Pricing European Convertible Bonds: Geometric Brownian motion vs. CGMY

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Outline

Introduction

- Convertible bonds are complex financial instruments, which despite their name, have more in common with derivatives than with conventional bonds.
- Over the last few years serious innovation has gone into adding contractual features to these products, which has resulted in more and more fundamental pricing challenges.
- The questions of how to deal with the coupled nature of the convertible bond and how to include credit risk into its value, are some of the problems.
- As the pricing techniques are moving away from geometric Brownian motion, it is important to look at the valuation of convertible bonds under different densities.
- We have chosen to compare the CGMY model prices of convertible bonds to the GBM model prices.
- In today's presentation, we will only look at 2 pricing techniques: the component model and a Monte Carlo method.

What is a convertible bond?

Definition

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- Lower credit companies who may not be able to access the straight debt market can often still issue convertible bonds.

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Advantages for investor:

- A convertible offers greater stability of income than regular stock.
- If the company does well, they can convert to equity and receive the benefits of holding stock.

Terms of a convertible bond I

- **Principal**: The face value of the convertible bond, i.e. the redemption value.
- **Coupon**: The annual interest rate as a percentage of the principal.
- **Coupon frequency**: The number of coupon payments per year.
- **Conversion Ratio**: The number of shares of the underlying stock that the convertible bond can be exchanged into. This ratio is usually determined at issue and is only changed to keep the total equity value constant, eg. when dividends or stock splits occur.
- **Conversion Price**: The price of each underlying share paid on conversion, assuming the bond principal is used to pay for the shares received.

$$\text{Conversion Price} = \frac{\text{Principal}}{\text{Conversion Ratio}} \tag{1}$$

- **Conversion Value**: The conversion value is generally determined on a daily basis as the closing price of the stock multiplied by the conversion ratio.
- **First Conversion Date**: The first date after issue at which the bond can be converted into stock. Sometimes there is a period after issue in which conversion is not allowed.

Terms of a convertible bond II

- **Call Provisions**: A call provision can be seen as a call option sold by the investor to the issuer. It gives the issuer the right to buy the bond back at the call price, which is specified in the call schedule, which gives the call price at each future time. Generally, convertible bonds are call protected for a certain number of years and only become callable after a certain date. This provision reduces the value of the bond.
- **Put provisions**: A put provision can be regarded as a put option sold by the issuer to the investor. It gives the investor the right to sell the bond back at the put price on certain dates prior to maturity. This provides the investor with extra downside protection and therefore increases the value of the bond.

Some further notation:

- $r_{t,T}$ = continuously compounded risk-free interest rate from t to T
- V_t = fair value of the convertible bond
- T = maturity of the convertible bond
- N = face value of the convertible bond
- S_t = price of the underlying equity at time t
- γ_t = conversion ratio at time t
- $\gamma_t S_t$ = conversion value at time t
- c = continuously compounded coupon rate of the bond
- $t_1, t_2, ..., t_n$ = coupon payment dates
- Ω_{call} = call period
- Ω_{conv} = conversion period
- Ω_{put} = put period

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The conversion value or equity value:

- is the equity portion of the convertible bond.
- It is the value of the convertible bond if it were to be converted into stock at the current market prices.
- It is found by multiplying the given stock price by the conversion ratio.

The lower bound of the convertible bond price is the maximum of these two values.

Lower bound

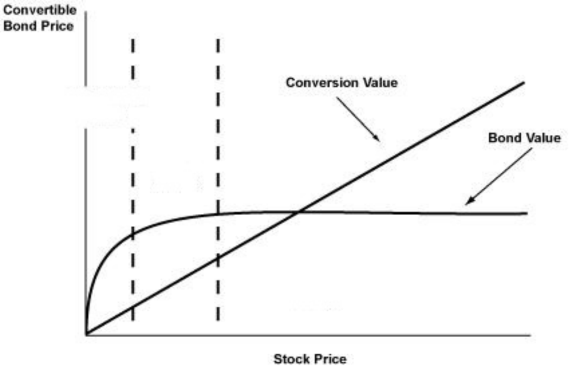

Figure: Lower Bound of the convertible bond price

Lévy-Khintchine formula

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions and let $T \in [0, \infty]$ denote the time horizon.

