THE APPLICATION OF ORDINARY DIFFERENTIAL OPERATORS TO THE STUDY OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

by

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A thesis submitted to the University of Dundee in application for the degree of Doctor of Philosophy

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Scotland, U.K.

November 1980

To

my wife Masela, my children Philip and Marie-Therese; and

to

my parents Romulus Otieno-Abonyo and Helena Obadha

PREFACE

In October 1977 I was admitted under Ordinance 12 as a full-time research student in the Department of Mathematics, University of Dundee, under the supervision of Professor W. N. Everitt.

DECLARATION

I declare that this thesis is my own work and that it has not been presented for a higher degree at any other university.

Vitalis Peter Onyango-Otieno

CERTIFICATE

This is to certify that Vitalis Peter Onyango-Otieno has complied with all the requirements for the submission of his Ph.D. thesis to the University of Dundee.

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ACKNOWLEDGEMENTS

I wish to express my sincere thanks to Professor W. N. Everitt, Baxter Professor of Mathematics at the University of Dundee, for his invaluable help, guidance and encouragement during the preparation of this thesis.

I should also like to thank the Association of Commonwealth Universities and the University of Nairobi (Kenya) for their financial support; and also the University of Dundee for the facilities made available to me.

Finally, I thank Miss Margaret Smith for her accurate typing of this thesis.

CONTENTS

INDEX OF SYMBOLS

$\{\cdot,\cdot,\ldots\}$, A, B, ...	notation for sets
$\in$, $\notin$	belongs to, does not belong to
$\{x \in A : p(x)\}$	denotes the set of all points x in the set for which the statement p(x) is true
$\subseteq$, $\subset$, $\not\subset$	notation for a subset, a proper subset and not a subset respectively
R	the set of all real numbers
R_+	the set $\{x \in R : x \geq 0\}$
(a,b)	an open interval on the real line R
[a,b]	a closed interval on the real line R
(a,b], [a,b)	semi-closed or semi-open intervals on the real line R
$[a,+\infty)$, $(a,+\infty)$, $(-\infty,a)$, $(-\infty,a]$	infinite intervals
$\mathbb{C}$	denotes the complex number field
$re[\cdots]$ ($im[\cdots]$)	denotes the real (imaginary) part of $[\cdots]$
max	notation for maximum value
$(x \in A)$	to be read as "for all x in the set A"
(pp $x \in A$)	to be read as "almost all x in the set A"
a.e.	to be read as "almost everywhere"
N_0	denotes the set $\{0,1,2,3,\ldots\}$
N_+	denotes the set $\{1,2,3,\ldots\}$
[r]	denotes largest integer less than or equal to the real number r
I	denotes identity operator
$(a)_n$	denotes $a(a+1)\ldots(a+n-1)$, $n \in N_0$, with $(a)_0 = 1$
n!	denotes $n\cdot(n-1)\cdot(n-2)\ldots 3\cdot 2\cdot 1$, $n \in N_0$, with $0! = 1$

$:=$	denotes "define"
$o(f(x))$	order of magnitude notation; if $\lim_{x \to x_0} g(x)/f(x) = 0$ in some neighbourhood of $x = x_0$, we write $g(x) = o(f(x))$
$O(f(x))$	order of magnitude of $f(x)$; when $x \to x_0$, we write $g(x) = O(f(x))$ if there exists $M \in R_0$, such that in a sufficiently small neighbourhood of $x = x_0$ everywhere $\lvert g(x) \rvert \le M \lvert f(x) \rvert$
$\sim$	denotes "approximately equal to" in formulas without explicit estimation of the error; we write $g(x) \sim f(x)$ as $x \to x_0$ if $\lim_{x \to x_0} g(x)/f(x) = 1$
$L(I)$	the set of all Lebesgue measurable functions $f : I \to R$, $I \subseteq R$, such that $\int_I \lvert f \rvert < \infty$
$L_{loc}(I)$	the set of all Lebesgue measurable functions $f : I \to R$, $I \subseteq R$, such that $\int_J \lvert f \rvert < \infty$ for any compact subinterval J of I
$C(I)$	the set of all complex-valued continuous functions defined on an interval I
$AC(I)$	the set of all complex-valued functions $f : I \to R$ such that f is absolutely continuous on the compact interval I of R; equivalently, if $f \in AC[a,b]$, then there exists $g \in L(a,b)$ and $\gamma \in R$ such that $f(x) = \gamma + \int_a^x g(t)dt$ $(x \in [a,b])$ (with $f'(x) = g(x)$ a.e. in $[a,b]$)
$AC_{loc}(I)$	the set of all complex-valued functions $f : I \to R$, $I \subseteq R$, such that f is absolutely continuous on any compact subinterval of I
$f = f(\cdot)$	denotes the function f
$f(x)$	denotes the function value at point x

$\{f_n$ or $f_n(\cdot)$, $n \in N_0\}$	denotes a sequence of functions
$f : A \to B$	stands for a function (mapping) from a set A to a set B
$'$, $''$	denote d/dx, d^2/dx^2 respectively
f_x	denotes the partial derivative $\partial f/\partial x$
§	to be read as "section"

ABSTRACT

This thesis is essentially concerned with the connections between the classical orthogonal polynomials and the differential operators associated with the differential equations of Gegenbauer, Legendre, Laguerre and Hermite; these differential equations are all of the form

$$M[y] := -(py')' + qy = \lambda wy \text{ on } (a,b) \qquad (A_1)$$

where the coefficients p, q and w are real-valued on an interval (a,b) of the real line, and λ is a complex-valued parameter.

In this thesis, differential operators generated by (A_1) are determined in both the so-called right- and left-definite cases, the analysis being based on the methods of Titchmarsh, Naimark, Pleijel and Everitt. In all cases the analysis is dependent upon representing solutions of the differential equation (A_1) in the form of contour integrals; this representation allows of a determination of the asymptotic form of the solutions of the equations in the neighbourhood of the end-point, whether regular or singular.

In the right-definite case, a differential operator T is determined directly from (A_1) by a definition of the form

$$Tf := w^{-1}M[f] \quad (f \in D(T)) \qquad (A_2)$$

for a suitably defined domain D(T) of T in the Hilbert function space $L_w^2(a,b)$.

However, for the left-definite case, a suitably determined resolvent function Φ is used to define a bounded self-adjoint operator A, whose inverse is the required self-adjoint "differential" operator S in an appropriate Hilbert function space $H_{p,q}^2(a,b)$, associated with the coefficients p and q.

In both cases, the spectra of the operators T and S are shown to e discrete and the corresponding eigenvectors turn out to be the lassical orthogonal polynomials in question. These results provide an lternative proof of the completeness of the classical orthogonal poly- omials in the appropriate space $L_w^2(a,b)$, and also in the space $^2_{p,q}(a,b)$, a property which does not seem to have been considered reviously in the books.

INTRODUCTION AND SUMMARY

INTRODUCTION

The subject of this work concerns the generation and completeness of classical orthogonal polynomials of Gegenbauer, Legendre, Laguerre and Hermite in an integrable-square function space, from the viewpoint of the Titchmarsh-Weyl theory of singular ordinary linear differential equations. This theory is concerned with the differential equation

$$M[y] := -(py')' + qy = \lambda wy \text{ on } (a,b), \qquad (E_1)$$

where the coefficients p, q and w are real-valued on an interval (a,b) of the real line, and λ is a complex-valued parameter.

It is known that the classical orthogonal polynomials are the only orthogonal polynomials which can be generated as solutions of differential equations of the type (E_1). This fact was first conjectured by J. Aczel (1953) and later proved by L. Feldman (1956) and in the stated form by P. Lesky (1962); see the references quoted in this paper of Lesky.

E. C. Titchmarsh, in his book (1962), has considered equation (E_1), but his account is based on the Liouville normal form of (E_1) (see §3.2 below), i.e.

$$-Y'' + QY = \lambda Y \text{ on } (A,B), \qquad (E_2)$$

which is a special case of (E_1), with p = w = 1 on (a,b). However, as is evident from his results in sections 4.2, 4.5 and 4.16, equation (E_2) does not of itself enjoy the property of having polynomial solutions. Throughout this thesis we work with an appropriate form of (E_1) which has the classical orthogonal polynomial directly as a solution.

Our interest throughout this work is to look at the integrable-square solutions of (E_1). In the light of a recent paper of W. N. Everitt (1980), if in (E_1) the coefficient w is non-negative on (a,b), then this is a so-called right-definite problem and is studied in the integrable-square space $L_w^2(a,b)$, where

$$L_w^2(a,b) = \{f : (a,b) \to \mathbb{C} : \int_a^b w(x)|f(x)|^2 dx < \infty\} . \qquad (E_3)$$

If in (E_1) it should happen that, whether w is of one sign or not, both p and q are non-negative, then the problem is called left-definite and is studied in the space $H_{p,q}^2(a,b)$, where

$$H_{p,q}^2(a,b) = \{f : (a,b) \to \mathbb{C} : f \in AC_{loc}(a,b), \\ q^{1/2}f \text{ and } p^{1/2}f' \in L^2(a,b)\} , \qquad (E_4)$$

for which the Dirichlet integral

$$\int_a^b \{p(x)|f'(x)|^2 + q(x)|f(x)|^2\}dx \qquad (E_5)$$

is finite, and is used to determine an appropriate inner-product.

The existence of such integrable-square solutions in the left-definite case (i.e. in $H_{p,q}^2(a,b)$) has been considered in recent years by Å. Pleijel (1969, 1971). Other contributions to the so-called left-definite form include papers by Shotwell (1971), Atkinson-Everitt-Ong (1974) and Everitt (1974, 1980).

With regard to the operator-theoretic aspects of (E_1), it is possible in the right-definite case - following the studies of Naimark (1968) and Glazman (see Akhiezer and Glazman (1963, Appendix II, §9)) - to determine explicitly self-adjoint differential operators T in terms of the differential expression M[y] in (E_1). This, however, depends on

CHAPTER ONE

ORTHOGONAL POLYNOMIALS

§1.0 Preliminary

This chapter contains a review of the properties of the orthogonal polynomials in general, and the classical orthogonal polynomials of Jacobi (including Gegenbauer, Legendre and Tchebishef), Hermite and Laguerre type in particular. We shall show that these polynomials are complete in the Hilbert function space $L^2_\alpha(a,b)$, where the interval (a,b) is, in general, either finite or infinite, and the function $\alpha(\cdot)$ is defined below. The standard textbook for this chapter is by G. Szegö (1975) to which we refer frequently. Other references include the books by Tricomi (1955), Sansone (1959), Alexits (1961), Lebedev (1961), Rainville (1963) and Freud (1955).

