A comparison of the use of risk measures and the application of the adjustment coefficient in calculating optimal reinsurance.

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Abstract

Reinsurance is a mechanism by which an insurance company can protect itself against the risk of losses by transferring the risk to other companies. A reinsurance arrangement could be considered optimal if it minimizes the probability of ruin. When an insurer effects reinsurance, they are required to pay a reinsurance premium. Therefore, the total cost to the insurer in the presence of reinsurance is the cost of meeting the retained loss in the event of a claim and paying the reinsurance premium. The purchase of reinsurance is therefore a compromise between expected gain and security. Reinsurance reduces the cedant's risk; on the other hand, it will reduce the expected gain of the cedant. Claims experience is assumed to follow a particular loss distribution. i.e. the exponential, pareto, gamma, lognormal, Weibull and burr distributions. This paper determines optimal reinsurance by use of risk measures such as VaR and CTE. The results are compared with the use of the adjustment coefficient in determining the optimal reinsurance strategy. Claims experience data is simulated through the Monte Carlo simulation techniques

Introduction

Research on reinsurance has predominantly centred upon the contract design problem between the insurer and reinsurer. (Arrow, 1963) considers the variance minimization and expected utility maximization of an insurer. (Jun Cai, 2008) studies optimal reinsurance contracts using a risk-measure-theoretical framework. (Centeno, 2002) measures the effect of reinsurance by the adjustment coefficient in the Sparre Andersen model.

In a reinsurance arrangement, the insurer transfers part of its loss to a reinsurer. Therefore, the insurer retains a loss. In return, the insurer is obligated to compensate the reinsurer for undertaking the risk by paying the reinsurance premium. Hence the sum of the retained loss and the reinsurance premium can be interpreted as the total cost of managing the risk in the presence of reinsurance. In this paper, we use the Value-at-risk (VaR) and conditional tail expectation (CTE) of an insurer's total cost as one criterion for determining the optimal reinsurance.

Researchers have long recognized the importance of measuring the risk of a portfolio of financial assets or securities. One technique advanced in the literature involves the use of value at risk models. Many institutions have adopted value at risk (VAR) to measure portfolio risk today. These models measure the market or price risk of a portfolio of financial assets i.e. the risk that the market value of the portfolio will decline as a result of changes in interest changes, foreign exchange rates, equity prices or commodity prices. The confidence levels specify the probability that losses of a portfolio could decline over a given period of time with a given probability as a result of changes in market prices or rates.

If the given period of time is one day and the given probability is 1% the value at risk measure would be an estimate of the decline in the portfolio value that could occur over the next trading day. One of the objectives of this research is to model the tails of loss severity distributions. This is a relevant area in reinsurance. Value at risk focuses on the tails of the distribution. (Jorion, 2000)

The other criterion involves considering the effect of reinsurance on the adjustment coefficient. Assumptions have to be made in this case such as the use of a compound Poisson process which will give us a value for the Poisson parameter. The value of the insurer's loading factor will also have to be assumed. When aggregate claims are a compound Poisson process, the adjustment coefficient is defined in terms of the Poisson parameter, the moment generating function of individual claim amounts and premium income per unit time.

The optimal retention, if it exists, is a function of the assumed loss distribution and the reinsurer's loading factor. (Jun Cai, 2008). This relates to the use of VaR and CTE risk measures in calculating the optimal reinsurance.

Optimal reinsurance can also be determined by considering the effect of reinsurance on the adjustment coefficient. If a reinsurance arrangement can be found that maximises the value of the adjustment coefficient, the upper bound for the probability of ruin will be minimized.

This research seeks to compare the use of VaR and CTE with the use of the adjustment coefficient in determining the optimal retention for an insurer. Under similar assumptions, do these criteria produce the same results?

Methodology

Data types and sources.

Monte – Carlo Simulation.

Consider when the claims follow Exponential, Pareto, Gamma, Lognormal, Weibull and Burr Distributions.

Generate X_1,X_2,……,X_{200}.

To generate the exponential random variable, we use the inverse transform method. The simulation of a Pareto variate can be conducted via the inverse transform method. The Burr variate is also generated using the inverse transform method. Similarly, Weibull variates can be generated using the inverse transform method. Simulation of the gamma distribution is not as straightforward as for the distributions presented above.

Fix the mean and variance so that they are constant for all the loss distributions.

Plot the loss severity distribution.

✓ Model design.

VaR – optimization and CTE – optimization

(Silvia Dedu, 2010)From an insurer's point of view, prudent risk management is to ensure that the risk measures associated with the insurer's cost after effecting reinsurance are as small as possible.

VaR – optimization.

In this section we analyze the optimal solution to the VaR – optimization. The survival function of the retained loss X_I is given by:

$S_X(x) = \begin{cases} S(x), & 0 < x < d \\ 0, & x > d \end{cases}$

If 0 < α < $S_X(d)$ or equivalently 0 < d < $S^{-1}_X(\alpha)$, then VaR_X(d,α) = d

If α > S_X (d) or equivalently d > $S^{-1}_X(\alpha)$, then VaR_{X_I}(d,α) = $S^{-1}_X(\alpha)$

Hence, the VaR of the retained loss X_I can be represented as:

$VaR_{X_I}(d,\alpha) = \begin{cases} d, & 0 < d < S_X^{-1} \\ S_X^{-1}(\alpha), & d > S_X^{-1}(\alpha) \end{cases}$

It follows that there exists a relationship between the VaR of the total cost and the VaR of the retained risk:

$VaR_T(d,\alpha) = \begin{cases} d + \delta(d), & 0 < d < S_X^{-1}(\alpha) \\ S_X^{-1}(\alpha) + \delta(d), & d > S_X^{-1}(\alpha) \end{cases}$

Before proceeding it is convenient to first define,

$$\rho^* = \frac{1}{1+\rho}$$

Which plays a critical role in the solutions to our optimization problems.

