$

Therefore,

$$M_n(x) \sim \exp\Big((g(r_0) - \mu_n)\frac{x}{\sigma_n} + \frac{1}{2}\Big(g(r_0) - h(r_0) - \frac{g_1^2(r_0)}{f_2}\Big)\frac{x^2}{\sigma_n^2}\Big) \tag{44}$$

as $n \to \infty$. Then Theorem 1 follows by using the asymptotic formulas for the mean and variance we proved in the first part of this section, together with Curtiss's Theorem [4].

3. Proof of Theorem 2

Here in this section $d = d(n)$ is an increasing function of n, and we assume that $d(n) \to \infty$ as $n \to \infty$. We shall keep the notations in Section 2 and note that the function $\phi(v, \tau)$, and thus also $g(\tau)$ and $h(\tau)$ are now functions of n as d is a function of n.

Let us first assume that $dn^{-1/4} \to \infty$, then it is sufficient to show that that the mean μ_n tends to 0, since we are dealing with a nonnegative random variable ϖ_n, and Markov's inequality will give us the desired result. Indeed, we still have

$$\mu_n \sim \frac{e^{nr}}{2\pi Q_n(1)}\int_{-r^{1+\beta}}^{r^{1+\beta}} \exp\Big(nit + f(r+it)\Big)g(r+it)dt,$$

where r is determined by the equation

$$n = \sum_\lambda \frac{\lambda}{e^{\lambda r} - 1}.$$

So it suffices to show that $g(r + it)$ goes to 0 uniformly in t ($|t| < r^{1+\beta}$). We have

$$|g(r+it)| \le \sum_\lambda e^{-d\lambda r}|1 - e^{-\lambda(r+it)}|,$$

for $\lambda \geq r^{-1}$ we have

$$\sum_{\lambda \geq r^{-1}} e^{-d\lambda r}|1 - e^{-\lambda(r+it)}| \ll \sum_{\lambda \geq r^{-1}} e^{-d\lambda r} \ll r^{-1}e^{-d},$$

and the latter is smaller than any power of n^{-1}. For $\lambda < r^{-1}$,

$$\sum_{\lambda < r^{-1}} e^{-d\lambda r}|1 - e^{-\lambda(r+it)}| \ll \sum_{\lambda} e^{-d\lambda r}(1 - e^{-\lambda r}) \ll \frac{1}{d^2 r}.$$

Therefore, $\mu_n \to 0$ as $n \to \infty$, and by Markov's inequality we have

$$\lim_{n\to\infty} \mathbb{P}\Big(\varpi_n \geq \epsilon\Big) = 0 \tag{45}$$

for any $\epsilon > 0$, which proves the convergence in probability to the degenerate random variable with support at 0.

Now for the remaining case $d = O(n^{1/4})$, we follow the lines in Section 2, but one needs asymptotic estimates for the functions $g_k(r)$ and $h_k(r)$. We cannot directly use Lemma 4 since d and r are somehow related and this might affect our estimates. But we use the same approach as in the proof of Lemma 4 in [6]. We have the Mellin transform

$$\mathcal{M}(g_k(r), s) = (-1)^k(d^{-s+k} - (d+1)^{-s+k})\Gamma(s)\zeta(s-k).$$

We want to show that the main term in the asymptotic formula of $g_k(r)$ is still

$$\frac{(-1)^k k!}{d(d+1)} r^{-(k+1)}.$$

That is the case if

$$\Big| \int_{c-i\infty}^{c+i\infty} (d^{-s+k} - (d+1)^{-s+k})\Gamma(s)\zeta(s-k)r^{-s}ds \Big| = o\Big(\frac{1}{d^2 r^{k+1}}\Big)$$

for $c < 1$. Let us just prove this for the case $k = 0$, and the other cases are obtained in a similar way. So let $0 < c < 1$; then we have

$$\begin{aligned}
&\Big| \int_{c-i\infty}^{c+i\infty} (d^{-s} - (d+1)^{-s})\Gamma(s)\zeta(s)r^{-s}ds \Big| \\
&\qquad \leq (dr)^{-c} \int_{c-i\infty}^{c+i\infty} \Big| \Big(1 - \frac{d^s}{(d+1)^s}\Big)\Gamma(s)\zeta(s) \Big| ds \\
&\qquad \ll \frac{1}{d^{c+1}r^c}.
\end{aligned}$$