§1.1 Systems of orthogonal functions

Definition 1.1.1

Let $\alpha := \alpha(\cdot)$ be a non-decreasing, non-constant function in the bounded (compact) interval $[a,b]$ of the real line R. (If $a = -\infty$ or $b = +\infty$, we require that

$$\left.\begin{aligned} \alpha(-\infty) &= \lim_{x\to-\infty}\alpha(x) \\ &\text{and} \\ \alpha(+\infty) &= \lim_{x\to+\infty}\alpha(x) \end{aligned}\right\} \qquad (1.1.1)$$

are finite.) Then (see Royden (1963; §12.3)) there is a unique Borel measure μ_α defined on all the Borel sets contained in $[a,b]$ such that for all $a_0, b_0 \in [a,b]$, with $a_0 \leq b_0$,

$$\mu_\alpha\{[a_0,b_0]\} = \alpha(b_0+) - \alpha(a_0-) .$$

If f is a non-negative Borel measurable function and α is defined as above, then we denote the Stieltjes-Lebesgue integral of f, with respect to α as

$$\int_a^b f(x)\,d\alpha(x) = \int_{[a,b]} f\,d\mu_\alpha \tag{1.1.2}$$

(see Royden (1963; §12.3)).

Now let $L^2_\alpha(a,b)$ denote the class of complex-valued Borel measurable functions f defined on $[a,b]$ such that

$$\int_a^b |f(x)|^2 d\alpha(x) = \int_{[a,b]} |f(x)|^2 d\mu_\alpha(x) < \infty . \tag{1.1.3}$$

Definition 1.1.2

Using (1.1.3), we define the norm $\|f\|_\alpha$ of f in $L^2_\alpha(a,b)$ as:

$$\|f\|_\alpha = \{\int_a^b |f(x)|^2 d\alpha(x)\}^{1/2} \tag{1.1.4}$$

and the corresponding scalar product of two functions $f,g \in L^2_\alpha(a,b)$ as

$$(f,g)_\alpha = \int_a^b f(x)\,\overline{g(x)}\,d\alpha(x) . \tag{1.1.5}$$

We may then establish the following results:

Lemma 1.1.3

(i) $L^2_\alpha(a,b)$ is a vector space over the complex field $\mathbb{C}$;

(ii) let $d(f,g) := \|f-g\|_\alpha \quad (f,g \in L^2_\alpha(a,b))$ denote the metric distance of f from g then $L^2_\alpha(a,b)$ is a metric space;

(iii) the space $L^2_\alpha(a,b)$ is a separable Hilbert space.

Proof: See Rudin (1976; Chapter 11, §11.35-43).

Remark 1.1.4

It is important to note that if $\alpha \in AC[a,b]$ with $w(x) := \alpha'(x)$ (pp $x \in [a,b]$ in the Lebesgue sense), then (1.1.5) reduces to

$$(f,g)_\alpha = \int_a^b w(x)f(x)\overline{g(x)}dx \ ; \qquad (1.1.6)$$

the function w, called the weight function, is non-negative and measurable in the Lebesgue sense, i.e. $w \in L(a,b)$ and

$$0 < \int_a^b w(x)dx < \infty \ .$$

Definition 1.1.5

(i) Let $a,b \in R$; a system of real functions $\{\psi_n; \ n \in N_0\}$ with $\psi_n \in L^2_\alpha(a,b)$ $(n \in N_0)$ is said to be orthogonal with respect to the measure $d\alpha(\cdot)$ on the compact interval $[a,b]$ if

$$\int_a^b \{\psi_n(x)\}^2 d\alpha(x) > 0 \quad (n \in N_0)$$

and

$$\left.\begin{aligned}&\int_a^b \psi_n(x)\psi_m(x)d\alpha(x) = 0 \quad (n,m \in N_0,\ n \neq m)\\ &\underline{\text{i.e.}}\ (\psi_n,\psi_m)_\alpha = 0\ .\end{aligned}\right\} \quad (1.1.7)$$

(ii) Put

$$\phi_n(x) = \frac{\psi_n(x)}{(\psi_n,\psi_n)_\alpha^{1/2}} \quad (x \in [a,b])$$

then

$$(\phi_n,\phi_n)_\alpha = 1 \quad (n \in N_0) \quad (1.1.8)$$

and the system $\{\phi_n,\ n \in N_0\}$ is said to be normal and orthogonal.

Definition 1.1.6

A system of real functions $\{\phi_n,\ n \in N_0\}$ in $L^2_\alpha(a,b)$ is said to be orthonormal if both (1.1.7) and (1.1.8) are satisfied, i.e. if

$$\int_a^b \phi_n(x)\phi_m(x)d\alpha(x) = \delta_{n,m} \quad (n,m \in N_0) \quad (1.1.9)$$

(where $\delta_{n,m}$ is the Kronecker delta).

One application of (1.1.9) is in proving the following property:

Lemma 1.1.7

The functions of any finite subset of an orthonormal system $\{\phi_n,\ n \in N_0\}$ are linearly independent, i.e.

$$\sum_{i=0}^{n} c_n\phi_n(x) = 0 \quad (x \in [a,b])$$

implies

$$c_i = 0, \ i = 0,1,2,\ldots,n \ .$$

Proof: See, for instance, Szegö (1975; §1.4).

The next result, due to Gram and Schmidt, gives a procedure for obtaining an orthogonal system from a set of linearly independent functions. In the case of complex-valued functions, see Maclane and Birkhoff (1967; p. 409).

Theorem 1.1.8

Let $\{f_i, \ i = 1,2,\ldots,n\}$ be a set of real-valued linearly independent functions in $L^2_\alpha(a,b)$: then a new set of mutually orthogonal functions $\{\psi_i, \ i = 1,2,\ldots,n\}$ can be constructed according to the following Gram-Schmidt process (for all $x \in [a,b]$):

$$\left.\begin{aligned} \psi_1(x) &= f_1(x) \\ \psi_2(x) &= f_2(x) - \frac{(f_2,\psi_1)_\alpha}{(\psi_1,\psi_1)_\alpha}\,\psi_1(x) \\ &\text{------------------------------} \\ \psi_n(x) &= f_n(x) - \sum_{j=1}^{n-1} \frac{(f_n,\psi_j)_\alpha}{(\psi_j,\psi_j)_\alpha}\,\psi_j(x) \ . \end{aligned}\right\} \qquad (1.1.10)$$

Proof: See Akhiezer and Glazman (1963; §8).

Example 1.1.9

An application of the formulas (1.1.10) and (1.1.8) to the following finite set of functions $\{1,x,x^2\}$ in the space $L^2(0,1)$ gives a new set of orthonormal functions, namely

$$\{1, \frac{x - 1/2}{1/2\sqrt{3}}, \frac{x^2 - x + 1/6}{1/6\sqrt{5}}\} .$$

Remark 1.1.10

The above definitions may be extended to the case of unbounded intervals in the sense of the relations (1.1.1); we retain the notation $L^2_\alpha(a,b)$ in this case also.

§1.2 Closure, completeness and Parseval's identity

Let $L^2_\alpha(a,b)$ be given, with the interval bounded or unbounded.

Definition 1.2.1 (Szegö (1975; §1.5))

A system of functions $\{f_n, n \in N_0\}$ in $L^2_\alpha(a,b)$ is said to be closed in $L^2_\alpha(a,b)$ if for $f \in L^2_\alpha(a,b)$ and every $\epsilon > 0$ there exists $\{c_i, i = 0,1,2,\ldots,n\}$ such that if we define

$$s_n(x) := \sum_{i=0}^{n} c_i f_i(x) \qquad (x \in [a,b])$$

then

$$\| f - s_n \|^2_\alpha = \int_a^b |f(x) - s_n(x)|^2 d\alpha(x) < \epsilon ; \qquad (1.2.1)$$

i.e. the linear manifold spanned by $\{f_n, n \in N_0\}$ is dense in $L^2_\alpha(a,b)$. This definition lends itself to the following minimum property:

Theorem 1.2.2

Let $f \in L^2_\alpha(a,b)$ *and let* $\{\phi_n,\ n \in N_0\}$ *be an orthonormal system in* $L^2_\alpha(a,b)$; *if*

$$s_n(x) = \sum_{k=0}^{n} c_k \phi_k(x) \quad (x \in [a,b]) ;$$

then

$$I_n(f) = \int_a^b |f(x) - s_n(x)|^2 d\alpha(x) \tag{1.2.2}$$

takes its minimum value if and only if

$$c_k = (f,\phi_k)_\alpha \ . \tag{1.2.3}$$

Note: s_n is called the best approximation of degree n of f, and I_n is the measure of accuracy of this approximation, both in $L^2_\alpha(a,b)$.

Proof: Let

$$a_k = (f,\phi_k)_\alpha = \int_a^b f(x)\phi_k(x)d\alpha(x),$$

then

$$\int_a^b |f(x)-s_n(x)|^2 d\alpha(x) = \int_a^b |f(x) - \sum_{k=0}^{n} c_k\phi_k(x)|^2 d\alpha(x)$$

$$= \int_a^b \{|f(x)|^2 - 2\sum_{k=0}^{n} c_k f(x)\phi_k(x) + \sum_{j,k=0}^{n} c_j c_k \phi_j(x)\phi_k(x)\} d\alpha(x) \ ;$$

but each of the last two terms on the right hand side is the sum of a finite number of integrable functions (see Titchmarsh (1978; §10.72). Hence, using the orthogonality of $\{\phi_n, n \in N_0\}$,

$$\int_a^b |f(x) - s_n(x)|^2 d\alpha(x) = \int_a^b |f(x)|^2 d\alpha(x) - 2\sum_{k=0}^{n} c_k a_k + \sum_{k=0}^{n} c_k^2$$

$$= \int_a^b |f(x)|^2 d\alpha(x) - \sum_{k=0}^{n} a_k^2 + \sum_{k=0}^{n} (c_k - a_k)^2$$

and the right-hand side is minimal if $c_k = a_k$, i.e. if $c_k = (f,\phi_k)_\alpha$; and conversely. Hence the best approximation of f is the $(n+1)^{th}$ partial sum of the formal "Fourier" series

$$s(x) = \sum_{k=0}^{\infty} (f,\phi_k)_\alpha \phi_\alpha(x) \; ; \qquad (1.2.4)$$

this completes the proof.

