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LAGUERRE POLYNOMIALS AND SINGULAR DIFFERENTIAL OPERATORS

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This paper is concerned with the connections between the orthogonal polynomials and the differential operators generated by the Laguerre differential equation

$$M[y] := -(p\,y')' + qy = \lambda wy \text{ on } (0, \infty) \qquad \ldots(A^*)$$

(where $p(x) = x^{\alpha+1}e^{-x}$, $q(x) = \dfrac{\alpha+1}{2}\, w(x)$, $w(x) = x^{\alpha}e^{-x}$,

$\alpha > -1$, $\lambda \in \mathbb{C}$),

in both the so-called right- and left-definite cases. In the right-definite case, a differential operator T_α is determined directly from (A^*) by a definition of the form $T_\alpha f := w^{-1} M[f]$ $(f \in D(T_\alpha))$ for a suitably defined domain of T_α in the space $L^2_w(0, \infty)$. However, for the left definite case, a suitably determined resolvent function Φ is used to define a bounded self-adjoint operator A_α whose inverse is the required self-adjoint "differential" operator S_α in the space $H^2_{p,q}(0, \infty)$. In both cases the spectra of T_α and S_α are shown to be discrete and the corresponding eigenvectors turn out to be the orthogonal polynomials of Laguerre. These results provide an alternative proof of the completeness of the Laguerre polynomials in the spaces $L^2_w(0,\infty)$ and $H^2_{p,q}(0,\infty)$.

1. INTRODUCTION

For convenience let $(0, \infty)$ denote the open interval of the real line, and let $N_0 = \{0, 1, 2, 3, \ldots\}$ be the set of non-negative intergers, and $N_+ = \{1, 2, 3, \ldots\}$, the set of positive intergers.

The Laguerre polynomials may be defined in a number of different but related ways[3] namely, by the Poisson generating function, the Rodrigues formula, the Gram-Schmidt orthogonalization, or by the Laguerre differential equation which for our purposes is written in the form (here $' \equiv d/dx$) :

$$x\,y''(x) + (\alpha + 1 - x)\,y'(x) + ny(x) = 0, \; (x \in (0, \infty), n \in N_0) \qquad \ldots(1.1a)$$

or

$$(x^{\alpha+1} e^{-x} y'(x))' + nx^{\alpha} e^{-x} y(x) = 0; (x \in (0, \infty), \ \alpha > -1, n \in N_0). \qquad \text{...(1.1b)}$$

This equation has singular points at 0 (for $\alpha > -1$) and at $+\infty$ where the leading coefficient vanishes and it has a non-trivial solution if and only if $n \in N_0$; the corresponding solutions are $L_n^{(\alpha)}(\,.\,)$, the Laguerre polynomials of order n, see Endelyi *et al.*[3]

Whichever form of definition is adopted it is important to show that the set of Laguerre polynomials $\{L_n^{(\alpha)}(\,.\,); n \in N_0, \alpha > -1\}$ is complete in $L_w^2 \ (0, \infty)$, (where $w(x) = x^{\alpha+1} e^{-x}, \alpha > -1$), [see Akhiezer and Glazman[1], (sections 8 and 9; and Szegö[17], (section 5.7).]

Our interest in this paper is to study the definition and completeness of the Laguerre polynomials in $L_w^2 \ (0, \infty)$ from the viewpoint of the Titchmarsh-Weyl theory of singular differential operators, associated with the differential equation

$$-(py')' + qy = \lambda \, w \, y \text{ on } (a, b) \qquad \text{...(1.2)}$$

where p, q, w are real-valued coefficients defined on the interval (a, b) and $\lambda \in \mathbb{C}$.

If in (1.2), $w \geqslant 0$ on (a, b), then this is called a right-definite problem and is studied in the space $L_w^2 \ (a, b)$ for which

$$\int_a^b w(x) \mid f(x) \mid^2 dx < \infty, \ (f \in L_w^2 \ (a, b)). \qquad \text{...(1.3)}$$

If in (1.2) it should happen that whether w is of one sign or not, both p and q are non-negative then the problem is called left-definite and is studied in the space

$$H_{p,q}^2 \ (a, b) := \{f : (a, b) \to \mathbb{C} : \ f \in AC_{loc} \ (a, b),$$
$$q^{1/2} f \text{ and } p^{1/2} \ f' \in L^2 \ (a, b)\} \qquad \text{...(1.4)}$$

for which

$$\int_a^b \{p(x) \mid f'(x) \mid^2 + q(x) \mid f(x) \mid^2\} \, dx < \infty. \qquad \text{...(1.5)}$$

The existence of such integrable—square solutions in the left-definite case has been considered in recent years by Pleijel[12–14], Shortwell[16], Atkinson *et al.*[2] and Everitt[5,5] among others.

For both right- and-left-definite cases the differential equation (1.2) may be classified as either 'limit-point' or 'limit-circle' at the end-point $a(b)$ according as to whether not all or all solutions of (1.2) are in the spaces $L^2_w\ (a, b)$ and $H^2_{p,q}\ (a, b)$ respectively in the neighbourhood of $a\ (b)$. For reference to the classification of the differential equation (1.2), see Naimark[8], (section 18.1) and Titchmarsh[18] (sections 2.1 and 2.19) for the right-definite case; and in the left-definite case, see Pleijel[15] (sections 1.4 and 1.5).

Now for the purpose of studying the Laguerre differential equation in both the right- and left-definite cases, we write (1.1) in the form

$$M[y] \equiv (x^{\alpha+1}e^{-x}y'(x))' + (\alpha + 1)/2\ x^{\alpha}e^{-x}\ y(x) = \lambda x^{\alpha}e^{-x}\ y(x) \quad ...(1.6)$$
$$(x \in (0, \infty),\ \lambda \in \mathbb{C})$$

so that in comparison with (1.2), $a = 0$, $b = +\infty$ and $p(x) = x^{\alpha+1}\ e^{-x}$,

$$q(x) = (\alpha + 1)/2,\ W(x) = x^{\alpha}\ e^{-x},\ (x \in (0, \infty),\ \alpha > 1). \quad ...(1.7)$$

The original study of Laguerre's equation in the right-definite case in $L^2_w\ (0, \infty)$, is due to Titchmarsh[18] (section 4.16). We note however that the analysis of Titchmarsh is essentially "classical" with no reference to the operator theoretic concepts. Also we note that for all $\lambda \in \mathbb{C}$, (and in particular, for $\lambda = (\alpha+1)/2$, $\alpha > -1$), the Laguerre's differential equation satisfies the following properties (for details see Theorem 3. 5, of Onyango-Otieno[10]).