To check the last line, for every real number t we have

$$|\Gamma(c+it)\zeta(c+it)| \leq c_1 e^{-c_2 t}$$

for some positive constants c_1 and c_2, and

$$\Big|1 - \frac{d^{c+it}}{(d+1)^{c+it}}\Big| = \Big|1 - \Big(1 - \frac{1}{d+1}\Big)^c e^{it\log(d/(d+1))}\Big|$$

which is $O(\max\{t/d, 1/d\})$ if $|t| \le \sqrt{d}$ and $O(1)$ for $|t| > \sqrt{d}$ as $n \to \infty$ where the implied constants are independent of t. Therefore we have

$$g(r) = \frac{1}{d^2 r} + O\Big(\frac{1}{d^{c+1} r^c}\Big)$$

as $(n, r) \to (\infty, 0)$. Similarly for $h(r)$, we have the Mellin transform

$$\mathcal{M}(h_k(r), s) = (-1)^k((2d)^{-s+k} - 2(2d+1)^{-s+k} + (2d+2)^{-s+k})\Gamma(s)\zeta(s-k).$$

Again for the case $k = 0$,

$$\begin{aligned}\Big|\frac{1}{(2d)^{c+it}} - 2\frac{1}{(2d+1)^{c+it}} + \frac{1}{(2d+2)^{c+it}}\Big| \\ \le \frac{1}{(2d)^c}\Big|1 - 2\Big(1 - \frac{1}{2d+1}\Big)^{c+it} + \Big(1 - \frac{1}{d+1}\Big)^{c+it}\Big|,\end{aligned}$$

then for $|t| \le \sqrt{d}$, the latter is a $O((t^2+1)/d^{c+2})$ and $O(1/d^c)$ for $|t| > \sqrt{d}$. Therefore

$$h(r) = \Big(\frac{1}{2d} - \frac{2}{2d+1} + \frac{1}{2d+2}\Big)r^{-1} + O\Big(\frac{1}{d^{c+2} r^c}\Big)$$

as $(n, r) \to (\infty, 0)$.

Therefore the mean and variance are asymptotically equal as n goes to infinity, more precisely:

$$\mu_n \sim \frac{\sqrt{6n}}{\pi d(d+1)} \quad \text{and} \quad \sigma_n^2 \sim \frac{\sqrt{6n}}{\pi d(d+1)}. \tag{46}$$

We shall now establish an asymptotic formula for $Q_n(u)$. Note first that the statement of Lemma 5 is valid in this case, and for Lemma 6 one may easily show (by using the same idea in the proof) that for a fixed positive integer j and sufficiently large n we have

$$\phi_j(u, r) = O(vr^{-1})$$

as $r \to 0$, and the implied constant is independent of n. Therefore, for some positive constant δ the following asymptotic formula still holds:

$$Q_n(v+1) = \frac{1}{\sqrt{2\pi(f_2(r) + \phi_2(v, r))}} e^{nr + f(r) + \phi(v,r)}(1 + O(n^{1/7})) \tag{47}$$

as $n \to \infty$ uniformly for $|v| \le \delta$, where $r = r(v, n)$ is the unique positive solution of the equation

$$n = -f_1(r) - \phi_1(v, r).$$

If $d = o(n^{1/4})$, then both the mean and variance tend to infinity so we shall consider the normalised random variable whose moment generating function can be expressed as

$$M_n(x) = e^{-x\mu_n/\sigma_n} \frac{Q_n(e^{x\varpi_n/\sigma_n})}{Q_n(1)}.$$

By the same arguments that we used to deduce Equation (42) we obtain the asymptotic formula

$$M_n(x) \sim \exp\Big(- \frac{x\mu_n}{\sigma_n} + g(r)(e^{x/\sigma_n} - 1)\Big) \tag{48}$$

as $n \to \infty$. Thus Equation (48) along with the formulas for the mean and variance implies

$$M_n(x) \sim e^{x^2/2}$$

which proves convergence to the normalised Gaussian distribution by using Curtiss's Theorem again.