Note that we have

$$I_n(f) = \int_a^b |f(x)|^2 d\alpha(x) - \sum_{k=0}^{n} c_k^2 \, ,$$

and since $I_n(f) \geq 0$ and $f \in L_\alpha^2(a,b)$, it follows that $\sum_{k=0}^{\infty} c_k^2$ is convergent, and we obtain Bessel's inequality

$$\sum_{k=0}^{\infty} c_k^2 \leq \int_a^b |f(x)|^2 d\alpha(x) \, . \qquad (1.2.5)$$

It may happen that

$$\lim_{n\to\infty} \int_a^b |f(x) - s_n(x)|^2 d\alpha(x) = 0 \, ,$$

in which case

$$\int_a^b |f(x)|^2 d\alpha(x) = \sum_{n=0}^{\infty} c_n^2 \qquad (1.2.6)$$

and (1.2.6) is called Parseval's equality. Clearly if (1.2.6) holds, then

$$\int_a^b |f(x) - s_n(x)|^2 d\alpha(x)$$

can be made as small as is desired. This leads to the following result:

Corollary 1.2.3

The orthonormal system $\{\phi_n, n \in N_0\}$ *is closed in* $L_\alpha^2(a,b)$ *if and only if Parseval's identity* (1.2.6) *holds for each* $f \in L_\alpha^2(a,b)$.

Another concept which is closely associated to the concept of closure is that of completeness:

Definition 1.2.4

An orthonormal system of functions $\{\phi_n, n \in N_0\}$ *is said to be complete in* $L_\alpha^2(a,b)$ *if for any* $f \in L_\alpha^2(a,b)$ *which satisfies*

$$(f,\phi_n)_\alpha = 0 \quad (n \in N_0)$$

then

$$f(x) = 0 \quad (\mu_\alpha \text{ - almost all } x \in [a,b]).$$

Since $L_\alpha^2(a,b)$ is separable (Lemma 1.1.3), it follows that every orthonormal system of vectors in $L_\alpha^2(a,b)$ consists of a finite or countable number of elements (Akhiezer-Glazman (1963; §9)).

Theorem 1.2.5

An infinite orthonormal sequence $\{\phi_n, n \in N_0\}$ is complete in $L^2_\alpha(a,b)$ if and only if the sequence is closed in $L^2_\alpha(a,b)$.

Proof: See Akhiezer-Glazman (1963; §9). Thus completeness in $L^2_\alpha(a,b)$ implies closure in $L^2_\alpha(a,b)$, which also implies Parseval's identity (Corollary 1.2.3), and conversely.

Now for a compact interval [a,b], the function $f \in L^2_\alpha(a,b)$ may also be approximated in the mean by a continuous function F (Szegö (1975; §1.5)) and, from the theorem of Weierstrass, F can be approximated by a polynomial ρ in the following sense: Let $F \in C[a,b]$, then, given $\epsilon > 0$, there exists a polynomial ρ on R such that

$$|F(x) - \rho(x)| < \frac{\epsilon}{2\sqrt{\mu_\alpha\{[a,b]\}}} \quad (x \in [a,b]).$$

Hence

Theorem 1.2.6

Let the interval [a,b] be compact, and let $f \in L^2_\alpha(a,b)$. Then for $\epsilon > 0$ there exists a polynomial ρ on R such that on [a,b]

$$\int_a^b |f(x) - \rho(x)|^2 d\alpha(x) < \epsilon^2 . \qquad (1.2.7)$$

Note that this theorem is false if the interval (a,b) is unbounded.

§1.3 Orthogonal polynomials

One important class of orthogonal systems consists of orthogonal polynomials $\{p_n = p_n(\cdot),\ n \in N_0\}$. To begin with, we consider the following definition:

Definition 1.3.1 (see Szegö (1975; §2.2))

Let α be defined on $[a,b]$ as before (see §1.1) and let the mapping $x \to x^n \in L^2_\alpha(a,b)$ $(n \in N_0)$, i.e.

$$\int_a^b |x|^{2n} d\alpha(x) < \infty \qquad (n \in N_0) \tag{1.3.1}$$

(where $n = 0$ gives $0 < \int_a^b d\alpha(x) < \infty$); then the "moments" M_n are defined as

$$M_n = \int_a^b x^n d\alpha(x) \ . \tag{1.3.2}$$

Now suppose we orthogonalise the set

$$\{x^n,\ n \in N_0\}$$

in the sense of Theorem 1.1.8 as in Example 1.1.9, then we obtain a set of polynomials

$$\{p_n,\ n \in N_0\} \ , \tag{1.3.3}$$

uniquely defined by the following conditions:

(a) p_n is a polynomial of precise degree n in which the coefficient of x^n $(n \in N_0)$ is positive;

(b) the system (1.3.3), after normalisation (see (1.1.8)), is orthonormal, i.e.

$$\int_a^b p_n(x)p_m(x)d\alpha(x) = \delta_{m,n} \quad (m,n \in N_0). \tag{1.3.4}$$

Now let the interval (a,b) be finite, let the distribution $d\alpha$ on [a,b] be given as above, and let $f \in L^2_\alpha(a,b)$; then, according to Theorem 1.2.6, the function f can be approximated in the mean of $L^2_\alpha(a,b)$, by a polynomial ρ_n of degree n ($n \in N_0$). But every polynomial ρ_n can be represented as a linear combination of the orthogonal polynomials $\{p_i,\ i = 0,1,\ldots,n\}$ (see Szegö (1975; §1.12)). Hence from Theorem 1.2.5 and Definition 1.2.1 we obtain:

Theorem 1.3.2

Let the above conditions hold; then

(a) the set of orthonormal polynomials $\{p_n,\ n \in N_0\}$ is closed in $L^2_\alpha(a,b)$;

(b) the set $\{p_n,\ n \in N_0\}$ is complete in $L^2_\alpha(a,b)$;

(c) if $f \in L^2_\alpha(a,b)$, then Parseval's formula (1.2.6), where the coefficients $\{c_n,\ n \in N_0\}$ are defined by

$$c_n = \int_a^b f(x)p_n(x)d\alpha(x) \quad ,$$

holds;

(d) if $f \in L^2_\alpha(a,b)$, and if the coefficients $\{c_n,\ n \in N_0\}$ satisfy

$$c_n = 0 \quad (n \in N_0)$$

then

$$f(x) = 0 \quad (\text{pp } x \in [a,b]).$$

Remark 1.3.3

(a) The assumption $c_n = 0$ in part (d) above is equivalent to the fact that

$$\int_a^b f(x)x^n d\alpha(x) = 0 \quad (n \in N_0). \tag{1.3.5}$$

The discussion of this condition is closely connected with the uniqueness of Stieltjes' problem of moments (see Polya-Szegö (1972; pp. 83, 173, Problems 138, 139)).

(b) In general, Theorem 1.3.2 does not hold if the interval (a,b) is unbounded (see Example 1.3.4 below). However, there are examples for which the interval (a,b) is unbounded but the results of Theorem 1.3.2 still hold (see Theorems 1.6.3 and 1.7.3).

Example 1.3.4

Let

$$f(x) = \sin(x^\mu \sin \pi\mu)$$

and

$$d\alpha(x) = \exp[-x^\mu \cos \pi\mu]dx \ , \quad 0 < \mu < 1/2;$$

then (see Polya-Szegö (1972), Vol. I, pp. 134, 332, Problem 153)

$$\int_0^\infty f(x)x^n d\alpha(x) = \int_0^\infty \sin(x^\mu \sin\alpha)\exp[-x^\mu \cos\alpha]x^n dx \qquad \text{where } \alpha = \pi\mu$$

$$= \frac{1}{\mu}\,\Gamma\left(\frac{n+1}{\mu}\right) \sin\frac{(n+1)\alpha}{\mu}$$

$$= \frac{1}{\mu}\,\Gamma\left(\frac{n+1}{\mu}\right) \sin(n+1)\pi \;=\; 0 \ .$$

Hence (1.3.5) is satisfied, and yet f is not a zero function. If we now choose any polynomial ρ then

$$\int_0^\infty \{f(x)\}^2 d\alpha(x) = \int_0^\infty f(x)\{f(x) - \rho(x)\} d\alpha(x) + \int_0^\infty f(x)\rho(x) d\alpha(x)$$

and the last integral vanishes on using (1.3.5). Hence

$$\int_0^\infty \{f(x)\}^2 d\alpha(x) = \int_0^\infty f(x)\{f(x) - \rho(x)\} d\alpha(x)$$

$$\leq \int_0^\infty |f(x)||f(x) - \rho(x)| d\alpha(x)$$

$$\leq \int_0^\infty |f(x) - \rho(x)| d\alpha(x) ,$$

i.e.

$$\int_0^\infty \{f(x)\}^2 d\alpha(x) \leq \{\int_0^\infty |f(x) - \rho(x)|^2 d\alpha(x) \int_0^\infty d\alpha(x)\}^{1/2}$$

$$< \infty .$$

Since $f \neq 0$ on $[0,\infty)$, the left-hand side does not vanish, hence the integrals

$$\int_0^\infty |f(x) - \rho(x)| d\alpha(x) \quad \text{and} \quad \int_0^\infty |f(x) - \rho(x)|^2 d\alpha(x)$$

cannot be made arbitrarily small by choice of ρ.

We conclude this section with a few remarks on the zeros of orthogonal polynomials (see Szegö (1975; §3.3 and §6)). Since p_n $(n \in N_0)$

is a polynomial of degree n, it follows from the fundamental theorem of algebra, that p_n has n zeros in $\mathbb{C}$.

Theorem 1.3.5

(a) The polynomial p_n $(n \in N_0)$ has n real simple zeros in the interior of $[a,b]$;

(b) the polynomials p_n and p_{n+1} $(n \in N_0)$ do not vanish simultaneously in $[a,b]$.

Proof: See Szegö (1975; §3.3) and, for part (b), Hochstadt (1971; §1.6).

§1.4 The Classical Orthogonal Polynomials

We assume from now on that $\alpha \in AC_{loc}(a,b)$; and that $w(x) = \alpha'(x)$ (pp $x \in (a,b)$). Note from Remark 1.1.5 that $w \in L_{loc}(a,b)$ and $w(x) > 0$ (pp $x \in (a,b)$); we also assume that all the moments are finite, i.e.

$$\int_a^b |x|^n w(x)dx < \infty \qquad (n \in N_0),$$

provided that $w \in L(a,b)$ since we may take $n = 0$.