Theorem 1.1—(a) For the right-definite case (i. e. in $L^2_w\ (0, \infty)$), eqn. (1.6) is

(i) regular at $x = 0$ if $-1 < \alpha < 0$

(ii) limit-circle at $x = 0$ if $0 \leqslant \alpha < 1$

(iii) limit-point at $x = 0$ if $1 \leqslant \alpha < \infty$

(iv) limit-point at $x = +\infty$ for all $\alpha > -1$.

(b) for the left-definite case (i. e. in $H^2_{p,q}\ (0, \infty)$), eqn. (1.6) is

(i) regular at $x = 0$ if $-1 < \alpha < 0$

(ii) limit-point at $x = 0$ if $0 \leqslant \alpha < \infty$

(iii) limit-point at $x = +\infty$ for all $\alpha > -1$.

(In the left-definite case, greater care must be taken in looking at real values of λ in order to determine the classification of the equation, see the example of Ong[9] (section 4).

Our aim in this paper is to study the right- and left-definite problems for the Laguerre differential equation with the methods of Titchmarsh[18] in mind. We link the Titchmarsh method with operator-theoretic results in the Hilbert function spaces $L^2_w\ (0, \infty)$ and $H^2_{p,q}\ (0, \infty)$.

This approach is adapted from a similar discussion on the Legendre differential equation in a recent paper by Everitt[5] (see also Onyango-Otieno[11]).

The paper is in five parts; after this introduction the second section considers the essential properties of the Legendre differential equation. The third and fourth sections are devoted to a study of the right-and left- definite cases; and five to certain remarks.

Notations

R-real field; $\mathbb{C}$-complex field

L-Lebesgue integration,

AC_{loc}.—local absolute continuity;

if D is a set of elements f then $(f \in D)$ is to read the set of all $f \in D$.

$(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$, $k \in N_0$, with $(\alpha)_0 = 1$.

2. The Laguerre Differential Equation

The standard form of Laguerre's equation is given in (1.1a), but for the purpose of considering both right- and left-definite problems the form (1.6) is to be prefered, namely

$$-(x^{\alpha+1}e^{-x}\, y'(x))' + (\alpha+1)/2\, x^{\alpha}e^{-x}y(x) = \lambda x^{\alpha}e^{-x}y(x) \qquad \ldots(1.8)$$

$$(x \in (0,\infty),\ \alpha > -1,\ \lambda \in \mathbb{C})$$

with the coefficients p, q and w as in (1.7).

Titchmarsh's[18] (section 4.16) the Laguerre differential equation is based on the Liouville normal form of (1.6) by putting $x = 2x^{1/2}$ (or $x = X^2/4$) and

$$Y(X) = \{X^{4\alpha+2}e^{-X^2/2}/4^{2\alpha+1}\}^{1/4}\, y(x(X)),\ (X \in (0,\infty)) \qquad \ldots(2.1)$$

to give

$$-Y''(X) + \left(\frac{X^2}{16} - \frac{\frac{1}{4}-\alpha^2}{X^2}\right)Y(X) = \lambda Y(X),\ (X \in (0,\infty)). \qquad \ldots(2.2)$$

However, it is evident from the results[18] (section 4.16) that this equation does not of itself enjoy the property of having polynomial solutions. Here we adapt the analysis of Titchmarsh to apply similar methods to the eqn. (1.6) which does have the Laguerre polynomials directly as solutions.

Now following the analysis in Onyango-Otieno[10] (section 7.1) a solution of (1.6) is of the form

$$Y_\alpha(x,\lambda) = \begin{cases} \dfrac{-\Gamma((1-\alpha)/2+\lambda)\,x^{-\alpha}}{2\pi i}\displaystyle\int_\infty^{(0+)} \frac{\exp[-z]\,.\,(z+x)^{\lambda+(\alpha-1)/2}}{(-z)^{\lambda-(\alpha-1)/2}}\,dz & \ldots(2.3a) \\ \quad (\text{re}\,(\lambda-(\alpha+1)/2 \neq -n,\ n\in N_+) & \\ \dfrac{x^{-\alpha}}{\Gamma((\alpha+1)/2-\lambda}\displaystyle\int_0^\infty \frac{\exp[-z]\,.\,(z+x)^{\lambda+(\alpha-1)/2}}{z^{\lambda-(\alpha-1)/2}}\,dz & \ldots(2.3b) \\ \quad (\text{otherwise}) & \end{cases}$$

where the contour of integration c, from $+\infty$, round 0 to $+\infty$, excludes the point $z = -x$ (note that $Y_\alpha(0,\lambda)$ remains defined as $x \to 0$, provided $\alpha > 0$) with

$$(-z)^{\lambda-(\alpha-1)/2} := \exp[(\lambda-(\alpha-1)/2.\ \log(-z)],\ 0 \leqslant \arg(-z) < 2\pi$$

and

$$(z+x)^{\lambda+(\alpha-1)/2} := \exp[(\lambda+(\alpha-1)/2.\ \log(z+x)].$$

(note that under these circumstances the integrand is analytic in x and z).

To show that (2.3) satisfies Laguerre's equation (1.6) put

$$Y_\alpha(x,\lambda) = \frac{-\Gamma((1-\alpha)/2+\lambda)}{2\pi i}\ V_\alpha(x,\lambda)$$

where

$$V_\alpha(x,\lambda) = x^{-\alpha}\int_\infty^{(0+)} \frac{\exp[-z]\,.\,(z+x)^{\lambda+(\alpha-1)/2}}{(-z)^{\lambda-(\alpha-1)/2}}\,dz.$$

Since $V_\alpha(.,\lambda)$ is analytic in x, it follows from a standard result in Titchmarsh[19] (section 2.83) that

$$V'_\alpha(x,\lambda) = \int_\infty^{(0+)} \frac{x^{-\alpha-1}\exp[-z]\,.\,(z+x)^{\lambda+(\alpha-3)/2}}{(-z)^{\lambda-(\alpha-1)/2}}$$

$$\times\ [(\lambda-(\alpha+1)/2)\,x-\alpha z]\,dz.$$

Similarly

$$(x^{\alpha+1}e^{-x}\ V'_\alpha(x,\lambda))' = e^{-x}\int_\infty^{(0+)} \frac{\exp[-z]\,.\,(z+x)^{\lambda+(\alpha-5)/2}}{(-z)^{\lambda-(\alpha-1)/2}}\ \{\alpha z^2$$

$$-(\lambda-(\alpha+1)/2)\,x^2-(\lambda-(\alpha+1)/2-\alpha)\,xz+(\lambda+(\alpha-1)$$
$$/2)\,(\lambda-(\alpha+1)/2)x+(1-\alpha)\,(\lambda+(\alpha-1)/2)\,z\}\ dz.$$