Theorem

The distribution f_X of a random variable X is infinitely divisible if and only if there exists a triplet (b, c, ν), with $b \in \mathbb{R}$, $c \in \mathbb{R}^+$ and a measure satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$ such that

$$\mathbb{E}[e^{iuX}] = exp\left[ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux - 1 - iux\mathbb{1}_{\{|x|<1\}}})\nu(dx)\right]. \quad (2)$$

- The triplet (b, c, ν) is called the Lévy or characteristic triplet,
- the exponent in (2) is denoted by $\kappa(u)$ and is called the *Lévy* or *characteristic exponent*,
- $b \in \mathbb{R}$ is called the *drift term*,
- $c \in \mathbb{R}^+$ the *diffusion coefficient*, and
- ν the *Lévy measure*.

CGMY process

Definition

The Lévy density of the CGMY process is given by

$$\kappa_{CGMY}(x) = \begin{cases} \frac{Ce^{-M|x|}}{|x|^{1+Y}} & \text{if } x > 0 \\ \frac{Ce^{-G|x|}}{|x|^{1-Y}} & \text{if } x < 0, \end{cases}$$

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Role of parameters:

- C can be seen as the measure of overall level of activity in the process,
- G and M control the rate of exponential decay on the right and left of the Lévy density, leading to skewed distributions when they are unequal, and
- Y is useful in characterising the monotonicity of the process including whether the process has finite or infinite activity and finite or infinite variation.

CGMY stock price process

- $X_{CGMY}(t, C, G, M, Y)$ = the infinitely divisible process of independent increments with the CGMY density.
- The characteristic function of the CGMY process is given by

$$\phi_{CGMY}(u, t, C, G, M, Y) = e^{tC\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]}.$$

- The CGMY stock price process is given by

$$S_t(\omega) = S_0 e^{(\mu+\omega)t + X_{CGMY}(t,C,G,M,Y)},$$

where μ is the mean rate of returns on the stock and ω is a 'convexity correction'.

Pricing options given the characteristic function

- If the density of our stock price process is known, then pricing options is easy as you just need to calculate the expected value.
- However, if only the characteristic function is known, then Carr et al. showed that the price of a European call option $C(T, K)$ with maturity T and strike K is given by

$$C(T,K) = \frac{e^{-\alpha log(K)}}{\pi} \int_0^\infty e^{-ivlog(K)} \rho(v) dv$$

where

$$\rho(v) = \frac{e^{-rT}\phi(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}$$

and α is a positive constant such that the αth moment of the stock price exists (typically a value of $\alpha = 0.75$ will do).
- Using fast Fourier transforms, it is possible to compute within seconds the complete option surface.

Component model I

The convertible bond is divided into

- a straight bond component denoted by B_t and
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Bond: The fair value of the bond is calculated using the standard formulae, i.e. the sum of the present value of its cashflows:

$$B_t = Ne^{-(r_{t,T}+\xi_t)(T-t)} + Nc\sum_{i=1}^{n} e^{-(r_{t_i,T}+\xi_{t_i})(t_i-t)}. \tag{3}$$

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Call: The call K_t can be priced either using the standard Black-Scholes formula or via the fast Fourier transform.

Component model II

Then the value of the convertible bond is just the sum of the two:

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- This model will only work on European convertible bonds.
- This kind of separation relies on restrictive assumptions like the absence of embedded options.
- The strike price is not known in advance, it is in fact a stochastic strike price as the value of the bond depends on the future development of interest rates and the future credit spread.