$VaR_T(d^*,\alpha) = \min\{VaR_T(d,\alpha)\}$

The resulting optimal retention d* ensures that the VaR of the total cost is minimized for a given risk tolerance level α.

The optimal retention d* >0 exists if and only if both:

$\alpha > \rho^* < S_X(0)$

$S_X^{-1}(\alpha) > S_X^{-1}(\rho^*) + \delta(S_X^{-1}(\rho^*))$

hold.

When the optimal retention d* exists, then d* is given by

$d^* = S_X^{-1}(\rho^*)$

and the minimum VaR of T is given by;

VaR_T(d*,α) = d*+δ(d*)

CTE – optimization.

The motivation of CTE is to limit the amount of loss instead of its probability of occurrence only.

The CTE of the total cost T can be decomposed as:

$CTE_T(d,\alpha) = E[X_I + \delta(d)|X_I + \delta(d) > VaR_T(d,\alpha)] = CTE_{X_I}(d,\alpha) + \delta(d)$

$CTE_{X_I}(d,\alpha) = E[VaR_{X_I}(d,\alpha) + X_I - VaR_{X_I}(d,\alpha)|X_I > VaR_{X_I}(d,\alpha)]$

$$= VaR_{X_I}(d,\alpha) + \frac{\int_{VaR_{X_I}}^{\infty} S_{X_I}(x)\,dx}{Pr\{X_I > VaR_{X_I}(d,\alpha)\}}$$

It follows that:

$$\int_{VaR_{X_I}(d,\alpha)}^{\infty} S_{X_I}dx = \int_{VaR_{X_I}(d,\alpha)}^{\infty} S_X(x)\,dx$$

$$= \begin{cases} 0, & 0 < d < S_X^{-1}(\alpha) \\ \int_{S_X^{-1}(\alpha)}^{d} S_X(x)dx, & d > S_X^{-1}(\alpha) \end{cases}$$

and

$$\Pr\{X_I > VaR_{X_I}(d,\alpha)\} = \Pr\{X_I = VaR_{X_I}(d,\alpha)\} + S_{X_I}(VaR_{X_I}(d,\alpha))$$

$$= \begin{cases} Pr\{X_I = d\} + S_{X_I}(d), & 0 < d < S_X^{-1}(\alpha) \\ Pr\{X_I = S_X^{-1}(\alpha)\} + S_{X_I}(S_X^{-1}(\alpha)), d > S_X^{-1}(\alpha) \end{cases}$$

$$= \begin{cases} PrX > d, 0 < d < S_X^{-1}(\alpha) \\ S_X(S_X^{-1}(\alpha)) = \alpha, d > S_X^{-1}(\alpha) \end{cases}$$

By combining the above equations, we derive an expression for $CTE_T(d,\alpha)$ as follows:

$$CTE_T(d,\alpha) = \begin{cases} d + \delta(d), & 0 < d < S_X^{-1}(\alpha) \\ S_X^{-1}(\alpha) + \delta(d) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^{d} S_X(x)dx, & d > S_X^{-1}(\alpha) \end{cases}$$

$CTE_T(d',\alpha) = \min\{CTE_T(d,\alpha)\}$

The optimal retention d’ from the above optimization focuses on the right tail risk by minimizing the expected loss of extreme events.

The following theorem states the necessary and sufficient conditions for the existence of the optimal retention to the CTE – optimization.

a. The optimal retention d’ > 0 exists if and only if

$0 < \alpha < \rho^* < S_X(0)$

b. When the optimal retention d’ > 0, then

$d' = S_X^{-1}(\rho^*)$ if $\alpha < \rho^*$

d'>$S_X^{-1}(\rho^*)$ if α = ρ^*

Conclusion.

If the optimal solution exists, both CTE – optimization and VaR – optimization yield the same optimal solution.

Therefore according to this framework (Jun Cai, 2008), the optimal retention,d, is given by:

d = $S_X^{-1}(\rho^*)$

where: d is the optimal retention.

S_X^{-1} is the inverse of the loss distribution function

$\rho^* = \frac{1}{1+\rho}$, where ρ is the reinsurer's loading factor.

The value obtained for d is then indicated on the graph.

The Adjustment Co-efficient.

(Waters, 1979), studied the behavior of the adjustment coefficient as a function of the retention level either for quota – share reinsurance or for excess of loss reinsurance. If the reinsurance premium is calculated according to the expected value principle with loading coefficient α, then the optimal retention is attained at the unique point M satisfying

M = R^{-1}ln(1 + α)

Where R is the adjustment coefficient of the retained risk.

The value of R for a compound Poisson process is calculated as the unique positive root of:

M_X(R) = 1 + (1 + θ)m_1R

Where θ is the loading co – efficient for the insurer's premium.

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