Hence

$$(x^{\alpha+1}e^{-x} V'_\alpha \ (x, \lambda))' + (\lambda - (\alpha + 1)/2)\, x^\alpha\, e^{-x}\, V_\alpha\, (x, \lambda)$$

$$= (\lambda + (\alpha - 1)/2)e^{-x} \int_\infty^{(0+)} \frac{\exp[-z] \,.\, (z + x)^{\lambda+(\alpha-5)/2}}{(-z)^{\lambda-(\alpha-1)/2}} [z^2 + xz + (\lambda - (\alpha + 1)/2)\, x + (1 - \alpha)\, z]\, dz$$

$$= (\lambda + (\alpha - 1)/2)e^{-x} \int_\infty^{(0+)} d/dz\, \{\exp[-z].\, (z + x)^{\lambda+(\alpha-3)/2}\, (-z)^{-\lambda+(\alpha+1)/2}\}\, dz = 0$$

(because the expression $\{......\}$ tends to zero as $z \to +\infty$) and this is the condition that Y_α (., λ) should satisfy eqn. (1.6).

To obtain a second solution Z_α (., λ) of (1.6) werec all its Liouville normal form (2.2), namely

$$Y''(X) + (\lambda - X^2/16 + (1/4 - \alpha^2)/X^2)\; Y(X) = 0 \qquad ...(2.2)$$

$$(X \in (0, \infty),\ \lambda \in ¢)$$

Put

$$X = 2x^{1/2} \text{ and } Y(x) = x^{-1/4}\, U(x); \qquad ...(2.4)$$

then with $d/\,dX = x^{1/2}\, d/dx$, (2.2) now becomes

$$U''(x) + (-1/4 + \lambda/x + (1 - \alpha^2)/4x^2)\; U(x) = 0. \qquad ...(2.5)$$

$$(x \in (0, \infty),\ \lambda \in ¢)$$

which is a confluent hypergeometric equation. It has as one of its solutions[18] (section 4.16), the series

$$U_\alpha (x, \lambda) = x^{(\alpha+1)/2}e^{-x/2} \left\{1 + \sum_{k=1}^{\infty} \frac{((1 + \alpha)/2 - \lambda)_k}{(\alpha + 1)_k} x^k/k! \right\} \qquad ...(2.6)$$

$$(\alpha > -1,\ x \in (0, \infty)).$$

It follows from (2.1) that $Y(x) = x^{\alpha/2+1/4}\, e^{-x/2}\, y(x)\ (x \in (0, \infty))$

$= x^{-1/4}\, U(x)$ (see 2.4).

Hence

$$y(x) = x^{-(\alpha+1)/2}e^{x/2}\; U(x).$$

Put $Z_\alpha(x, \lambda) := y(x)$; then from (2.6)

$$Z_\alpha(x, \lambda) = 1 + \sum_{k=1}^{\infty} \frac{((\alpha+1)/2 - \lambda)_k}{(\alpha+1)_k} \cdot \frac{x^k}{k!}, (\alpha > -1, x \in (0, \infty)). \quad \ldots(2.7)$$

On substituting this into (1.6) and then comparing the coefficients of x^k $(k \in N_0)$, $Z_\alpha(., \lambda)$ is shown to be a solution of Laguerre's differential equation (1.6).

Note that the other solution of (2.5) turns out to be

$$W_\alpha(x, \lambda) = x^{(\alpha+1)/2} e^{x/2} Y_\alpha(x, \lambda) \quad \ldots(2.8)$$

where $Y_\alpha(., \lambda)$ is given by the integral (2.3)

In terms of confluent hypergeometric functions ${}_1F_1(a; c; z)$ and $U(a; c; z)$, the solutions Y_α and Z_α may be expressed as

$$Y_\alpha(x, \lambda) = U((\alpha+1)/2 - \lambda; \alpha+1; x) \quad \ldots(2.9)$$

where

$$U(a; c; x) = \frac{x^{1-c}}{\Gamma(a)} \int_0^{\infty} z^{a-1} (x+z)^{c-a-1} \exp[-z]\, dz \quad \ldots(2.10)$$

$$(\text{re } a > 0, x > 0)$$

(with $a = (1+\alpha)/2 - \lambda$), $c = \alpha + 1$ and $z = x$), and

$$Z_\alpha(x, \lambda) = {}_1F_1((\alpha+1)/2 - \lambda; \alpha+1; x) \quad \ldots(2.11)$$

where

$${}_1F_1(a; c; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!}, (c \neq -n, n \in N_0). \quad \ldots(2.12)$$

The asymptotic forms of the solutions Y_α and Z_α in the neighbourhood of the singular end-points 0 and $+\infty$ can be calculated on using (2.9) and (2.12), and the asymptotic forms of the confluent hypergeometric functions $U(a; c; x)$ and ${}_1F_1(a; c; x)$ as $x \to 0$ and $x \to +\infty$; for details see Onyango-Otieno[10], (section 7.2).

In the case of Y_α and Z_α we find that as $x \to 0$

$$Z_\alpha(x, \lambda) = 1 + \sum_{k=1}^{N} \frac{((\alpha+1)/2 - \lambda)_k}{(\alpha+1)_k} \cdot \frac{x^k}{k!} + O(|x|^{N+1}) \ (\alpha > -1) \quad \ldots(2.13)$$

$$Z'_\alpha\ (x, \lambda) = \frac{(\alpha + 1)/2 - \lambda}{\alpha + 1} \left\{1 + \sum_{k=1}^{N} \frac{((\alpha + 3)/2 - \lambda)_k}{(\alpha + 2)_k} \cdot \frac{x^k}{k!}\right\}$$

$$+ O(|x|^{N+1}) \qquad \ldots(2.14)$$

$$\left.\begin{aligned} Y_\alpha (x, \lambda) &= \frac{\Gamma(-\alpha)}{\Gamma((1-\alpha)/2 - \lambda)} + O(|x|^{-\alpha}), (\text{if } -1 < \alpha < 0) \\ &= \frac{1}{\Gamma(\frac{1}{2} - \lambda)}\left[\log x + \frac{\Gamma'(\frac{1}{2} - \lambda)}{\Gamma(\frac{1}{2} - \lambda)} - 2\gamma\right] + O(|x \log x|) \\ &\quad (\text{if } \alpha = 0 \text{ and } \lambda \neq \tfrac{1}{2} + n, n \in N_0, \gamma = \text{Euler's constant}) \\ &= \frac{\Gamma(\alpha)}{\Gamma((\alpha+1)/2 - \lambda)} x^{-\alpha} + O(1), (\text{if } 0 < \alpha < 1) \\ &= \frac{\Gamma(1)}{\Gamma(1 - \lambda)} x^{-1} + O(|\log x|), (\text{if } \alpha = 1) \\ &= \frac{\Gamma(\alpha)}{\Gamma((\alpha+1)/2 - \lambda)} x^{-\alpha} + O(|x|^{\alpha - 1}), \text{if } \alpha > 1) \end{aligned}\right\} \qquad \ldots(2.15)$$