Consider now the set $\{p_n,\ n \in N_0\}$ of orthogonal polynomials, and w defined as above on (a,b), finite or infinite. Then (see Szegö (1975; §2.4)) we have the following:

Definition 1.4.1

Let $a = -1$, $b = +1$ and $w(x) = (1-x)^\alpha(1+x)^\beta$ $(x \in (-1,1))$, with $\alpha > -1$, $\beta > -1$. Then the orthogonal polynomials $\{p_n,\ n \in N_0\}$ where

$$p_n := P_n^{(\alpha,\beta)}(\cdot) \quad \text{on} \quad (-1,1) \tag{1.4.1}$$

are called the Jacobi polynomials.

Some special cases of (1.4.1) are:

(a) the Gegenbauer (or Ultraspheric) polynomials $\{C_n^{\nu}(\cdot),\ n \in N_0\}$, if $\alpha = \beta = \nu - 1/2$;

(b) the Legendre polynomials $\{P_n(\cdot),\ n \in N_0\}$ if $\alpha = \beta = 0$; and

(c) the Tchebichef polynomials of the first kind $\{T_n(\cdot),\ n \in N_0\}$ if $\alpha = \beta = -1/2$ and of the second kind $\{U_n(\cdot),\ n \in N_0\}$ if $\alpha = \beta = +1/2$.

Definition 1.4.2

Let $a = 0$, $b = +\infty$ and let $w(x) = e^{-x}x^{\alpha}$ $(x \in (0,\infty))$ with $\alpha > -1$; then the orthogonal polynomials

$$\{L_n^{(\alpha)}(\cdot),\ n \in N_0\} \tag{1.4.2}$$

are called the Laguerre polynomials.

Definition 1.4.3

Let $a = -\infty$, $b = +\infty$ and $w(x) = e^{-x^2}$ $(x \in (-\infty,\infty))$, then the orthogonal polynomials

$$\{H_n(\cdot),\ n \in N_0\} \tag{1.4.3}$$

are called the Hermite polynomials.

The polynomials of Jacobi (including Gegenbauer, Legendre and Tchebichef), Laguerre and Hermite are known as the classical orthogonal polynomials. Note that in all cases $w(x) > 0$ (pp $x \in (a,b)$). They share a number of properties, a few of which we highlight here.

Properties of classical orthogonal polynomials (Erdelyi (1953; §10.6)

Let $\{p_n, n \in N_0\}$ denote any set of classical orthogonal polynomials, then

(a) $\{p_n, n \in N_0\}$ are the only orthogonal polynomials which satisfy the generalized Rodrigues' formula:

$$p_n(x) = \frac{1}{k_n w(x)} \frac{d^n}{dx^n} \{w(x)p(x)\} \quad (x \in (a,b),\ n \in N_0) \qquad (1.4.4)$$

where p is a given polynomial, w is the associated weight function and k_n is a constant;

(b) the polynomials $\{p_n, n \in N_0\}$ are the coefficients of a convergent series

$$G(x,t) = \sum_{n=0}^{\infty} p_n(x)t^n \qquad (1.4.5)$$

where $|t|$ is sufficiently small, and $G(\cdot,t)$ is called the generating function of the polynomials $\{p_n, n \in N_0\}$. Note, however, that this property is not restricted only to classical orthogonal polynomials, for example the Bessel functions $J_n(\cdot)$ of integral order also have a generating function

$$\exp[\tfrac{x}{2}(t-t^{-1})] = \sum_{n=-\infty}^{\infty} J_n(x)t^n \qquad (1.4.6)$$

(see Birkhoff and Rota (1969; §3.4, Ex. B8));

(c) $\{p_n, n \in N_0\}$ are the only orthogonal polynomials for which the derivatives $\{p'_n, n \in N_0\}$ form a system of orthogonal polynomials (Krall (1936)).

In addition to these properties, we also have the following result:

Theorem 1.4.4 (P. Lesky (1962))

The classical orthogonal polynomials are the only polynomials which satisfy a Sturm-Liouville differential equation

$$P(x)y''(x) + Q(x)y'(x) + (T(x)-\lambda)y(x) = 0 \qquad (1.4.6)$$

where the coefficients P, Q and T are fixed over the interval $(a,b) \subseteq R$ and $\lambda \in R$. This result was first conjectured by J. Aczel (1953) and proved by L. Feldman (1956), M. Mikolas (see Freud (1971; Chapter I)) and in the stated form by P. Lesky (1962).

Later on - and this is the main concern of this thesis - we shall show that the classical orthogonal polynomials can also be generated as the eigenfunctions of an associated differential operator.

§1.5 The Jacobi Polynomials

Let $a = -1$, $b = 1$ and let $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $p(x) = (1-x^2)^n$; then from (1.4.4) p_n $(n \in N_0)$ now becomes

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\} \qquad (1.5.1)$$

$$(x \in (-1,1))$$

where $k_n = (-1)^n 2^n n!$ $(n \in N_0)$.

Using Leibnitz's rule for the n^{th} derivative, then

$$P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{r=0}^{n}\binom{n+\alpha}{r}\binom{n+\beta}{n-r}(x-1)^{n-r}(x+1)^r$$

$$(x \in [-1,1],\ n \in N_0).$$

Hence

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \quad (n \in N_0) \qquad (1.5.2)$$

and

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \quad (x \in [-1,1],\ n \in N_0). \quad (1.5.3)$$

The generating function $G(\cdot,t)$ in this case is

$$\frac{2^{\alpha+\beta}}{R(1-t+R)^{\alpha}(1+t+R)^{\beta}} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n, \quad (1.5.4)$$

where $|t|$ is sufficiently small, $R = (1-2xt+t^2)^{1/2}$ and R is assumed to be a regular analytic function of t. The proof of (1.5.4) may be found in Szegö (1975; §4.4), where we consider the branch which gives $R(0) = 1$; and also the expressions $(---)^{\alpha}$ and $(---)^{\beta}$ must be taken positive for $t = 0$.

Theorem 1.5.1

Let $\alpha > -1$, $\beta > -1$; then the polynomials $\{P_n^{(\alpha,\beta)}(\cdot),\ n \in N_0\}$ satisfy the differential equation

$$(1-x^2)y''(x) + [\beta - \alpha - (\alpha+\beta+2)x]y'(x) + \lambda_n y(x) = 0 \quad (1.5.5)$$

or equivalently

$$((1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x))' + \lambda_n(1-x)^{\alpha}(1+x)^{\beta}y(x) = 0 \quad (1.5.6)$$

where $x \in (-1,1)$ and $\lambda_n = n(n+\alpha+\beta+1)$ $(n \in N_0)$.

Proof: See Szegö (1975; §4.2 and 4.6) and also Pleijel (1975).

One important application of this theorem is in proving the following result:

Theorem 1.5.2

Let $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ $(x \in (-1,1))$, $\alpha > -1$ *and* $\beta > -1$. *Then the polynomials* $\{P_n^{(\alpha,\beta)}(\cdot),\ n \in N_0\}$ *are orthogonal in* $(-1,1)$ *with respect to* $w(\cdot)$, *i.e.*

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\,dx = 0 \tag{1.5.7}$$

$$(m,n \in N \text{ and } m \neq n)$$

but if $m = n$ *then*

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \{P_n^{(\alpha,\beta)}(x)\}^2 dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)} . \tag{1.5.8}$$

Proof: See Rainville (1963; §135).

Thus the two relations (1.5.7) and (1.5.8) imply that the system $\{\phi_n,\ n \in N_0\}$, where

$$\phi_n(x) = \sqrt{\frac{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}\, P_n^{(\alpha,\beta)}(x) \tag{1.5.9}$$

is orthonormal in $L_w^2(-1,1)$.

Also we note here that the interval $(-1,1)$ is finite, hence the polynomials $\{P_n^{(\alpha,\beta)}(\cdot),\ n \in N_0\}$ retain the properties of Theorem 1.3.2 and in particular we have the following:

Theorem 1.5.3

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}(\cdot),\ n \in N_0\}$ *associated with the weight function* $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ *over* $(-1,1)$ *are complete in* $L_w^2(-1,1)$.

The two particular cases of Jacobi polynomials, viz. the Gegenbauer polynomials $\{C_n^{\nu}(\cdot),\ n \in N_0\}$ and the Legendre polynomials $\{P_n(\cdot),\ n \in N_0\}$, may be defined respectively as

$$C_n^{\nu}(x) = \frac{(2\nu)_n P_n^{(\nu-1/2,\nu-1/2)}(x)}{(\nu+1/2)_n} \tag{1.5.10}$$

$$(x \in (-1,1),\ \nu > -1/2,\ n \in N_0)$$

and

$$P_n(x) = P_n^{(0,0)}(x) \quad (n \in N_0,\ x \in (-1,1)). \tag{1.5.11}$$

It follows from (1.5.10) and (1.5.11) that, on putting $\alpha = \beta = \nu - 1/2$ or $\alpha = \beta = 0$ in §1.5, both Gegenbauer and Legendre polynomials retain all the properties of Jacobi polynomials in §1.5.

One of the generating functions, essentially different from (1.5.4) if $\alpha = \beta = \nu - 1/2$, and which is sometimes given as the definition of Gegenbauer polynomials is

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n \quad (|t| < 1,\ \nu \neq 0) \tag{1.5.12}$$

(see Szegö (1975; §4.7) for the proof of (1.5.12)).

Using this, Szegö (1975; §4.82) has shown that if $x = \cos\theta$, $0 < \theta < \pi$, $t = \exp[i\phi]$, $0 \le \phi < \theta$ and $0 < \nu < 1$, then

$$C_n^{\nu}(\cos\theta) = \frac{1}{\pi}\int_0^{\pi} F(\phi)\,|2\cos\phi - 2\cos\theta|^{-\nu}\,d\phi \tag{1.5.13}$$

where

$$F(\phi) = \begin{cases} \cos\,(n+\nu)\phi & \text{if } 0 \le \phi < \theta \\ \cos\,[(n+\nu)\phi - \nu\pi] & \text{if } \theta < \phi \le \pi\ . \end{cases}$$

The integral (1.5.13) is a generalization of the Dirichlet-Mehler integral for Legendre polynomials:

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi} F(\phi)\,|2\cos\phi - 2\cos\theta|^{-1/2}\,d\phi \quad (n \in N_0) \qquad (1.5.14)$$

where $F(\phi)$ is given by (1.5.13) on putting $\nu = 1/2$. There are many proofs of (1.5.14) (see, for example, Whittaker-Watson (1935; §15.231), Askey (1969, 1975), Henrici (1971), Szegö (1975; §4.8) and Van de Watering (1968)). The restriction $0 < \nu < 1$ ensures that the integral (1.5.13) converges at $\phi = \pm\theta$; this restriction on ν can be removed if the integral (1.5.13) is a contour integral in the complex plane (see Chapter Six).