If $d \sim \alpha n^{1/4}$, then both the mean and variance tend to a constant $\frac{\sqrt{6}}{\pi\alpha^2}$. We want to estimate the probability generating function, so let us fix a sufficiently small $\delta > 0$ and assume that $|u-1| \leq \delta$. Then we have

$$\mathbb{E}(u^{\varpi_n}) = \frac{Q_n(u)}{Q_n(1)} \sim \exp\Big(n(r-r_0) + f(r) - f(r_0) + \phi(u-1,r)\Big)$$

as $n \to \infty$, here we use the same notations as in the previous section. First we need to estimate the difference $r - r_0$, so let $|\eta| \leq |u-1| \leq \delta$. Then

$$\begin{aligned}\frac{\partial}{\partial v}\phi_1(v, r(\eta))\Big|_{v=\eta} &= -\sum_\lambda \frac{\lambda e^{-d\lambda r(\eta)}\Big(d - (d+1)e^{-\lambda r(\eta)}\Big)}{(1+\eta e^{-d\lambda r(\eta)} - \eta e^{-(d+1)\lambda r(\eta)})^2} \\ &\ll g_1(r(\eta)) \\ &\ll \frac{1}{r_0}.\end{aligned}$$

Hence $|r - r_0| \ll n^{-1}$. So, since $f_1(r_0) = -n$,

$$n(r-r_0) + f(r) - f(r_0) \ll f_2(r_0)n^{-2} \ll n^{-1/2}$$

and

$$\begin{aligned}\phi(u-1,r) &= (u-1)g(r) + O(n^{-1/4}) \\ &= \frac{\sqrt{6}}{\pi\alpha^2}(u-1) + o(1).\end{aligned}$$

Finally we deduce that

$$\mathbb{E}(u^{\varpi_n}) \sim e^{\frac{\sqrt{6}}{\pi\alpha^2}(u-1)}$$

as $n \to \infty$, which proves the convergence to the Poisson distribution with parameter $\frac{\sqrt{6}}{\pi\alpha^2}$. This completes the proof of Theorem 2.

4. Generalisation

As we mentioned in the introduction we shall see how these results change when we deal with partitions into elements of an arbitrary sequence λ. So from now on λ is a sequence of nondecreasing positive integers $(\lambda_1, \lambda_2, \lambda_3 \cdots)$ such that λ_k tends to infinity when k tends to infinity. The notations $\sum_\lambda$ and $\prod_\lambda$ now stand for the sum and product taken over the sequence λ. Then we have the following theorem:

Theorem 8. *If the sequence λ satisfies the conditions (M1) to (M3) of the Meinardus scheme then the number of parts of multiplicity d in a random λ-partition of n is asymptotically normally distributed where the mean and variance are given by the asymptotic formulas:*

$$\mu_n \sim \left(\frac{1}{d^\alpha} - \frac{1}{(d+1)^\alpha}\right) \frac{\Gamma(\alpha)A}{(A\zeta(\alpha+1)\Gamma(\alpha+1))^{\alpha/(\alpha+1)}} n^{\alpha/(\alpha+1)}$$

and

$$\sigma_n^2 \sim \Big(\frac{1}{d^\alpha} - \frac{1}{(d+1)^\alpha} - \frac{1}{(2d)^\alpha} + \frac{2}{(2d+1)^\alpha} - \frac{1}{(2d+2)^\alpha} - \left(\frac{1}{d^\alpha} - \frac{1}{(d+1)^\alpha}\right)^2 \frac{\alpha}{(\alpha+1)\zeta(\alpha+1)}\Big) \frac{A\Gamma(\alpha)n^{\alpha/(\alpha+1)}}{(A\zeta(\alpha+1)\Gamma(\alpha+1))^{\alpha/(\alpha+1)}}$$

respectively, if $d = o(n^{\alpha/(\alpha+1)^2})$.

If $d \sim an^{\alpha/(\alpha+1)^2}$, then the limiting distribution is Poisson with parameter

$$\frac{A\Gamma(\alpha+1)}{a^{\alpha+1}\Big(A\zeta(\alpha+1)\Gamma(\alpha+1)\Big)^{\alpha/(\alpha+1)}}.$$

And if $dn^{-\alpha/(\alpha+1)^2} \to \infty$, then the limiting distribution is degenerate at zero.