$$\left.\begin{aligned} Y'_\alpha\ (x, \lambda) &= \frac{-((\alpha+1)/2 - \lambda)\,\Gamma(\alpha+1)}{\Gamma((\alpha+3)/2 - \lambda)} x^{-\alpha-1} + O(1), \\ &\qquad (\text{if } -1 < \alpha < 0) \\ &= \frac{-((\alpha+1)/2 - \lambda)\,\Gamma(\alpha+1)}{\Gamma((\alpha+3)/2 - \lambda)} x^{-\alpha-1} + O(|\log x|) \\ &\qquad (\text{if } \alpha = 0) \\ &= \frac{-((\alpha+1)/2 - \lambda)\,\Gamma(\alpha+1)}{\Gamma((\alpha+3)/2 - \lambda)} x^{-\alpha-1} + O(|x|^{\alpha}) \\ &\qquad (\text{if } \alpha > 0). \end{aligned}\right\} \qquad \ldots(2.16)$$

Also as $x \to +\infty$, (with $\alpha > -1$ and $\lambda \in \mathbb{C}$)

$$Z_\alpha (x, \lambda) = \frac{\Gamma(\alpha + 1)}{\Gamma((\alpha + 1)/2 - \lambda)} x^{-(\lambda + (\alpha+1)/2)} e^x [1 + O(|x|^{-1})] \qquad \ldots(2.17)$$

$$Z'_\alpha\ (x, \lambda) = \frac{((\alpha + 1)/2 - \lambda)}{(\alpha + 1)} \frac{\Gamma(\alpha + 2)}{\Gamma((\alpha + 3)/2 - \lambda)} x^{-(\lambda + (\alpha+1)/2)} e^x$$

$$[1 + O(|x|^{-1})] \qquad \ldots(2.18)$$

$$Y_\alpha (x, \lambda) = \sum_{k=0}^{N} \frac{(-1)^k ((\alpha + 1)/2 - \lambda)_k ((1 - \alpha)/2 - \lambda)_k}{k!} x^{-k+\lambda-(\alpha+1)/2}$$

$$+ O(|x|^{-N+\lambda-(\alpha+3)/2}) \qquad \ldots(2.19)$$

and

$$Y'_\alpha\ (x,\lambda) = -\left(\frac{\alpha+1}{2}-\lambda\right)\sum_{k=0}^{N}\frac{(-1)^k((\alpha+3)/2-\lambda)_k\,((1-\alpha)/2-\lambda)_k}{k!}$$

$$x^{-k+\lambda-(\alpha+3)/2} + O(\,|\,x\,|^{\,-N+\lambda-(\alpha+5)/2}). \qquad \ldots(2.20)$$

On the basis of these asymptotic results, the limit-point, limit-circle classification in Theorem 1.1 of Laguerre's differential equation in $L^2_w\ (0,\infty)$ and $H^2_{p,q}\ (0,\infty)$ can be verified, and is seen to hold for all $\lambda \in \mathbb{C}$. Note that the linear independence of Y_α and Z_α is a consequence of the linear independence of U and ${}_1F_1$ (Erdelyi *et al.*[8]).

3. The Right Definite Case

This section is partly adapted from Titchmarsh's method[18] (section 4.16).

Again consider the Laguerre differential equation in the form

$$-(x^{\alpha+1}e^{-x}\ y'\ (x))' + \frac{\alpha+1}{2}\,x^\alpha e^{-x}y\,(x) = \lambda\,x^\alpha e^{-x}y\,(x)$$

$$(x \in (0,\infty),\ \lambda \in \mathbb{C},\ \alpha > -1)$$

since the solutions Y_α and Z_α of (1.6) are linearly independent, it follows from the asymptotic results (2.13)—(2.16) that as $x \to 0$, the Wronskian in the form

$$[Z_\alpha\,(.,\lambda), Y_\alpha\,(.,\lambda)]\,(x) = p\,(x)\,(Z_\alpha\,(.,\lambda)\,\bar{Y}'_\alpha\ (.,\lambda) - Z'_\alpha\ (.,\lambda)$$

$$\bar{Y}_\alpha\,(.,\lambda))(x)$$

$$\sim p\,(x)\Big[-\lambda\,((\alpha+1)/2-\lambda)\ \frac{\Gamma\,(\alpha+1)}{\Gamma\,((\alpha+3)/2-\lambda)}\ x^{-\alpha-1}$$

$$-((\alpha+1)/2-\lambda)\ \frac{\Gamma\,(\alpha)}{\Gamma\,((\alpha+1)/2-\lambda)}\ x^{-\alpha}\Big]$$

$$= -e^{-x}\frac{\Gamma\,(\alpha+1)}{\Gamma\,((\alpha+1)/2-\lambda}\Big[1+\frac{((\alpha+1)/2-\lambda)}{\alpha}\ x\Big]$$

(provided $\alpha > -1$ and $\alpha \neq 0$). Thus as $x \to 0$

$$[Z_\alpha\,(.,\lambda),\ Y_\alpha\,(.,\lambda)]\,(x) \sim -\frac{\Gamma\,(\alpha+1)}{\Gamma\,((\alpha+1)/2-\lambda)}.$$

Since the Wronskian is not dependent on x, the left-hand side is a constant and therefore equals the right-hand side. Let

$$\omega\,(\lambda) := \frac{\Gamma\,(\alpha+1)}{\Gamma\,((\alpha+1)/2-\lambda)},\ (\lambda \neq (\alpha+1)/2+n,\ n \in N_0) \qquad \ldots(3.1)$$

$$\psi_0\,(x,\lambda) := Y_\alpha\,(x,\lambda),\ \ \phi_0\,(x,\lambda) := Z_\alpha\,(x,\lambda).$$

The Green's function in this case is given by

$$G(x, t; \lambda) = \begin{cases} -\dfrac{\psi_0(x, \lambda)\, \phi_0(t, \lambda)}{\omega(\lambda)}, & 0 < t < x < \infty \\ -\dfrac{\phi_0(x, \lambda)\, \psi_0(t, \lambda)}{\omega(\lambda)}, & 0 < x < t < \infty \end{cases}$$

and from (3.1) the eigenvalues are given by

$$\lambda_n = n + (\alpha + 1)/2 \; (n \in N_0, \alpha > -1). \qquad \text{...(3.2)}$$

Anticipating the definition of the operator T_α below, let

$$P\sigma(T_\alpha) = \{\lambda_n = n + \alpha + 1/2, \;\; n \in N_0, \;\; \alpha > -1). \qquad \text{...(3.3)}$$

By a similar analysis as in Titchmarsh[18], (section 4.16) we can show that the eigenfunctions corresponding to eigenvalues $P\sigma(T_\alpha)$ are given by

$$\phi_n(x) = \left(\frac{n!}{\Gamma(n + \alpha + 1)}\right)^{1/2} . \; L_n^{(\alpha)}(x), \; (n \in N_0, \;\; x \in (0, \infty)) \qquad \text{...(3.4)}$$

where $\{L_n^{(\alpha)}(\,.\,), \; n \in N_0\}$ are the Laguerre polynomials.

