The Malliavin derivative and application to pricing and hedging a European exchange option

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Outline

Pricing and hedging contingent claims in complete markets

- Consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P) \quad t \geq T$
- Asset prices X_t are $\mathcal{F}_t-$ adapted
- For a European contingent claim $F(\omega) = f(X_T)$, for some (measurable) function f, the price of F is $v(t,T) = E_Q\left[e^{-\int_t^T \rho(s)ds} f(X_T)|\mathcal{F}_t\right]$
- For an American option $v(t,T) = ess\ sup_{\tau \in \mathcal{T}_{t,T}} E_Q\left[e^{-\int_t^\tau \rho(s)ds} f(X_\tau)|\mathcal{F}_t\right]$
- For geometric Brownian motion with constant coefficients $dX(t) = X(t)[\rho dt + \sigma d\tilde{B}(t)]$ with respect to Q, the solution when $f(x) = (x-K)^+$ is known and equal to $v(t,T) = X_t N(d_1) - Ke^{\rho(T-t)} N(d_2)$ where $d_{1,2} = \frac{\ln(\frac{X_t}{K}) + (\rho \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and N(x) is the cumulative distribution of N(0,1)

Black-Scholes Equation

- In the same case, the Black-Scholes equation becomes a boundary value problem
$$\begin{cases} \frac{\partial v}{\partial t} + \rho \frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} = \rho v \\ v(0,t) = 0 \\ v(T,x) = (x-K)^+ \end{cases}$$
- By a suitable transformation the Black-Scholes equation becomes the heat equation $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial s^2}$ with corresponding boundary conditions.
- numerical methods to solving the price

Hedging

Hedging a contingent claim implies

- looking for a self financing portfolio Θ .
- The value of the portfolio at any time $t \geq 0$ is $V(t) = V^{\Theta(t)}(t)$.
- If $V(T) = F(\omega)$ then the contingent claim is attainable in the market.

Price

- The manufacturing cost $V^{\Theta}(t)$ should then equal v(t,T) .
- This is the so called Harrison-Pliska result.
- Any market such that every contingent claim is attainable is called complete.
- In a complete market, there exists only one risk neutral measure Q, found through Girsanov Theorems

Martingale Representation Theorem

- If we assume that $X_t \in R^n$ and that $\tilde{B}(t) \in R^n$ (recipe for completeness), then the value of portfolio $\Theta \in \mathbb{R}^n$ is $V(t) = V(0) + \int_0^t \Theta(s)dX(s)$.
- It can be shown that $e^{-\rho T}V(T) = z + \int_0^T \phi(t)d\tilde{B}(t) = z + \int_0^T \sum_{j=1}^n \phi_j(t)d\tilde{B}_j(t) \quad z \in \mathbb{R} \quad (**)$
- Thus $E_Q[e^{-\rho T}V(T)] = z$ and $\phi(t)$ is related to $\Theta(t)$ in a special way. It turns out that $z = E_Q[e^{-\rho T}F(\omega)$ in a complete market and $z = v(0, x)$ is the price at time 0 of the contingent claim.
- Therefore a market is complete iff there exists $z \in \mathbb{R}$ and $\Theta(.)$ such that $(**)$ is satisfied.
- How do we find $\Theta(.)$?

Delta Hedging

If the market is Markovian, in the sense that the stock price processes is given by

$$dY(t) = b(Y(t))\,dt + \sigma(Y(t))\,dB(t), \quad Y(0) = y \in \mathbb{R}^n \tag{1}$$

then let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function . Then $h(Y(T)) = E_Q^y[h(Y(T))] + \int_0^T \phi(t,\omega)d\tilde{B}(t)$ where $\phi = (\phi_1, \ldots, \phi_n)$ is such that

$$\phi_j(t,\omega) = \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left(E_Q^y[h(Y(T-t))]\right)_{y=Y(t)} \sigma_{ij}(Y(t)), \quad 1 \le j \le m \tag{2}$$

Donsker-delta function approach

Let $(S) = (S)(\mathbb{R})$ be the Hida space of test functions and $(S)^* = (S)^*(\mathbb{R})$ be its dual, which is the space of *tempered* distributions.Now, for $\omega \in (S)^*$ and $\phi \in S$, let $\omega(\phi) :=< \omega, \phi >$ denote the *action* of ω on ϕ, then by the Bochner-Minlows theorem , there exists a probability measure P on $(S)^*$ such that

$$\int_{(S)^*} e^{i<\omega,\phi>} dP(\omega) = e^{-\frac{1}{2}||\phi||^2}; \ \phi \in S \tag{3}$$

where $||\phi||^2 = \int_{\mathbb{R}} |\phi(x)|^2 dx = ||\phi||^2_{L^2(\mathbb{R})}$. In this case P is called the white noise probability measure and $((S)^*, \mathcal{B}, P)$ is the white noise probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is the family of all Borel subsets of $(S)^*$.