We shall not present the proof of this theorem since it is essentially the same as for ordinary partitions, and conditions (M1) to (M3) provide us with the necessary tools we need. More precisely conditions (M1) and (M2) allow us to apply the Mellin transform method to obtain asymptotic estimates for the functions f, ϕ, g, h and their derivatives. The condition (M3) is needed for the tail estimates in the saddle point method (just like in Lemma 5). See for instance [8] for a similar use of these conditions.

The result in Theorem 8 works for fairly large varieties of sequences of positive integer. For example all integer valued polynomials, where $\lambda_n = P(n)$ with an additional technical condition: $\gcd(P(n) : n \in \mathbb{Z}) = 1$, satisfies the Meinardus scheme and so Theorem 8 applies. However, there are some interesting sequences that fail to satisfy the conditions in this theorem, see for instance [10]. One obvious example that one could think of is the case of prime partitions. Condition (M3) is satisfied by the sequence of primes, as shown by Roth and Székeres, see [12]. But the Dirichlet series associated to the sequence of primes has a logarithmic singularity at 1, so clearly (M1) is not satisfied, but one can still use the Mellin transform method by using a Hankel contour, see [11] for a similar use of this technique. The final result reads as follows:

Theorem 9. *The number of parts with multiplicity d in a random prime partitions is:*

- *asymptotically normally distributed with mean and variance*

$$\mu_n \sim \frac{1}{\pi d(d+1)}\sqrt{\frac{12n}{\log n}}$$

and

$$\sigma_n^2 \sim \left(\frac{1}{\pi d(d+1)} - \frac{1}{2\pi d(d+1)(2d+1)} - \frac{3}{\pi^3 d^2(d+1)^2}\right)\sqrt{\frac{12n}{\log n}}$$

respectively, if $d = o((n/\log n)^{1/4})$,

- *Poisson with parameter* $\frac{\sqrt{12}}{a^2\pi}$ *if* $d \sim a(n/\log n)^{1/4}$,
- *degenerate at zero for* $d(n/\log n)^{-1/4} \to \infty$.

5. Parts with multiplicity d or more in λ-partitions

The number of ascents of size d or more in a random partition of an integer n has already been treated in [2] for fixed d, a result that can be expressed in the language of multiplicities since there is a one-to-one correspondence between partitions having parts multiplicity d and partitions with ascents of size d. So for completeness we shall give a generalisation of this result for λ-partitions. For this case we have the bivariate generating function

$$Q^*(u,z) = \prod_{\lambda} \left(\tfrac{1+(u-1)z^{\lambda d}}{1-z^{\lambda}}\right), \tag{49}$$

where the product is taken over the sequence λ. The logarithm

$$\phi^*(v,\tau) = \sum_{\lambda} \log(1 + ve^{-\lambda d\tau}), \tag{50}$$

is for our purposes actually easier to handle than the function $\phi(v,\tau)$ but the technique remains the same. There is a slight change though: a phase transition occurs when $d \sim an^{1/(1+\alpha)}$, and the limiting distribution in this case is not Poisson. This can be shown by the following simple argument: the probability generating function can be expressed as

$$\frac{Q_n^*(u)}{Q_n^*(1)} \sim \frac{e^{nr}}{2\pi Q_n^*(1)} \int_{-r^{1+\beta}}^{r^{1+\beta}} e^{\phi^*(u-1,r+it)} \exp\Big(nit + f(r+it)\Big)dt \tag{51}$$

as $n \to \infty$, where r is the unique positive solution of the equation

$$n = \sum_{\lambda} \frac{\lambda}{e^{r\lambda} - 1},$$

and β is an arbitrary constant such that $\frac{\alpha}{3} < \beta < \frac{\alpha}{2}$. It follows that

$$r \sim \Big(A\zeta(\alpha+1)\Gamma(\alpha)\Big)^{1/(\alpha+1)} n^{-1/(\alpha+1)}.$$