We now leave the classical study of the Laguerre differential equation (1.6) and consider the associated differential operator in this the right definite case.

Consider the differential expression

$$M[f](x) := -\;(x^{\alpha+1}e^{-x} f'(x))' + (\alpha + 1)/2\; x^{\alpha}e^{-x} f(x), \; (x \in (0, \infty),$$
$$\alpha > -1)$$

for any $f: (0, \infty) \to ¢$, with f and $f' \in AC_{loc}(0, \infty)$; and recall from Theorem (1.1) that in the right-definite case, $M[f]$ is

(i) regular at 0 if $-1 < \alpha < 0$

(ii) limit-circle at 0 if $0 \leqslant \alpha < 1$

(iii) limit-point at 0 if $1 \leqslant \alpha < \infty$, and

(iv) limit-point at ∞ for all $\alpha > -1$.

With this in mind, we first consider the boundary conditions for both the regular and limit-circle cases :

At 0, with $-1 < \alpha < 0$ (i. e. regular case) we impose a regular boundary condition of the form, see Titchmarsh[18] (section 1.6)

$$[f, \phi_0](0) = 0, \;\text{ or equivalently, }\; (pf')(0) = 0. \qquad \text{...(3.5)}$$

However, for the limit-circle case, $0 \leqslant \alpha < 1$, at 0 we impose the Titchmarsh-Naimark type of bounday conditions :

$$\lim_{x \to 0+} \cdot [f, \phi_0 (., \lambda)] \; (x) = 0. \qquad ...(3.6)$$

For the limit-point case at 0, (with $1 \leqslant \alpha < \infty$) and at $+\infty$, we require the following :

Lemma 3.1—A necessary and sufficient condition for $M[f]$ to be limit-point at 0 $(1 \leqslant \alpha < \infty)$ and at $+\infty$ $(\alpha > -1)$ is that

$$\lim_{x \to 0+} [f, g] (x) = \lim_{x \to 0+} p (x) [f (x) g' (x) - f' (x) g (x)] = 0$$

$$(f, g \in \nabla_\alpha, \; 1 \leqslant \alpha < \infty) \qquad ...(3.7)$$

and

$$\lim_{x \to +\infty} [f, g] \; (x) = 0, \; (f, g \in \nabla_\alpha, \; \alpha > -1) \qquad ...(3.8)$$

where ∇_α is a linear manifold of L^2_w $(0, \infty)$ defined by

$$\nabla_\alpha : = \{ f : (0, \infty) \to \not{c} : f, p f' \in AC_{loc} (0, \infty)$$

and

$$f, \; w^{-1} M [f] \in L^2_w \; (0, \infty)\}. \qquad ...(3.9)$$

Both Everitt[4] and Naimark[8] (section 17.4) have proved the general form of this result for $w(x) = 1$; however, the extension to general weight functions $w(x)$ presents no additional difficulty[10].

Using (3,5), (3.6) and (3.9), we also define the following linear manifolds of L^2_w $(0, \infty)$, namely

$$D_\alpha : = \{ f \in \nabla_\alpha : \text{(i)} \;\; (p f') (0) = 0 \text{ if } -1 < \alpha < 0$$

and

$$\text{(ii)} \;\; \lim_{x \to 0} \; [f, \phi_0 (., \lambda)] (x) = 0, \text{ if } 0 \leqslant \alpha < 1\} \qquad ...(3.10)$$

and

$$D_\alpha : = \nabla_\alpha \; (1 \leqslant \alpha < \infty). \qquad ...(3.11)$$

Now let T_α be a linear operator defined in L^2_w $(0, \infty)$ by

$$T_\alpha f : = w^{-1} M [f] \; (f \in D (T_\alpha)) \qquad ...(3.12)$$

where $D (T_\alpha) : = D_\alpha$ is the domain of T_α. Then on using the formula

$$\int_\alpha^\beta w \{w^{-1} M [f] \cdot \bar{g} - f . w^{-1} \overline{M [g]}\} = [f, g] \Big|_\alpha^\beta \qquad ...(3.13)$$

for all compact intervals $[\alpha, \beta] \subset (0, \infty)$ and then taking the limit as $\alpha \to 0^+$ and $\beta \to +\infty$ follows from (3.5), (3.6) and Lemma 3.1 (depending on whether $-1 < \alpha < 0$, $0 \leqslant \alpha < 1$ or $1 \leqslant \alpha < \infty$) that

$$(T_\alpha f, g)_w = (f, T_\alpha g)_w \qquad \text{...(3.14)}$$

where $(.,.)_w$ is the inner product in $L^2_w\ (0, \infty)$,

If $C_0^\infty(0,\infty)$ represents all infinitely differentiablelfunctions with compact support in $(0, \infty)$ then clearly $C_0^\infty\ (0, \infty) \subset D\,(T_\alpha)$; hence $D\,(T_\alpha)$ is dense in $L^2_w\,(0, \infty)$, From this result and (3.14) it follows that T_α is symmetric in $L^2_w\ (0, \infty)$.