§1.6 The Laguerre Polynomials

Let $a = 0$, $b = +\infty$ and let $w(x) = x^{\alpha}e^{-x}$, $\alpha > -1$ and $p(x) = x^n$ ($n \in N_0$, $x \in (0,\infty)$); then Rodrigues' formula (1.4.4) now becomes

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^{x}}{n!}\frac{d^n}{dx^n}[x^{n+\alpha}e^{-x}] \qquad (1.6.1)$$

where $k_n = n!$ ($n \in N_0$), $x \in (0,\infty)$ and $\{L_n^{(\alpha)}(\cdot),\ n \in N_0\}$ are the Laguerre polynomials. The first few polynomials are

$$L_0^{(\alpha)}(x) = 1\ , \quad L_1^{(\alpha)}(x) = 1 + \alpha - x$$

$$L_2^{(\alpha)}(x) = \frac{1}{2}[x^2 - 2(2+\alpha)x + (\alpha+2)(\alpha+1)]$$

and in general (using Leibnitz's rule)

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)k!(n-k)!}\,x^k \qquad (1.6.2)$$

$$(x \in (0,\infty),\ \alpha > -1)$$

with

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} \quad (n \in N_0). \tag{1.6.3}$$

The generating function in this case is

$$(1-t)^{-\alpha-1}\exp[-\frac{xt}{1-t}] = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n \tag{1.6.4}$$

$$(x \in (0,\infty),\ |t| < 1,\ \alpha > -1).$$

Laguerre's polynomials also have the following property:

<u>Theorem 1.6.1</u>

<u>The polynomial</u> $p(x) = L_n^{(\alpha)}(x)$ <u>is a solution of Laguerre's differential equation</u>

$$(x^{\alpha+1}e^{-x}y'(x))' + nx^{\alpha}e^{-x}y(x) = 0 \tag{1.6.5}$$

$$(x \in (0,\infty),\ \alpha > -1,\ n \in N_0).$$

<u>Proof</u>: See Section 7.1, where we consider the equation (1.6.5) with n replaced by $\lambda - \frac{\alpha+1}{2}$. Using this result, we can then prove the following:

<u>Theorem 1.6.2</u>

<u>The polynomials</u> $\{L_n^{(\alpha)}(x),\ n \in N_0\}$ <u>are orthogonal over the interval</u> $(0,\infty)$ <u>with respect to the weight function</u> $w(x) = x^{\alpha}e^{-x}$ $(x \in (0,\infty),\ \alpha > -1)$, <u>i.e.</u>

$$\int_0^{\infty} x^{\alpha}e^{-x}L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx = 0 \quad (m,n \in N_0,\ m \neq n). \tag{1.6.6}$$

But if m = n, then

$$\int_0^\infty x^\alpha e^{-x}\{L_n^{(\alpha)}(x)\}^2 dx = \frac{\Gamma(n+\alpha+1)}{n!} \qquad (\alpha > -1,\ n \in N_0). \tag{1.6.7}$$

Proof: See Sansone (1959; §4.1).

We indicated in part (b) of Remark 1.3.3 that Theorem 1.3.2 cannot be extended in general to the case of unbounded intervals, i.e. the resulting polynomials $\{p_n,\ n \in N_0\}$ are not necessarily complete in $L^2_\alpha(a,b)$ if the interval (a,b) is unbounded. However, the next two results show that the generalized Laguerre polynomials $\{L_n^{(\alpha)}(\cdot),\ n \in N_0\}$ defined over $(0,\infty)$ are complete in $L^2_w(0,\infty)$, where

$$w := w(x) = x^\alpha e^{-x} \qquad (x \in (0,\infty),\ \alpha > -1)$$

is the associated weight function.

Theorem 1.6.3 (Szegö (1975; §5.7))

The system $\{e^{-x/2}x^{\alpha/2}x^n,\ n \in N_0,\ \alpha > -1\}$ is closed in $L^2(0,\infty)$ (this is equivalent to the closure of $\{e^{-x/2}x^{\alpha/2}L_n^{(\alpha)}(x),\ n \in N_0,\ \alpha > -1\}$ in $L^2(0,\infty)$).

Proof: First we note that $\{(\log 1/y)^{\alpha/2}y^n,\ n \in N_0\}$ is closed in $L^2(0,1)$. This is a consequence of Theorem 1.3.2 because $(\log 1/y)^\alpha \in L(0,1)$, i.e.

$$\int_0^1 |\log 1/y|^\alpha dy < \infty \quad \text{if} \quad \alpha > -1,$$

for if $x = \log 1/y$, then

$$y = e^{-x} \ , \quad dy = -e^{-x}dx$$

and

$$\int_0^1 |\log 1/y|^{\alpha} dy = \int_0^{\infty} x^{\alpha} e^{-x} dx$$

$$< \infty \quad \text{if} \quad \alpha > -1 \ .$$

Now let $e^{-x/2}x^{\alpha/2}f(x) \in L^2(0,\infty)$, i.e.

$$\int_0^{\infty} e^{-x}x^{\alpha}|f(x)|^2 dx < \infty \qquad (\alpha > -1),$$

and put $x = \log 1/y$, then

$$\int_0^1 (\log 1/y)^{\alpha}|f(\log 1/y)|^2 dy < \infty \ ,$$

<u>i.e.</u> $(\log 1/y)^{\alpha/2}f(\log 1/y) \in L^2(0,1)$, and by Theorem 1.2.6 there exists a function of the form $(\log 1/y)^{\alpha/2}\rho(y)$ such that, for $\epsilon > 0$,

$$\int_0^1 (\log 1/y)^{\alpha}[f(\log 1/y) - \rho(y)]^2 dy < \epsilon$$

(where ρ is a polynomial), i.e.

$$\int_0^{\infty} e^{-x}x^{\alpha}[f(x) - \rho(e^{-x})]^2 dx < \epsilon \ .$$

What remains now is to show that if $m \in N_0$ then for every $\delta > 0$ there exists a polynomial p such that

CHAPTER THREE

DIFFERENTIAL EQUATIONS

§3.0 Preliminary

The object of this chapter is to examine the differential equation

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \quad (x \in (a,b),\ \lambda \in \mathbb{C}), \tag{3.0.1}$$

which forms the basis of our study in the remaining chapters.

We begin by considering equation (3.0.1) with particular reference to the concept of quasi-derivatives. This is followed by showing that under the so-called Liouville transformation, equations (3.0.1) and (2.1.3) are equivalent and certain properties remain invariant.

Next we introduce the concepts of right- and left-definiteness in the Hilbert function spaces $L^2_w(a,b)$ and $H^2_{p,q}(a,b)$ respectively.

Finally we classify the end-points of the interval (a,b) as being regular, limit-circle or limit-point, in terms of the solutions of (3.0.1) lying in the space $L^2_w(a,b)$ or $H^2_{p,q}(a,b)$; we then give a complete classification of Jacobi, Laguerre and Hermite equations in both right- and left-definite cases.

§3.1 Symmetric Differential Equations

Let M[y] denote the formally symmetric, second-order differential expression defined by

$$M[y](x) := -(p(x)y'(x))' + q(x)y(x) \quad (x \in (a,b)) \tag{3.1.1}$$

for suitably differentiable complex-valued functions y. The coefficients p and q are real-valued Lebesgue measurable on (a,b), satisfying

$$\left.\begin{array}{ll}\text{(i)} & p(x) > 0 \quad (\text{pp } x \in (a,b)) \quad \text{and} \quad p^{-1} \in L_{loc}(a,b); \\ \text{(ii)} & q \in L_{loc}(a,b)\ .\end{array}\right\} \qquad (3.1.2)$$

(Note: We write p^{-1} for the reciprocal function with value $\{p(x)\}^{-1}$ for all $x \in (a,b)$, and not the inverse function of p, which may not exist.)

Let w be a real-valued, Lebesgue-measurable function on (a,b), such that

$$\left.\begin{array}{ll}\text{(i)} & w(x) > 0 \quad (\text{pp } x \in (a,b)) \\ \text{(ii)} & w \in L_{loc}(a,b)\ ;\end{array}\right\} \qquad (3.1.3)$$

and consider the linear second-order differential equation

$$M[y](x) := -(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \qquad (3.1.4)$$
$$(x \in (a,b),\ \lambda \in \mathbb{C});$$

or, in an equivalent notation,

$$M[y] := -(py')' + qy = \lambda wy \ \text{ on } \ (a,b) \quad (\lambda \in \mathbb{C}). \qquad (3.1.4)'$$

Definition 3.1.1

With the basic conditions (3.1.2) and (3.1.3) satisfied, equation (3.1.4) is said to be regular, in the sense of Naimark (1967; §15.1), if the interval (a,b) is finite and $p^{-1},q,w \in L(a,b)$; otherwise it is said to be singular. In particular, the end-point $x = a$ is regular if $a > -\infty$, and $p^{-1},q,w \in L(a,\beta)$ where $\beta < b$; otherwise a is a singular end-point. A similar definition obtains at the end-point b.

Definition 3.1.2

It may be the case that p and q are not differentiable on (a,b). Then for M[y] to make sense, we require that

$$y^{[i]} \in AC_{loc}(a,b) \quad (i = 0,1), \tag{3.1.5}$$

where $y^{[i]}$ (i = 0,1,2) denote the quasi-derivatives of y defined by

$$y^{[0]} = y \quad \text{on} \quad (a,b),$$

$$y^{[1]} = py' \quad \text{on} \quad (a,b),$$

and $$y^{[2]} = qy^{[0]} - (y^{[1]})'$$

$$= qy - (py')',$$

i.e.

$$y^{[2]} = M[y] \quad \text{on} \quad (a,b).$$

Thus M[y] makes sense for a given function y if $y^{[1]}$ exists and $y^{[0]}$, $y^{[1]}$ satisfy (3.1.5) (see Naimark (1967; §15.2)).

Having defined the equation (3.0.1), we now show how it is related to equation (2.1.3). To do this we use the so-called Liouville transformation; see, for example, the account in Birkhoff and Rota (1969; §10.1), and Everitt (1972).