The construction of a version of the Brownian motion then is a direct consequence of the Bochner-Minlows theorem in that if $\phi(t) = \begin{cases} 1 \text{ if } s \in [0,t] \\ 0 \text{ if } s \notin [0,t] \end{cases}$ then clearly $||\phi||^2_{L^2(\mathbb{R})} = t$ and thus $\int_{(S)^*} e^{i<\omega,\phi>} dP(\omega) = e^{-\frac{1}{2}||\phi||^2} = e^{-\frac{1}{2}t}$ so that immediately we conclude that $<\omega,\phi>= B(t)$ where B(t) is normal with mean 0 and variance t. One can easily prove that $B(t)$ is really a standard Brownian motion

Definition

Let $Y : \Omega \to \mathbb{R}$ be a random variable which belongs to $(S)^*$. Then a continuous function $\delta(Y - .) = \delta_Y(.) : \mathbb{R} \to (S)^*$ is called a Donsker delta function of Y if it has the property that $\int_{\mathbb{R}} \delta_Y(y) g(y) dy = g(Y)$ for all (measurable) $g : \mathbb{R} \to \mathbb{R}$ such that the integral converges.

Theorem

[3] The Donsker delta function of B(t) is

$$\delta(B(t)-y)=\delta_{B(t)}(y)=\frac{1}{\sqrt{2\pi t}}\exp^{\diamond}\left[-\frac{(y-B(t))^{\diamond 2}}{2t}\right]\in(S)^{*}$$

Malliavin Derivative

Definition

Assume that $F : \Omega \to \mathbb{R}$ has a directional derivative in all directions γ of the form $\gamma(t) = \int_0^t g(s)ds$ where $g \in L^2([0, T])$ for fixed T, in the strong sense that $D_\gamma F(\omega) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(\omega + \epsilon\gamma) - F(\omega)]$ exists in $L^2(\Omega)$ and assume further that there exists $\psi(t, \omega) \in L^2([0, T] \times \Omega)$ such that $D_\gamma F(\omega) = \int_0^T \psi(t, \omega)g(t)dt$,
then we say that F is differentiable and we call $D_t F(\omega) = \psi(t, \omega) \in L^2([0, T] \times \Omega)$ the Malliavin derivative of F.

- $D_t B(s) = 1_{t \le s} = \begin{cases} 1 \text{ if } t \le s \\ 0 \text{ otherwise} \end{cases}$
- Chain rule yields that, $D_t\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma B(T)}\right) = \sigma S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma B(T)}$.

Generalized CHO formula

Theorem (The generalized Clark-Ocone-Haussmann formula)

Suppose that $F \in \mathcal{D}_{1,2}$ and assume that the following conditions hold:

1. $E_Q[||F||_{L^2(Q)}] < \infty$
2. $E_Q\left[\int_0^T ||D_tF||^2_{L^2(Q)} dt\right] < \infty$
3. $$E_Q\left[||F||_{L^2(Q)} \cdot \int_0^T \left(\int_0^T D_t u(s,\omega) dB(s) + \int_0^T D_t u(s,\omega).u(s,\omega)\right)^2 dt\right] <$$ ∞

then

$F(\omega) = E_Q[F] +$
$\int_0^T E_Q\left[\left(D_tF - F\int_t^T D_t u(s,\omega) d\tilde{B}(s) | \mathcal{F}_t d\tilde{B}(t)\right) | \mathcal{F}_t\right] d\tilde{B}(t)$

where $u(s,\omega)$ is the Girsanov kernel, Q is the equivalent

The market portfolio

Stock:

$$dX_0(t) = \rho(t)X_0(t)dt \tag{4}$$

Bonds:

$$dX_i(t) = \alpha_i(t,\omega)dt + \sum_{j=1}^{n}\sigma_{ij}(t,\omega)dB_j(t), \quad X_i(0) = x_i \tag{5}$$

where α_i is the appreciation rate of security number i and σ_{ij} is the volatility coefficient of the Brownian motion $B_j(t)$ in security i.