If we also define

$$\Phi : (0, \infty) \times (\mathbb{C} - R) \times L^2_w\ (0, \infty) \longrightarrow \mathbb{C}$$

by[18] (section 2. 6)

$$\Phi\,(x, \lambda; f) := \int_0^\infty w\,(t)\, G\,(x, t\,;\lambda)\, f\,(t)\, dt. \qquad \text{...(3.15)}$$

Then

$$\Phi\,(., \lambda; f) \in D\,(T_\alpha)$$
$$M\,[\Phi] = \lambda\, w\, \Phi + w\, f \text{ on } (0, \infty). \qquad \text{...(3.16)}$$

From this result it may be shown that

$$(T_\alpha + iI)\, D\ (T_\alpha) = L^2_w\ (0, \infty).$$

Thus T_α is a self-adjoint (unbounded) differential operator in the space $L^2_w\ (0, \infty)$. We summarise these properties as follows :

Theorem 3.2—(i) $D\,(T_\alpha)$ is dense in $L^2_w\ (0, \infty)$;

(ii) T_α is a symmetric operator in $L^2_w(0,\infty)$;

(iii) T_α is a self-adjoint (unbounded) differential operator in $L^2_w\ (0, \infty)$,

As for the spectrum of the operator T_α; this consists of the set

$$P\,\sigma\,(T_\alpha) = \{\lambda_n = n + (\alpha + 1)/2,\, n \in N_0,\, \alpha > -1\}$$

see (3.3), each point of which is a simple eigenvalue with the corresponding eigenvectors, the Laguerre polynomials $\{L_n^{(\alpha)}\,(\,.\,); n \in N_0\}$; clearly $L_n^{(\alpha)}\,(\,.\,) \in D\ (T_\alpha), (n \in N_0)$.

For any $\mu \notin P\sigma(T_\alpha)$, we can show as in the proof of part (iii) of Theorem 3.2, that $(T_\alpha - \mu I)\, D(T_\alpha) = L^2_w\,(0, \infty)$, i. e. μ is in the resolvent set of T_α [see Akhiezer and Glazman[1], Section 43]. Hence spectrum of T_α is discrete.

The general spectral theory of self-adjoint operators[1] (chapter V1), now yields the completeness of the set of Laguerre polynomials in $L^2_w\,(0, \infty)$ as the set of eigen-vectors of a self-adjoint operator T_α in $L^2_w\,(0, \infty)$ with a simple, discrete spectrum. The normalized eigenvectors $\{\phi_n(\,.\,), n \in N_0\}$ of T_α, where

$$\phi_n(\,.\,) = \sqrt{\frac{n!}{\Gamma(n + \alpha + 1)}} \cdot L_n^{(\alpha)}(\,.\,), (n \in N_0),$$

then form an orthonormal basis in $B^2_w\,(0, \infty)$.

4. The Left Definite Case

Again we consider the Laguerre differential equation (1.6), namely

$$M[y](x) = -(x^{\alpha+1} e^{-x} y'(x))' + \alpha + 1/2\, x^\alpha e^{-x} y(x) = \lambda x^\alpha e^{-x} y(x) \quad ...(1.6)$$

$$(x \in (0, \infty), \lambda \in ¢, \alpha > -1.$$

As in section 1 above, we define $H^2_{p,q}\,(0, \infty) = H^2_{p,q}$ as the Hilbert function space

$$H^2_{p,q}\,(0, \infty) = \{ f : (0, \infty) \to ¢ : f \in AC_{loc}(0, \infty), q^{1/2} f$$

and

$$p^{1/2} f' \in L^2(0, \infty)\} \quad ...(1.4)$$

with inner-product

$$(f, g)_H = \int_0^\infty \{p\, f'\, \bar{g}' + q\, f\, \bar{g}\} \quad ...(1.5)$$

and norm $\| f \|_H$; where

$$p(x) = x^{\alpha+1} e^{-x},\; q(x) = (\alpha + 1)/2\, w(x),\; w(x) = x^\alpha e^{-x}$$

$$(x \in 0, \infty), \alpha > -1).$$

We note here that by replacing the Hilbert space $L^2_w\,(0, \infty)$ with $H^2_{p,q}\,(0, \infty)$, and with a similar analysis to that of the right-definite case we obtain the same results in in (3.2) and (3.4). For more details see Onyango-Otieno[10,11] (section 7.3 and section 4 respectively).

As in section 3 above, we construct the resolvent function $\widetilde{\Phi}$; in fact we can identify $\widetilde{\Phi}$ with Φ of (3.15)

$$\widetilde{\Phi}\,(x, \lambda; f) = \Phi\,(x, \lambda; f) \qquad \ldots(4.1)$$

but now defined for $x \in (0, \infty)$, $\lambda \in \not{c} - \left\{n + \frac{\alpha + 1}{2}; n \in N_0, \alpha > -1\right\}$ and all $f \in H^2_{p,q}\ (0, \infty)$. It is convenient to define

$$\Psi : (0, \infty) \times H^2_{p,q}\ (0, \infty) \to \not{c} \text{ by}$$

$$\Psi\,(x; f) : = \widetilde{\Phi}\,(x, 0; f). \qquad \ldots(4.2)$$

Then following from (3.16)

$$M\,[\Psi\,(x, f)] = w\,(x) f(x), \ (x \in (0, \infty)). \qquad \ldots(4.3)$$

We consider now a linear operator A_α defined on $H^2_{p,q}\ (0, \infty)$ by

$$(A_\alpha f)\,(x) = \Psi\,(x; f), (x \in (0, \infty)) \qquad \ldots(4.4)$$

for all $f \in H^2_{p,q}\ (0, \infty)$. Our aim is to show that A_α satisfies the following properties.

Theorem 4.1—(i) A_α is a bounded linear operator on $H^2_{p,q}\ (0, \infty)$; (ii) A_α is a symmetric operator; (iii) A_α has an inverse operator A_α^{-1} .

To prove this we require :

Lemma 4.2—Let $f \in H^2_{p,q}\ (0, \infty)$, $w\,(x) = x^\alpha e^{-x}\ (\alpha > -1)$ then

(i) $\Psi\,(\,.\,; f) \in L^2_w\ (0, \infty)$;

(ii) $\lim\limits_{x \to 0} p\,(x)\,\Psi'\,(x; f)\,g\,(x) = 0, \left(g \in H^2_{p,q}\right)$;

(iii) $\lim\limits_{x \to +\infty} p\,(x)\,\Psi'\,(x; f)\,g\,(x) = 0, \ (g \in H^2_{p,q})$

(iv) $\Psi\,(.\,; f) \in H^2_{p,q}\ (0, \infty)$.

PROOF : (i) This follows form the definition (4.1) and (4.2) of Ψ and the asymptotic properties of the solutions Y_α and Z_α of Laguerre's equation; see (2.13), (2.15), (2.17) and (2.19).