§3.2 Liouville's Transformation

Consider the differential equation (3.0.1)

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \tag{3.0.1}$$

$$(x \in (a,b),\ \lambda \in \mathbb{C}).$$

In addition to conditions (3.1.2) and (3.1.3), let

$$p \text{ and } p', \ w \text{ and } w' \in AC_{loc}(a,b). \tag{3.2.1}$$

Let

$$X := X(x) = C + \int_c^x \sqrt{\frac{w(t)}{p(t)}}\,dt \quad (x \in (a,b)), \tag{3.2.2}$$

where $a < c < b$, and C is any real number.

Also let

$$\begin{aligned} Y(X) &= \{p(x)w(x)\}^{1/4} y(x) \quad (x \in (a,b)) \\ &= \{p(x(X))\cdot w(x(X))\}^{1/4} y(x(X)) \quad (X \in (A,B)) \end{aligned} \tag{3.2.3}$$

where $A = X(a)$, $B = X(b)$ with $A < C < B$; and $x = x(X)$ $(X \in (A,B))$ is the uniquely determined inverse function of $X = X(x)$ $(x \in (a,b))$. In addition, we define P, Q_0 and W on (A,B) by

$$P(X) = p(x(X)) \ , \quad Q_0(X) = q(x(X)) \ ,$$

$$\text{and} \quad W(X) = w(x(X)) \quad (X \in (A,B)).$$

Let $y'(x) = dy/dx$ and $Y'(X) = dY/dX$; then we obtain the following result:

Theorem 3.2.1

Under the Liouville substitutions (3.2.2) *and* (3.2.3), *equation* (3.0.1) *above transforms into the normal form* (see (2.1.3)):

$$-Y''(X) + Q(X)Y(X) = \lambda Y(X) \quad (X \in (A,B),\ \lambda \in \mathbb{C}), \tag{3.2.4}$$

where

$$Q(X) = \frac{q(x(X))}{w(x(X))} + \frac{1}{4}\left[\frac{p''(x(X))}{p(x(X))} + \frac{w''(x(X))}{w(x(X))}\right]$$

$$+ \frac{p'(x(X))w'(x(X))}{8p(x(X))w(x(X))} - \frac{3}{16}\left[\frac{p'^2(x(X))}{p^2(x(X))} + \frac{w'^2(x(X))}{w^2(x(X))}\right]$$

i.e.

$$Q(X) = \frac{Q_0(X)}{W(X)} + \frac{1}{4}\left[\frac{P''(X)}{P(X)} + \frac{W''(X)}{W(X)}\right] + \frac{P'(X)W'(X)}{8P(X)W(X)}$$

$$- \frac{3}{16}\left[\frac{P'^2(X)}{P^2(X)} + \frac{W'^2(X)}{W^2(X)}\right] \qquad (X \in (A,B)). \qquad (3.2.5)$$

Proof: From (3.2.2),

$$dX = \sqrt{\frac{w(x)}{p(x)}}\,dx$$

and

$$\frac{d}{dx} = \sqrt{\frac{p(x(X))}{w(x(X))}}\,\frac{d}{dX}$$

$$= \sqrt{\frac{W(X)}{P(X)}}\,\frac{d}{dX} := \sqrt{\frac{W}{P}}\,\frac{d}{dX} ;$$

hence

$$- \frac{d}{dX}\left[p(x)\frac{dy(x)}{dx}\right] = -\sqrt{\frac{W}{P}}\,\frac{d}{dX}\left[P\sqrt{\frac{W}{P}}\,\frac{d}{dX}(YP^{-1/4}W^{-1/4})\right]$$

$$= -P^{-1/4}W^{3/4}Y'' - S(P,W)Y, \qquad (3.2.6)$$

where

$$S(P,W) = \frac{3}{16}(P^{-9/4}W^{3/4}(P')^2 + P^{-1/4}W^{-5/4}(W')^2)$$

$$- \frac{1}{8}P^{-5/4}W^{-1/4}P'W' - \frac{1}{4}(P^{-5/4}W^{3/4}P'' + P^{-1/4}W^{-1/4}W'') .$$

Also

$$q(x)y(x) = P^{-1/4}W^{-1/4}Q_0 Y \tag{3.2.7}$$

and

$$\lambda w(x)y(x) = \lambda P^{-1/4}W^{3/4}Y \ ; \tag{3.2.8}$$

substituting (3.2.6), (3.2.7) and (3.2.8) in (3.0.1) and then dividing by $P^{-1/4}W^{3/4}$, we obtain (3.2.4).

Remark 3.2.3

It follows from (3.2.5) that if p satisfies the condition (3.1.2) and w satisfies (3.1.3), and if in addition p and w satisfy (3.2.1), then Q satisfies the condition (ii) of (3.1.2) but on (A,B), i.e.

$$Q \in L_{loc}(A,B) \ .$$

Also, under (3.2.2) and (3.2.3), the initial conditions on a solution $y(\cdot,\lambda)$ of (3.0.1) at $x = c$ is transformed into an initial condition on $Y(\cdot,\lambda)$ at $X = C$ as follows:

$$\left.\begin{aligned} Y(C,\lambda) &= \{p(c)w(c)\}^{1/4}y(c,\lambda) \\ Y'(C,\lambda) &= \tfrac{1}{4}\{p^{3/4}(c)w^{-5/4}(c)w'(c) + p^{-1/4}(c)p'(c)w^{-1/4}(c)\}y(c,\lambda) \\ &\quad + p^{3/4}(c)w^{-1/4}(c)y'(c,\lambda). \end{aligned}\right\} \tag{3.2.9}$$

Furthermore, for $x_0 \in [c,b)$ and $X_0 = X(x_0) \in [C,B)$, if $y(\cdot,\lambda)$ is a solution of (3.0.1) and $Y(\cdot,\lambda)$ is the corresponding transformed solution of (3.2.4) then (see Birkhoff and Rota (1969; §10.9, Corollary 1)):

$$\int_c^{x_0} w(x)|y(x,\lambda)|^2 dx = \int_C^{X_0} w(x(X))|y(x(X))|^2 \sqrt{\frac{p(x(X))}{w(x(X))}}\, dX$$

$$= \int_C^{X_0} W(X)\,|Y(X,\lambda)|^2 P^{-1/2}(X) W^{-1/2}(X) \cdot P^{1/2}(X) W^{-1/2}(X)\,dX ,$$

i.e.

$$\int_c^{x_0} w(x)\,|y(x,\lambda)|^2 dx = \int_C^{X_0} |Y(X,\lambda)|^2 dX ; \qquad (3.2.10)$$

so that integrable-square solutions remain invariant in this sense. This result leads to the following conclusion:

Corollary 3.2.4

The transformed equation (3.2.4) is regular or limit-point or limit-circle at $X = B$ if and only if the original equation (3.0.1) is regular or limit-point or limit-circle at $x = b$ respectively; similarly for the end-points $x = a$ and $X = A$.

Now let the solutions $y_1(x)$, $y_2(x)$ of (3.0.1) be transformed, under Liouville's substitution (3.2.2) and (3.2.3), into $Y_1(X)$, $Y_2(X)$ respectively, the latter being the solutions of (3.2.4); then

$$\int_a^b w(x) y_1(x) y_2(x)\,dx = \int_a^b y_1(x) y_2(x) \sqrt{p(x)w(x)} \cdot \sqrt{\frac{w(x)}{p(x)}}\,dx$$

$$= \int_A^B Y_1(X) Y_2(X)\,dX \qquad (3.2.11)$$

(where $x \in (a,b)$, $X \in (A,B)$, and w is the weight function in (3.0.1)). We can then infer from (3.2.11) the following result:

Corollary 3.2.5

Liouville's substitutions (3.2.2) and (3.2.3) transform functions orthogonal with weight $w(x)$, $x \in (a,b)$, into functions orthogonal with unit weight (i.e. $W(X) = 1$, $X \in (A,B)$).

Thus equations (3.0.1) and (2.1.3) are equivalent under Liouville's transformation in the sense that certain properties remain invariant.

Example 3.2.6

I Consider Jacobi's equation

$$-((1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x))' = \lambda(1-x)^{\alpha}(1+x)^{\beta}y(x) \qquad (3.2.12)$$

($x \in (-1,1)$, $\alpha > -1$, $\beta > -1$ and $\lambda \in \mathbb{C}$). Using Liouville's substitution (3.2.2) and (3.2.3), let

$$X(x) := \int_0^x \sqrt{\frac{w(t)}{p(t)}}\,dt \quad (x \in (-1,1))$$

$$= \int_0^x \frac{dt}{\sqrt{1-t^2}} \quad (x \in (-1,1))$$

i.e. $X(x) = \sin^{-1}x$, so that

$$x(X) := x = \sin X \quad (X \in (-\pi/2,\pi/2));$$

and let

$$Y(X) := \{(1-x)^{\alpha+1}(1+x)^{\beta+1}(1-x)^{\alpha}(1+x)^{\beta}\}^{1/4}y(x) \quad (x \in (-1,1))$$

$$= \{(1-\sin X)^{2\alpha+1}(1+\sin X)^{2\beta+1}\}y(\sin X) \quad (x \in (-\pi/2,\pi/2));$$

then (3.2.12) is transformed into

$$-Y''(X) + Q(X)Y(X) = \lambda Y(X) \quad (\lambda \in \mathbb{C},\ X \in (-\pi/2,\pi/2), \qquad (3.2.13)$$

where (see (3.2.5))

$$Q(X) = -\frac{(\alpha+\beta+1)^2}{4} - \frac{1-2(\alpha^2+\beta^2)}{4}\sec^2 X - \frac{\beta^2-\alpha^2}{2}\tan X \sec X$$

$$(\alpha > -1,\ \beta > -1).$$

The two special cases of Gegenbauer and Legendre equations are obtained on putting $\alpha = \beta = \nu - 1/2$ and $\alpha = \beta = 0$ respectively in (3.2.13).