or

$$d\hat{X}(t) = \alpha(t)dt + \sigma(t)dB(t), \quad \hat{X}(0) = \hat{x}_0 \tag{6}$$

Value of portfolio

$$V^{\Theta}(t) = V^{\Theta}(t,\omega) = V(0) + \int_0^t \theta_0 dX_0(s) + \sum_{i=1}^{n} \int_0^t \theta_i(s) dX_i(s).$$

or

$dV^{\Theta}(t) = \rho(t)V^{\Theta}(t)dt + \Gamma(t)\sigma\left[\sigma^{-1}(\alpha - \rho\mathbb{I})dt + dB(t)\right]$

We expect

$F(\omega) = e^{-\rho T}V^{\Theta}(T)$

By letting $G(\omega) = e^{-\rho T}F(\omega)$ and applying the generalized CHO formula to G, we have

$$G(\omega) = E_Q[G] + \int_0^T E_Q\left[\left(D_t G - G\int_t^T D_t u(s,\omega)d\tilde{B}(s)|\mathcal{F}_t d\tilde{B}(t)\right)|\mathcal{F}_t\right.$$

By uniqueness due to the Martingale Representation Theorem, we get

$$V(0) = V^{\Theta}(0) = E_Q[G] \tag{8}$$

and

$$e^{-\rho t}\Gamma(t)\sigma = E_Q\left[\left(D_t G - G\int_t^T D_t u(s,\omega)d\tilde{B}(s)|\mathcal{F}_t d\tilde{B}(t)\right)|\mathcal{F}_t\right] \tag{9}$$

where as before $\Gamma(t) = \left(\theta_1, \ldots, \theta_n\right)^{Tr}$ and Tr means transpose. Therefore

$$\Gamma(t) = e^{-\rho(T-t)}\sigma^{-1}E_Q\left[\left(D_t G - G\int_t^T D_t u(s,\omega)d\tilde{B}(s)|\mathcal{F}_t d\tilde{B}(t)\right)|\mathcal{F}_t\right]$$

Proposition

Let X_1 and X_2 be two independent standard normal random variables and let $\lambda \in \mathbb{R}$. Define a probability measure P^{λ} equivalent to P with density

$$\frac{dP^{(\lambda)}}{dP} = e^{\lambda X_1 - \frac{1}{2}\lambda^2}.$$

Then the random Gaussian variable $X_1 - \lambda$ and X_2 are independent standard variables with respect to $P^{(\lambda)}$.

Corollary

Let X_1 and X_2 be as given in Proposition 1 and let y_1, y_2, λ_1 and λ_2 be real numbers. Then

$$E_P\left[(S_1 - S_2)^+\right] =$$

$$e^{y_1+\frac{1}{2}\lambda_1^2}\Phi\left(\frac{y_1 - y_2 + \lambda_1^2}{\sqrt{\lambda_1^2+\lambda_2^2}}\right) - e^{y_2+\frac{1}{2}\lambda_2^2}\Phi\left(\frac{y_1 - y_2 - \lambda_2^2}{\sqrt{\lambda_1^2+\lambda_2^2}}\right)$$

where $S_1 = e^{\lambda_1 X_1 + y_1}$ *and* $S_2 = e^{\lambda_2 X_2 + y_2}$

Proposition

Let X_1 and X_2 be two independent $m-$dimensional normal random vectors each with mean equal to the zero vector and covariance matrix equal to the identity matrix and let $\vec{u} \in \mathbb{R}^m$ be any non-zero vector. Define a probability measure $P^{(u)} = Q$, equivalent to P with density $dP^{(u)}(\omega) = e^{\vec{u}X_1 - \frac{1}{2}||u||^2} dP(\omega)$, where $||.||$ is the usual norm in $\mathbb{R}^m$.
Then $X_1 - \vec{u}$ and X_2 are independent Gaussian vectors with zero mean (vector) and covariance matrix equal to the identity.