(ii) Let $g \in H^2_{p,q}(0, \infty)$ and recall that $p(x) = x^{\alpha+1}e^{-x}$ ($\alpha > -1$);

then

$$g(x) = g(a) - \int_x^a g'(t)\,dt \quad (0 < a < \infty)$$

and

$$|g(x)| \leqslant |g(a)| + |\int_x^a g'(t)\,dt$$

$$\leqslant |g(a)| + \{\int_x^a p^{-1}(t)\,dt \int_x^a p(t)\,|g'(t)|^2\,dt\}^{1/2}$$

$$\leqslant |g(a)| + \{\int_x^a p^{-1}(t)\,dt\}^{1/2}\|g\|_H$$

since $p^{-1}(x) = x^{-\alpha-1}e^x \sim x^{-\alpha-1}$ ($x \to 0$, $\alpha > -1$), it follows that for some positive constant K_0,

$$|\int_x^a t^{-\alpha-1}e^t\,dt| \leqslant K_0 |\int_x^a t^{-\alpha-1}\,dt| = K_\alpha/\alpha\,(x^{-\alpha} - a^{-\alpha}).$$

Hence

$$g(x) = O(|x|^{-\alpha/2}(x \to 0). \qquad \text{...(4.5)}$$

Also

$$\int_a^x t^{-\alpha-1}e^t\,dt = [e^t\,t^{-\alpha-1}]_a^x + (\alpha+1)\int_a^x e^t\,t^{-\alpha-2}dt$$

and

$$\int_a^x e^t\,t^{-\alpha-2}\,dt \sim e^x\,x^{-\alpha-2} \quad (x \to +\infty);$$

hence for some positive constant k_1,

$$|\int_a^x e^t\,t^{-\alpha-1}\,dt| \leqslant e^x|x|^{-\alpha-1} + k_1 e^x|x|^{-\alpha-2}$$

so that

$$g(x) = e^{x/2}\,x^{-(\alpha+1)/2}(1 + O(|x|^{-1/2})), \ (x \to +\infty). \qquad \text{...(4.6)}$$

Also, from (4.1), (4.2) and the asymptotic properties of the solutions of the differential equation (1.6) we obtain

$$\int_a^x w\phi_0\,(.,0)\,f = O(|x|^{(\alpha+1)/2})\ x \to 0),$$

so that

$$\frac{p\psi_0'\ (.\ 0)\ g}{\omega\ (0)} \int_0^x w\phi_0\ (.,\ 0)\ f = 0\ (\ |\ x\ |\ ^{\alpha+1} e^{-x}\ |\ x\ |\ ^{-\alpha-1}\ |\ x\ |\ ^{(\alpha+1)/2})$$

$$= O\ (e^{-x}\ |\ x\ |\ ^{1/2})\ (x \to 0)$$

$$= o\ (1)\ (x \to 0)$$

(where $\omega\ (0) = -\ \Gamma\ (\alpha + 1)/\ \Gamma\ ((\alpha + 1)/2;\ \alpha > -\ 1$ see (3.1)).

Similarly

$$\frac{p\phi_0'\ (.,0)g}{\omega\ (0)} \int_x^\infty w\psi_0\ (.,\ 0)\ f = O\ (\ |\ x\ |\ pc/e^{-3x/2})\ (x \to 0)$$

$$= o\ (1),\quad (x \to 0).$$

Hence

$p\ (x)\ \Psi'\ \ (x; f\ )\ g\ (x) = o\ (1)\ \ (x \to 0)$ which proves part (ii).

(iii) Also for this part, the asymptotic properties of the solutions of Laguerre's differential equation give

$$\int_0^x w\phi_0\ (.,\ 0)\ \ f = O\ (e^{x/2}\ |\ x\ |\ ^{-1/2}),\ (x \to +\ \infty)$$

and

$$\int_x^\infty w\ \psi_0\ (.,\ 0)\ \ f = O\ (e^{-x/2}\ |\ x\ |\ ^{-1/2}),\ \ x \to +\ \infty)$$

hence

$$\frac{p\psi_0'\ \ (.,\ 0)\ g}{\omega\ (0)} \int_0^x w\phi_0\ (.,\ 0)\ f = O\ (\ |\ x\ |\ ^{\alpha+1}\ e^{-x}\ |\ x\ |\ ^{-(\alpha+3)/2)}\ |\ x\ |\ ^{-(\alpha+1)/2}$$

$$e^{x/2}\ |\ x\ |\ ^{-1/2}\ e^{x/2})$$

$$= O\ (\ |\ x\ |\ ^{-3/2}),\ (x \to \infty)$$

$$= o\ (1),\ (x \to \infty).$$

Similarly,

$$\frac{p\phi_0'\ \ (.,\ 0)\ g}{\omega\ (0)} \int_x^\infty w\psi_0\ (.,\ 0)\ f = O\ (\ |\ x\ |\ ^{-1/2})\ (x \to \infty)$$

$$= o\ (1),\ (x \to \infty).$$

Hence

$p(x)\,\Psi'(x;f)\,g(x) = o(1)\ (x \to \infty)$ and this proves part (iii).

(iv) From (4.3) we have

$$\int_s^t w f \overline{\Psi}(.;f) = \int_s^t M[\Psi(.;f)]\,\overline{\Psi}(.;f),\ f\,(\in H^2_{p,q} \text{ and } 0<s<t<\infty)$$

$$= \int_s^t \{-(p\,\Psi'(.;f))' + q\,\Psi(.,f)\}\,\overline{\Psi(.;f)}$$

$$= [-p\Psi'(.;f)\,\overline{\Psi(.;f)}\,]_s^t + \int_s^t \{p\,|\,\Psi'(.;f)\,|^2 + q\,|\,\Psi(.;f)|^2\}.$$

Let $s \to 0+$ and $t \to +\infty$, then from parts (ii) and (iii) above we obtain

$$\int_0^\infty w f\overline{(\Psi\,.;f)} = \int_0^\infty \{p\,|\,\Psi'(.;f)\,|^2 + |\,\Psi(.;f)\,|^2\}$$

$$= \|\,\Psi(.\,;f)\,\|^2_H\,.$$

The Cauchy-Schwarz inequality applied to the left-hand side gives

$$\|\Psi(.;f)\|^2_H = |\int_0^\infty w f\,\overline{\Psi(.;f)}\,|$$

$$\leqslant \{\int_0^\infty w\,|f|^2 \int_0^\infty w\,|\,\Psi(.;f)\,|^2)^{1/2} \qquad \ldots(4.7)$$

$$< \infty$$

(because $f \in H^2_{p,q}(0,\infty) \subset L^2_w(0,\infty)$, and $\Psi(.;f) \in L^2_w(0,\infty)$ see part (i) above); hence $\Psi(.;f) \in H^2_{p,q}(0,\infty)$ and this completes the proof of the lemma.