Monte Carlo method I

- Since convertible bonds are typically American in style, a technique to find the optimal stopping time needs to be added to the usual Monte Carlo method.
- In the case of a vanilla convertible bond, at every conversion time, the investor compares the payoff from immediate conversion to the expected present value of future payoffs from the bond to decide whether he should convert or not.
- Kind & Wilde and Lvov et al. suggested ways to find the optimal stopping times, based on the least squares regression approach suggested by Longstaff & Schwartz.

Optimal exercise decisions

- Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and
- a discrete number of stopping times, $t_0 \leq t_1 \leq ... \leq t_n$ with $t_0 = 0$ and $t_n = T$ and n equal to the number of days until maturity.
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At each timestep there are five possible outcomes:

1. The convertible bond continues to exist without being called or converted.
2. The convertible bond is called by the issuer.
3. The convertible bond is converted by the holder.
4. The convertible bond is called by the issuer and subsequently converted by the holder. This is called forced conversion.
5. The holder exercises the put option.

Monte Carlo method II

- Define Payoff(t_i) to be the cashflow received at time t_i.
- If there is no optimal stopping time before maturity, then Payoff(T) $= \max(\gamma_T S_T, N)$.
- Once an optimal stopping time has been found, all the values in the stock price path after this time are set to zero.
- Let $\tau_i^\star$ be the stopping time found in path i after the backward induction procedure.
- $c(\tau_i^\star)$ is the present value of all coupons received and accrued interest gained in the period $[t_0, \tau_i^\star]$.

The total cashflows received from a convertible bond investment at time $\tau_i^\star$ is given by

$$\text{Total Cashflow} = \text{CF}_{total}(\tau_i^\star) = \text{Payoff}(\tau_i^\star) + c(\tau_i^\star).$$

Monte Carlo method III

Once all the optimal stopping times and corresponding payoffs have been determined, the price of the convertible bond is given by the average discounted payoffs over all simulated paths:

$$V_t = \frac{1}{n}\sum_{i=1}^{n} e^{-r_{t,\tau_i^\star}(\tau_i^\star - t)}\, CF_{total}(\tau_i^\star). \tag{4}$$

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To do all of the above, it is necessary to estimate the continuation value. By no arbitrage arguments, the continuation value needs to be equal to the expected value of the future cashflows under the risk-neutral measure, denoted by $\mathbb{Q}$:

$$F(\omega, t_i) = E^{\mathbb{Q}}\Big[\sum_{j=i+1}^{n} e^{r_{t_i,t_j}(t_j - t_i)}\ CF_{total}(t_j) \mid \mathcal{F}_{t_i}\Big] \tag{5}$$

under the condition that exercise is only possible after t_i.

Simplifications

- We only look at European convertible bonds with no call or put provisions.
- The convertible bond will pay constant coupons at regular times and will be exchangeable into a certain number of shares at the discretion of the investor.
- Interest rates and volatilities are assumed to be constant.
- We work with a conversion ratio instead of a conversion price.

Parameters

The following parameters will be used to illustrate the advantages and challenges of implementing the models:

- $S_0 = 50$
- $\sigma_S = 30\%$
- $d_S = 3\%$
- $r = 8\%$
- $N = 100$
- $\kappa = 1$
- $\gamma = 2$
- $c = 10\%$ NACS

CGMY parameters

We first used from Carr et al.:

- $C = 5$
- $G = 5.6$
- $M = 8.6$
- $Y = 0.5$

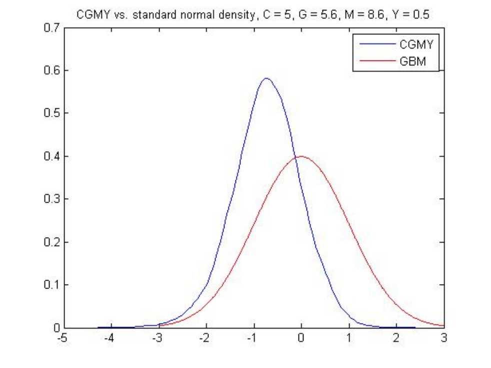


Figure: Comparing the densities.