Corollary

Let X_1 and X_2 be as in Proposition 2 and let y_1 and y_2 be real numbers. If $\vec{u}_1$ and $\vec{u}_2$ are $m-$dimensional vectors, then

$$E_P\left[(S_1 - S_2)^+\right] =$$
$$e^{y_1+\frac{1}{2}||u_1||^2}\Phi\left(\frac{y_1-y_2+||u_1||^2}{\sqrt{||u_1||^2+||u_2||^2}}\right) - e^{y_2+\frac{1}{2}||u_2||^2}\Phi\left(\frac{y_1-y_2-||u_2||^2}{\sqrt{||u_1||^2+||u_2||^2}}\right)$$

where $S_1 = e^{y_1+\vec{u}_1 X_1}$ and $S_2 = e^{y_2+\vec{u}_2 X_2}$ and $||.||$ denotes the usual norm in $\mathbb{R}^m$

Proposition

The price of the European exchange option is given by

$$V(0) = X_1(0)\Phi\left(\frac{\ln\left(\frac{X_1(0)}{X_2(0)}\right) + \frac{T}{2}\sum_{j=1}^{2}(\sigma_{2j}^2 + \sigma_{1j}^2)}{\sqrt{T\sum_{j=1}^{2}(\sigma_{1j}^2 + \sigma_{2j}^2)}}\right)$$

$$-X_2(0)\Phi\left(\frac{\ln\left(\frac{X_1(0)}{X_2(0)}\right) - \frac{T}{2}\sum_{j=1}^{2}(\sigma_{2j}^2 + \sigma_{1j}^2)}{\sqrt{T\sum_{j=1}^{2}(\sigma_{1j}^2 + \sigma_{2j}^2)}}\right)$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$ *is the cumulative distribution function of the standard normal distribution.*

We now calculate the hedging portfolio $\Theta = \Big(\theta_0(t), \theta_1(t), \theta_2(t)\Big)$.
For this two dimensional case, thanks to the CHO formula, we get, from (9), that
$\Gamma(t) = e^{-\rho(T-t)}\sigma^{-1}E_Q[D_tF|\mathcal{F}_t]$, where, as before
$\sigma^{-1} = I\begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$,
with
$I = (\sigma_{22}\sigma_{11} - \sigma_{12}\sigma_{21})^{-1}$ and $\Gamma(t) = \begin{pmatrix} \theta_1(t) & \theta_2(t) \end{pmatrix}$.
Now $D_tF = (\sigma_{11}\ ,\sigma_{12})^T X_1(T)1_D - (\sigma_{21}\ ,\sigma_{22})^T X_2(T)1_D$ where
$D = \{\omega : X_1(T,\omega) > X_2(T,\omega)\{$. Therefore
$E_Q[D_tF|\mathcal{F}_t] =$
$(\sigma_{11}\ ,\sigma_{12})^T E_Q[X_1(T)1_D|\mathcal{F}_t] - (\sigma_{21}\ ,\sigma_{22})^T E_Q[X_2 1_D|\mathcal{F}_t]$.
We thus have the following result

Proposition

The perfect hedge $\Theta(t)$ *is given by*

$$\theta_1(t) = \frac{1}{\Delta}\left[X_1(t)(\sigma_{11}\sigma_{22} - \sigma_{12}^2)\Phi(d_1) - X_2(t)(\sigma_{22}\sigma_{21} - \sigma_{12}\sigma_{22})\Phi(d_2)\right]$$

and

$$\theta_2(t) = \frac{1}{\Delta}\left[X_1(t)(\sigma_{11}\sigma_{12} - \sigma_{11}\sigma_{21})\Phi(d_1) - X_2(t)(\sigma_{11}\sigma_{22} - \sigma_{21}^2)\Phi(d_2)\right]$$

where $\Delta = \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}$,

$$d_1 = \frac{\ln\left(\frac{X_1(t)}{X_2(t)}\right) + \frac{T-t}{2}\sum_{j=1}^{2}(\sigma_{2j}^2 + \sigma_{1j}^2)}{\sqrt{(T-t)\sum_{j=1}^{2}(\sigma_{1j}^2 + \sigma_{2j}^2}}$$

and

$$d_2 = \frac{\ln\left(\frac{X_1(t)}{X_2(t)}\right) - \frac{T-t}{2}\sum_{j=1}^{2}(\sigma_{2j}^2 + \sigma_{1j}^2)}{\sqrt{(T-t)\sum_{j=1}^{2}(\sigma_{1j}^2 + \sigma_{2j}^2}}$$

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