Moreover, we have

$$\phi^*(u-1, r+it) = \phi^*(u-1, r) + O(r^{\beta})$$

uniformly for $|t| \le r^{1+\beta}$. Therefore for a fixed real number u we have

$$\frac{Q_n^*(u)}{Q_n^*(1)} \to \prod_{\lambda} (1 + (u-1)e^{-\lambda\kappa})$$

as $n \to \infty$, where

$$\kappa = a(A\zeta(\alpha+1)\Gamma(\alpha+1))^{1/(\alpha+1)}.$$

Hence the final result reads as follows:

Theorem 10. *If the sequence λ satisfies the conditions (M1) to (M3) of the Meinardus scheme, then the number of parts of multiplicity d or more in a random λ-partition of n is asymptotically normally distributed, where the mean and variance are given by the asymptotic formulas:*

$$\mu_n \sim \frac{1}{d^\alpha} \frac{\Gamma(\alpha)A}{(A\zeta(\alpha+1)\Gamma(\alpha+1))^{\alpha/(\alpha+1)}} n^{\alpha/(\alpha+1)}$$

and

$$\sigma_n^2 \sim \Big(\frac{1}{d^\alpha} - \frac{1}{(2d)^\alpha} - \frac{\alpha}{d^{2\alpha}(\alpha+1)\zeta(\alpha+1)}\Big) \frac{A\Gamma(\alpha)n^{\alpha/(\alpha+1)}}{(A\zeta(\alpha+1)\Gamma(\alpha+1))^{\alpha/(\alpha+1)}}$$

respectively, if $d = o(n^{1/(\alpha+1)})$.

If $d \sim an^{1/(\alpha+1)}$, then the limiting distribution is a series of Bernoulli variables

$$\sum_\lambda \mathrm{Be}(e^{-\lambda\kappa}),$$

this series converges almost surely.

If $dn^{-1/(\alpha+1)} \to \infty$ then the limiting distribution is degenerate at zero.

For the case of ordinary partitions, $\alpha = 1$, and this gives the result proved in [2, last section]. But also, for $d = 1$, the number of parts having multiplicity d or more is equal to the total number of distinct parts in random λ-partitions, a case that has already been treated in [7] and [8].

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References

[1] George E. Andrews. *The theory of partitions.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[2] Charlotte Brennan, Arnold Knopfmacher, and Stephan Wagner. The distribution of ascents of size d or more in partitions of n. *Combin. Probab. Comput.*, 17(4):495–509, 2008.

[3] Sylvie Corteel, Boris Pittel, Carla D. Savage, and Herbert S. Wilf. On the multiplicity of parts in a random partition. *Random Structures Algorithms*, 14(2):185–197, 1999.

[4] J. H. Curtiss. A note on the theory of moment generating functions. *Ann. Math. Statistics*, 13:430–433, 1942.

[5] Paul Erdös and Joseph Lehner. The distribution of the number of summands in the partitions of a positive integer. *Duke Math. J.*, 8:335–345, 1941.

[6] Philippe Flajolet, Xavier Gourdon, and Philippe Dumas. Mellin transforms and asymptotics: harmonic sums. *Theoret. Comput. Sci.*, 144(1-2):3–58, 1995. Special volume on mathematical analysis of algorithms.

[7] William M. Y. Goh and Eric Schmutz. The number of distinct part sizes in a random integer partition. *J. Combin. Theory Ser. A*, 69(1):149–158, 1995.

[8] Hsien-Kuei Hwang. Limit theorems for the number of summands in integer partitions. *J. Combin. Theory Ser. A*, 96(1):89–126, 2001.

[9] Arnold Knopfmacher and Augustine O. Munagi. Successions in integer partitions. *Ramanujan J.*, 18(3):239–255, 2009.

[10] Manfred Madritsch and Stephan Wagner. A central limit theorem for integer partitions. *Monatsh. Math.*, 161(1):85–114, 2010.

[11] Dimbinaina Ralaivaosaona. On the number of summands in a random prime partition. *Monatshefte für Mathematik*. to appear.

[12] K. F. Roth and G. Szekeres. Some asymptotic formulae in the theory of partitions. *Quart. J. Math., Oxford Ser. (2)*, 5:241–259, 1954.

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