II Consider Laguerre's equation

$$-(x^{\alpha+1}e^{-x}y'(x))' = \lambda x^{\alpha}e^{-x}y(x) \quad (x \in (0,\infty),\ \lambda \in \mathbb{C}), \qquad (3.2.14)$$

with $\alpha > -1$; then a similar application of Liouville's substitutions (3.2.2) and (3.2.3) gives

$$X = 2x^{1/2},$$

i.e.

$$x = \frac{X^2}{4},$$

and

$$Y(X) = \left\{\frac{X^{4\alpha+2}e^{-X^2/2}}{4^{2\alpha+1}}\right\}^{1/4} y(x(X)) \quad (X \in (0,\infty)). \qquad (3.2.15)$$

Equation (3.2.14) now transforms into

$$-Y''(X) + \left(\frac{X^2}{16} - \frac{\alpha+1}{2} - \frac{1/4-\alpha^2}{X^2}\right)Y(X) = \lambda Y(X) \quad (X \in (0,\infty)).\ (3.2.16)$$

III Finally, we consider Hermite's equation

$$-(e^{-x^2}y'(x))' = \lambda e^{-x^2}y(x) \quad (x \in (-\infty,\infty),\ \lambda \in \mathbb{C}); \qquad (3.2.17)$$

then the substitutions (3.2.2) and (3.2.3) give $X = x$ and $Y(X) = \{e^{-x^2}.e^{-x^2}\}^{1/4}y(x)$, i.e.

$$Y(X) = e^{-X^2/2}y(x(X)) \quad (X \in (-\infty,\infty));$$

and so (3.2.10) transforms into

$$-Y''(X)-(1-X^2)Y = \lambda Y(X) \quad (X \in (-\infty,\infty),\ \lambda \in \mathbb{C}). \qquad (3.2.18)$$

We consider now the solutions of (3.0.1) in the spaces $L^2_w(a,b)$ and $H^2_{p,q}(a,b)$ which are associated with the notions of "right-definiteness" and "left-definiteness", respectively.

§3.3 The right-definite case

Consider the differential equation (3.0.1), viz.:

$$M[y] := -(py')' + qy = \lambda wy \quad \text{on} \quad (a,b) \qquad (3.0.1)$$

and $\lambda \in \mathbb{C}$, with p,q satisfying the conditions (3.1.2) and w satisfying (3.1.3).

Definition 3.3.1

We use the term "right-definite" to describe the solutions of (3.0.1) in $L^2_w(a,b)$, with w satisfying the conditions (3.1.3).

Then, according to the general theory of Titchmarsh and Weyl (see, §2.1) at a singular end-point a(b), either

(a) M[y] is <u>limit-point</u> at $x = a(b)$, <u>i.e.</u> exactly one solution of (3.0.1) is in $L^2_w(a,c)$ $(L^2_w(c,b))$, $a < c < b$.

(b) M[y] is <u>limit-circle</u> at $x = a(b)$, <u>i.e.</u> all solutions of (3.0.1) are in $L^2_w(a,c)$ $(L^2_w(c,b))$.

Note that if $-\infty < a < b < \infty$ and M[y] is regular at all points of [a,b], then all solutions of (3.0.1) are necessarily in $L^2_w(a,b)$.

Let $\theta(\cdot,\lambda)$ and $\phi(\cdot,\lambda)$ ($\lambda \in \mathbb{C}$) be the solutions of (3.0.1) and let c, $a < c < b$, be a regular point of M[y], such that for $\lambda \in \mathbb{C}$ and $\alpha \in (-\pi/2,\pi/2]$

$$\theta^{[0]}(c,\lambda) = \cos\alpha \ , \quad \theta^{[1]}(c,\lambda) = \sin\alpha \tag{3.3.1}$$

$$\phi^{[0]}(c,\lambda) = -\sin\alpha \ , \quad \phi^{[1]}(c,\lambda) = \cos\alpha \ . \tag{3.3.2}$$

Then $\{\theta(\cdot,\lambda),\phi(\cdot,\lambda)\}$ forms a basis for all solutions of (3.0.1). The general theory of Titchmarsh and Weyl (see §2.1) now states that there are solutions $\psi_1(\cdot,\lambda)$ and $\psi_2(\cdot,\lambda)$ of (3.0.1) of the form

$$\psi_1(x,\lambda) = \theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda) \qquad (x \in (a,b),\ \lambda \in \mathbb{C} - R)$$

and

$$\psi_2(x,\lambda) = \theta(x,\lambda) + m_2(\lambda)\phi(x,\lambda) \qquad (x \in (a,b),\ \lambda \in \mathbb{C} - R)$$

such that

$$\left.\begin{array}{ll} \text{(i)} & m_1(\cdot), m_2(\cdot) \text{ are regular in } \mathbb{C} - R \\ \text{(ii)} & \psi_1(\cdot,\lambda) \in L^2_w(a,c) \\ \text{(iii)} & \psi_2(\cdot,\lambda) \in L^2_w(c,b) \ . \end{array}\right\} \tag{3.3.3}$$

The Green's function in this case is

$$G(x,t,\lambda) = \begin{cases} -\dfrac{\psi_2(x,\lambda)\psi_1(t,\lambda)}{\omega(\lambda)} & (t \le x) \\ -\dfrac{\psi_1(x,\lambda)\psi_2(t,\lambda)}{\omega(\lambda)} & (t > x) \end{cases} \tag{3.3.4}$$

where

$$\omega(\lambda) = (pW(\psi_1(\cdot,\lambda),\psi_2(\cdot,\lambda)))(x)$$

$$= m_1(\lambda) - m_2(\lambda) \quad (\lambda \in \mathbb{C} - R). \tag{3.3.5}$$

From the general theory of these equations (see Titchmarsh (1962; §2.18)), the eigenvalues (if any) associated with the equation (3.0.1) are derived from the zeros and poles of $m_1(\lambda) - m_2(\lambda)$.

In the case of a discrete spectrum, and following (2.1.10), we define the function $\Phi : (a,b) \times (\mathbb{C} - R) \times L^2_w(a,b) \to \mathbb{C}$ by

$$\Phi(x,\lambda;f) := -\int_a^b w(t)G(x,t,\lambda)f(t)dt \ ; \tag{3.3.6}$$

then an application of the Titchmarsh method (see §2.1) to (3.3.6) now gives the eigenfunctions $\{\phi_n,\ n \in N_0\}$ associated with the differential equation (3.0.1), the completeness of $\{\phi_n,\ n \in N_0\}$ in $L^2_w(a,b)$ and the series expansion of $f \in L^2_w(a,b)$ in terms of these eigenfunctions (see details for the examples considered in Chapters Six, Seven and Eight below).

§3.4 The left-definite case

We begin with the following definition:

Definition 3.4.1 (Adams (1975; §1.24))

A normed space X *is said to be imbedded in the normed space* Y, *and we write*

$$X \subset Y ,$$

provided

(i) X *is a vector subspace of* Y, *and*

(ii) *the identity mapping* I *defined on* X *by* $Ix = x \ (x \in X)$ *is continuous*.

Now consider the differential equation (3.0.1), *i.e.* $M[y] := -(py')' + qy = \lambda wy$ on (a,b). In addition to the conditions (3.1.2), namely $p > 0$ on (a,b) and $q \in L_{loc}(a,b)$, let

$$q \geq 0 \text{ on } (a,b) . \tag{3.4.1}$$

Let $H^2_{p,q}(a,b)$ denote the function space defined by

$$H^2_{p,q}(a,b) := \{f : (a,b) \to \mathbb{C} : f \in AC_{loc}(a,b), \\ q^{1/2}f \in L^2(a,b) \text{ and } p^{1/2}f' \in L^2(a,b)\} . \tag{3.4.2}$$

If we endow $H^2_{p,q}(a,b)$ with the scalar product

$$(f,g)_H = \int_a^b \{pf'\overline{g}' + qf\overline{g}\} \quad (f,g \in H^2_{p,q}(a,b)) \tag{3.4.3}$$

and the norm by the finite Dirichlet integral

$$\|f\|_H = \{\int_a^b [p|f'|^2 + q|f|^2]\}^{1/2} \quad (f \in H^2_{p,q}(a,b)); \tag{3.4.4}$$

then, with p and q satisfying above conditions, the following properties hold in $H^2_{p,q}(a,b)$ (note that we no longer require $w > 0$ on (a,b)):

Theorem 3.4.2

(a) $H^2_{p,q}(a,b)$ is a Hilbert space with respect to the norm (3.4.4);

(b) $H^2_{p,q}(a,b)$ is continuously imbedded into $L^2_w(a,b)$ in the sense of Definition 3.4.1.

Proof:

(a) It is easy to check with the help of (3.4.4) that $H^2_{p,q}(a,b)$ is a scalar product (or pre-Hilbert) space; so we shall omit it.

Now let (f_k) $(k \in N_0)$ be a Cauchy sequence in $H^2_{p,q}(a,b)$, then for some $\epsilon > 0$

$$\| f_j - f_k \|^2_H = \int_a^b [p|f_j' - f_k'|^2 + q|f_j - f_k|^2] < \epsilon \qquad (j,k \in N_0)$$

i.e.

$$\int_a^b p|f_j' - f_k'|^2 < \frac{\epsilon}{2} \quad \text{and} \quad \int_a^b q|f_j - f_k|^2 < \frac{\epsilon}{2}$$

i.e. $(p^{1/2} f_k')$ and $(q^{1/2} f_k)$ $(k \in N_0)$ are Cauchy sequences in $L^2(a,b)$. Since $L^2(a,b)$ is complete, there exists $p^{1/2} f'$ and $q^{1/2} f$ in $L^2(a,b)$ such that $\lim_{k\to\infty} p^{1/2} f_k' = p^{1/2} f'$ and $\lim_{k\to\infty} q^{1/2} f_k = q^{1/2} f$, where f' is the derivative of f on (a,b), i.e. $f \in AC_{loc}(a,b)$. Then, by relation (3.4.2), we have $f \in H^2_{p,q}(a,b)$, and so the above limit processes hold in $H^2_{p,q}(a,b)$, *i.e.* for some $\epsilon > 0$

$$\int_a^b \{|p^{1/2}f_n' - p^{1/2}f'|^2\} < \frac{\epsilon}{2} \quad (n \in N_0)$$

and

$$\int_a^b |q^{1/2}f_n - q^{1/2}f|^2 < \frac{\epsilon}{2} ;$$

or

$$\|f_n - f\|_H^2 = \int_a^b [p|f_n' - f'|^2 + q|f_n - f|^2] < \epsilon ,$$

i.e. $\lim_{n\to\infty} f_n = f$, and so $H^2_{p,q}(a,b)$ is complete, which proves the assertion.

(b) This is a consequence of Sobolev's imbedding theorem (see Adams (1975; §3.2, 5.2, 5.4), Aubin (1979; §7.5) and Hutson-Pym (1980; §11.3)).