Proof of Theorem 4.1 : (i) From (4.7) and

$$\|\,\Psi\,(.;f)\,\|^2_H \leqslant \{\int_0^\infty w\,|f|^2 \int_0^\infty w\,|\,\Psi(.;f)\,|^2\}^{1/2}$$

$$\leqslant \frac{2}{\alpha+1}\,\{\int_0^\infty [p\,|f'|^2 + q\,|f|^2]\int_0^\infty [p\,|\,\Psi'(.;f)\,|^2$$

$$+q\,|\,\Psi(.;f)\,|^2]\}^{1/2}\ (\alpha > -1)$$

[recall that $p(x) = x^{\alpha+1}e^{-x}$, $q(x) = (\alpha+1)/2\,w(x)$, $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, $x \in (0,\infty)$]

i. e.

$$\|\Psi(.;f)\|_H^2 \leqslant \frac{2}{\alpha+1}\|f\|_H\|\Psi(.;f)\|_H$$

or [see (4.4)]

$$\|A_\alpha f\|_H = \|\Psi(.;f)\|_H \leqslant 2/(\alpha+1)\|f\|_H$$

i. e. A_α is bounded in $H^3_{p,q}(0,\infty)$.

(ii) Let $f, g \in H_{p,q}(0,\infty)$;

then

$$A_\alpha(f,g)_H = (\Psi(.;f), g)_H$$

$$= \lim_{\substack{s\to 0\\ t\to\infty}} \int_s^t \{p\,\Psi'(.;f)\,\bar{g}' + q\,\Psi(.;f)\,\bar{g}\}$$

$$(0 < s < t < \infty)$$

$$= \int_0^\infty \{-(p\,\Psi'(.;f))' + q\,\Psi(.;f)\}\,\bar{g},$$

(on using Lemma 4.2)

$$= \int_0^\infty M[\Psi(.;f)]\bar{g}$$

$$= \int_0^\infty w f \bar{g} \quad \text{[see (4.3)]}$$

$$= \int_0^\infty f.\,\overline{M[\Psi(.;g)]}$$

$$= (f, A_\alpha g)_H$$

(on reversing the argument); thus A_α is symmetric. Since A_α is bounded it follows that A_α is self-adjoint.

(iii) Let $A_\alpha f = 0$, $(f \in H^2_{p,q}(0,\infty))$

i. e. $\Psi(x:f) = 0$ $(x \in (0,\infty))$; then from (4.3)

$$0 = M[\Psi(.;f)] = wf \text{ on } (0,\infty)$$

i. e. $f = 0$ in $H^2_{p,q}(0,\infty)$. Thus A_α has an inverse operator A_α^{-1} and this completes the proof.

Now define an operator $S_\alpha : D(S_\alpha) \subset H^2_{p,q} \to H^2_{p,q}$ by

$$\left.\begin{aligned} D(S_\alpha) &:= \{A_\alpha f : f \in H^2_{p,q}(0,\infty)\}, \\ \text{and} \quad S_\alpha f &:= A_\alpha^{-1} f \,(f \in D(S_\alpha)). \end{aligned}\right\} \qquad \ldots(4.8)$$

A standard result in Hilbert space theory[1] (section 4.1, corollary to Theorem 1), implies that S_α is self-adjoint (bounded or unbounded); indeed S_α must be unbounded since, from tha properties of Ψ, it follows that if $f \in D(S_\alpha)$ then $f' \in AC_{loc}(0,\infty)$, and so $D(S_\alpha)$ is strictly contained in $H^2_{p,q}(0,\infty)$, even though $D(S_\alpha)$ is dense in $H^2_{p,q}(0,\infty)$; it can be shown that if S_α is bounded then $D(S_\alpha) = H^2_{p,q}(0,\infty)$, a contradiction. Thus S_α is an unbounded self-adjoint operator. Also an analysis similar to that in Everitt[6] (section 4) and Onyango-Otieno[11] (section 4) now shows that S_α has a simple discrete spectrum

$$\sigma(S_\alpha) = P\sigma(S_\alpha) = \{n + (\alpha + 1)/2;\ n \in N_0,\ \alpha > -1\}$$

identical with $P\sigma(T_\alpha)$ of the operator T_α introduced in (3.3), and the corresponding eigenvectors are the Laguerre polynomials $\{L_n^{(\alpha)}(\,.\,),\ n \in N_0,\ \alpha > -1\}$. The spectral theory for self-adjoint operators in a Hilbert space, see [Akhiezer and Glazman[1], chapter VI], now implies that the Laguerre polynomials form a complete orthogonal set in $H^2_{p,q}(0,\infty)$, and hence in $L^2_w(0,\infty)$ because the set $H^2_{p,q}(0,\infty)$ is dense in $L^2_w(0,\infty)$.

5. Remark on Operators T_α and S_α

(i) It is of interst to note that we could have defined the operator T_α in the same way as the operator S_α in section 4. For with the resolvent function defined by (3.15). let the operator B_α be defined on $L^2_w(0,\infty))$

$$(B_\alpha f)(x) := \Phi(x, 0, f),\ (x \in (0,\infty),\ f \in L^2_w(0,\infty)).$$

Then with a similar analysis to that in section 4 we can prove that B_α is a bounded symmetric operator on $L^2_w(0,\infty)$ into $L^2_w(0,\infty)$, that the inverse B_α^{-1} exists and the operator T_α as defined in section 3 satisfies

$$T_\alpha = B_\alpha^{-1}.$$

This gives therefore an alternative way of determining the operator T_α in the right-definite case.

(ii) It is also of some interest to compare the operators T_α and S_α as defined in sections 3 and 4 respectively.

T_α is an unbounded self-adjoint differential operator in $L^2_w(0, \infty)$ with a simple discrete spectrum $(n + (\alpha + 1)/2, n \in N_0, \alpha > -1)$ and corresponding eigenvectors $\{L_n^{(\alpha)}(\,.\,), n \in N_0\}$.

S_α is an unbounded, self-adjoint operator in $H^2_{p,q}$ $(0, \infty)$ with the same simple, discrete spectrum and corresponding eigenvectors; however, we hesitate to call S_α a dffferential operator for the reasons given below.

T_α and its domain $D(T_\alpha)$ is defined directly in terms of the differential expression $M[y] = -(p y')' + qy$.

For S_α, the situation is different; we defined S_α as the inverse A_α^{-1} of a bounded symmetric operator in $H^2_{p,q}(0, \infty)$. While we can say something about the elements of $D(S_\alpha)$, it does not seem possible to characterise the opetator S_α directly in terms of $M[y]$. The definition of S_α as $S_\alpha := A_\alpha^{-1}$ depends upon a general theorem in Hilbert space theory[1] (section 41) which provides for the existencc of S_α, but does not give a constructive definition in general. Thus S_α appears as a differential operator only in an indirect sense in comparison to T_α.

(iii) We also note that the Laguerre differential equation is but a confluent case of the Jacobi differential equation; and by applying the limiting relations associated with their polynomial solutions[7,10] the foregoing results can be extended to the case of Jacobi's differential equations.

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