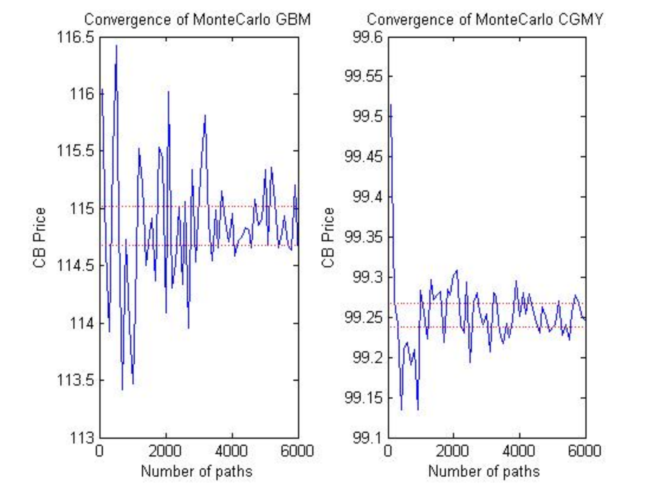


Figure: Conversion rate of Monte Carlo methods.

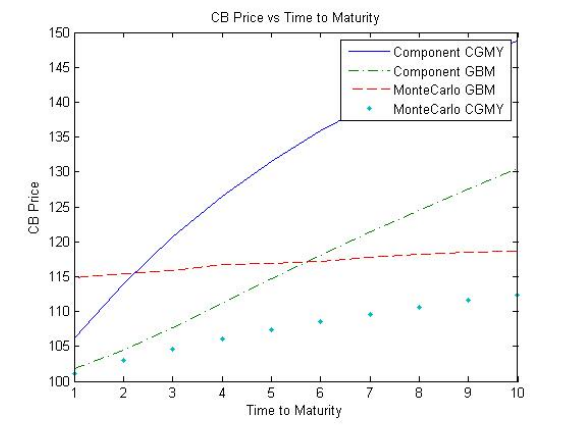
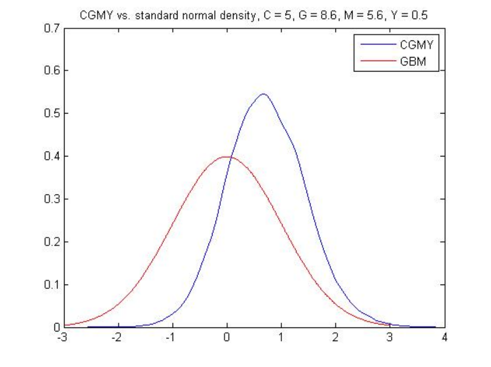


Figure: Component and Monte Carlo pricing techniques with coupons using a negatively skewed CGMY process.

We interchanged the values for *G* and *M*, thereby making it positively skewed.



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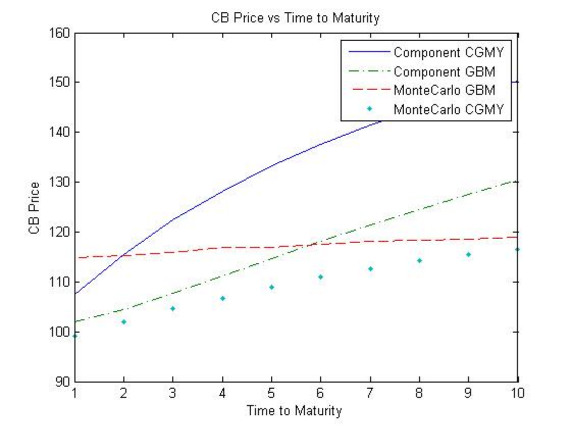

Figure: Component and Monte Carlo pricing techniques with coupons using a positively skewed CGMY process.

Conclusion

- This is only the beginning into the research of using the CGMY model to price convertible bonds.
- We now need to extend these model to price American convertible bonds, which include both call and put provisions.
- Credit risk, which was not considered in this presentation, also needs to be introduced into the pricing.

Thank You

The End.