Definition 3.4.3

We use the term left-definite (or polar) when the differential equation (3.0.1), i.e. $M[y] = \lambda wy$ is considered in the space $H^2_{p,q}(a,b)$. In this case the coefficient w may not be of one sign on (a,b) (see (3.1.3)); if so, then it may be necessary to require that for some $k \in R_+$

$$|w(x)| \le kq(x) \quad (x \in (a,b)) \tag{3.4.5}$$

(see Everitt (1972; §3)). We do not consider such cases when w is indefinite in this thesis.

Now consider the solutions of (3.0.1) in $H^2_{p,q}(a,b)$; then, adapting the general theory of Titchmarsh and Weyl for the space $H^2_{p,q}(a,b)$ (see Atkinson-Everitt-Ong (1974)), we obtain a similar classification to that of §3.3, <u>i.e.</u> either

(a) M[y] is limit-point at x = a(b), <u>i.e.</u> exactly one solution of $M[y] = \lambda wy$ is in $H^2_{p,q}(a,c)$ ($H^2_{p,q}(c,b)$) .

(b) M[y] is limit-circle at x = a(b), <u>i.e.</u> all solutions of $M[y] = \lambda wy$ are in $H^2_{p,q}(a,c)$ ($H^2_{p,q}(c,b)$).

Let $\theta(\cdot,\lambda)$ and $\phi(\cdot,\lambda)$ be the two linearly independent solutions of $M[y] = \lambda wy$, satisfying the conditions (3.3.1) and (3.3.2). Then, by the general theory of Titchmarsh and Weyl (§2.1), there are solutions $\Psi_1(x,\lambda) = \theta(x,\lambda) + M_1(\lambda)\phi(x,\lambda)$ and $\Psi_2(x,\lambda) = \theta(x,\lambda) + M_2(\lambda)\phi(x,\lambda)$ $(x \in (a,b),\ \lambda \in \mathbb{C} - R)$, such that

$$\left.\begin{array}{ll} \text{(i)} & M_1(\cdot) \text{ and } M_2(\cdot) \text{ are regular in } \mathbb{C} - R, \\ \text{(ii)} & \Psi_1(\cdot,\lambda) \in H^2_{p,q}(a,c),\ a < c < b, \\ \text{(iii)} & \Psi_2(\cdot,\lambda) \in H^2_{p,q}(c,b)\ . \end{array}\right\} \qquad (3.4.6)$$

<u>Note</u>:

It is possible from (3.3.3) and (3.4.6) that $M_i(\lambda) = m_i(\lambda)$, $i = 1,2$, but the choice of each may be different; for if M[y] is <u>limit-point</u> in L^2_w , but <u>limit-circle</u> in $H^2_{p,q}$, then $m_i(\cdot)$ is unique, while $M_i(\cdot)$ is <u>not</u>.

Let the Green's function in this case be defined as

$$G(x,t,\lambda) = \begin{cases} -\dfrac{\Psi_2(x,\lambda)\Psi_1(x,\lambda)}{(pW(\Psi_1(\cdot,\lambda),\Psi_2(\cdot,\lambda)))(x)} & (t \le x) \\[2ex] -\dfrac{\Psi_1(x,\lambda)\Psi_2(t,\lambda)}{(pW(\Psi_1(\cdot,\lambda),\Psi_2(\cdot,\lambda)))(x)} & (t > x) \end{cases} \qquad (3.4.7)$$

ith $(pW(\Psi_1(\cdot,\lambda),\Psi_2(\cdot,\lambda)))(x) = M_1(\lambda) - M_2(\lambda) \quad (\lambda \in \mathbb{C} - \mathbb{R})$. The igenvalues (if any) of (3.0.1) can now be deduced (see §3.3) from the eros or the poles of $M_1(\lambda) - M_2(\lambda)$.

In the case of a discrete spectrum, we define as in (3.3.6) the function $\Phi : (a,b) \times (\mathbb{C} - \mathbb{R}) \times H^2_{p,q}(a,b) \to \mathbb{C}$ by

$$\Phi(x,\lambda,f) = -\int_a^b G(x,t,\lambda)w(t)f(t)dt \quad (x \in (a,b)); \qquad (3.4.8)$$

then an application of Titchmarsh's method (see §2.1) to (3.4.8) gives the eigenfunctions $\{\phi_n, n \in N_0\}$, associated with $M[y]$, the completeness of $\{\phi_n, n \in N_0\}$ in $H^2_{p,q}(a,b)$, and the series expansion of f in $H^2_{p,q}(a,b)$ in terms of these eigenfunctions.

We end this chapter with a complete limit-point limit-circle classification of the Jacobi, Laguerre and Hermite equations.

§3.5 Limit-point, Limit-circle Classifications

We recall from part (c) of remark 2.1.1 that limit-point, limit-circle classifications of the equation

$$M[y] := -(py')' + qy = \lambda wy \quad \text{on} \quad (a,b) ,$$

holds for all $\lambda \in \mathbb{C}$.

I We start with Jacobi's case:

Theorem 3.5.1

Consider Jacobi's equation

$$-((1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x))' + \frac{(1+\alpha+\beta)^2}{4}(1-x)^{\alpha}(1+x)^{\beta}y(x)$$

$$= \lambda(1-x)^{\alpha}(1+x)^{\beta}y(x) \qquad (3.5.1)$$

$$(x \in (-1,1),\ \alpha > -1,\ \beta > -1,\ \lambda \in \mathbb{C}),$$

where λ in (3.2.12) is replaced by $\lambda - (1+\alpha+\beta)^2/4$ and

$$\left.\begin{aligned} p(x) &= (1-x)^{\alpha+1}(1+x)^{\beta+1} \\ q(x) &= \frac{(\alpha+\beta+1)^2}{4}(1-x)^{\alpha}(1+x)^{\beta} \quad (x \in (-1,1)) \\ w(x) &= (1-x)^{\alpha}(1+x)^{\beta}\ . \end{aligned}\right\} \qquad (3.5.2)$$

Writing (3.5.1) as $M_J[y] = \lambda(1-x)^{\alpha}(1+x)^{\beta}y(x)$ (where the subscript J stands for Jacobi), we have the following:

(a) for the right-definite case, with $L^2_w(-1,1)$ as the underlying space

(i) $M_J[y]$ is regular at $x = +1(-1)$ if $-1 < \alpha < 0$ $(-1 < \beta < 0)$;

(ii) $M_J[y]$ is limit-circle at $x = +1(-1)$ if $0 \le \alpha < 1$ $(0 \le \beta < 1)$;

(iii) $M_J[y]$ is limit-point at $x = +1(-1)$ if $1 \le \alpha < \infty$ $(1 \le \beta < \infty)$.

(b) for the left-definite case, with $H^2_{p,q}(-1,1)$ as the underlying space

(i) $M_J[y]$ is regular at $x = +1$ (-1) if $-1 < \alpha < 0$ $(-1 < \beta < 0)$;

(ii) $M_J[y]$ is limit-point at $x = +1$ (-1) if $0 \le \alpha < \infty$ $(0 \le \beta < \infty)$.

Proof:

(a) In the light of Remark 2.1.1, put in (3.5.1) $\lambda = \frac{1}{4}(1+\alpha+\beta)^2$, then

$$((1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x))' = 0 \quad (x \in (-1,1)) \qquad (3.5.3)$$

with

$$y'(x) = k_1(1-x)^{-\alpha-1}(1+x)^{-\beta-1}$$

and

$$y(x) = k_1\int_0^x (1-t)^{-\alpha-1}(1+t)^{-\beta-1}dt + k_2$$

where $k_1, k_2 \in R$ and $x \in (-1,1)$.

Writing $y(x) = k_1y_1(x) + k_2y_2(x)$ as the general solution of (3.5.3), with

$$\left.\begin{aligned} y_1(x) &= \int_0^x (1-t)^{-\alpha-1}(1+t)^{-\beta-1}dt \quad (x \in (-1,1)) \\ y_2(x) &= 1 \quad (x \in (-1,1). \end{aligned}\right\} \tag{3.5.4}$$

We observe that the solutions y_1, y_2 of (3.5.3) are linearly independent.

Now let $x \to +1$, then

$$(1-x)^{\alpha+1}(1+x)^{\beta+1} \sim 2^{\beta+1}(1-x)^{\alpha+1} \ ;$$

and for some $k \in R_+$

$$|y_1(x)| \le k\int_0^x |(1-t)^{-\alpha-1}|dt \ ,$$

<u>i.e.</u>

$$y_1(x) = \begin{cases} 0((1-x)^{-\alpha}) & (x \to +1, \ \alpha \neq 0) \\ 0(\log(1-x)) & (x \to +1, \ \alpha = 0). \end{cases} \tag{3.5.5}$$

Similarly

$$w(x) \sim 2^{\beta}(1-x)^{\alpha} \quad (x \to +1) \ ; \tag{3.5.6}$$

hence, for $\alpha \neq 0$, and some $k_3 \in R_+$,

$$\int_{-1}^{1} w(x)|y_1(x)|^2 dx \le k_3 \int_{-1}^{1} (1-x)^{-\alpha} dx < \infty ,$$

if $\alpha < 1$ and $\alpha \neq 0$. If $\alpha = 0$, then $w(x) = O(1)$ $(x \to +1)$, and

$$\int_{-1}^{1} |y_1(x)|^2 dx \le k_4 \int_{-1}^{1} [\log(1-x)]^2 dx \qquad (k_4 \in R_+)$$

$$= k_4 \int_{-\infty}^{\log 2} u^2 e^u du < \infty ;$$

hence $y_1 \in L^2_w(-1,1)$ if $-1 < \alpha < 1$. Similarly

$$\int_{-1}^{1} w(x)|y_2(x)|^2 dx \le k_5 \int_{-1}^{1} (1-x)^{\alpha} dx \qquad (k_5 \in R_+)$$

$$< \infty \quad \text{if} \quad \alpha > -1 ;$$

hence y_1 and $y_2 \in L^2_w(-1,1)$ if $-1 < \alpha < 1$. It appears therefore that at $x = +1$, $M_J[y]$ is limit-circle if $-1 < \alpha < 1$, and limit-point if $1 \le \alpha < \infty$.

Now, from (3.5.6), we have $w \in L(-1,1)$, if $\alpha > -1$, because

$$\int_{-1}^{1} (1-x)^{\alpha} dx < \infty \quad \text{if} \quad \alpha > -1 ;$$

also $p^{-1}(x) \sim 2^{-\beta-1}(1-x)^{-\alpha-1}$ $(x \to +1)$, and

$$\int_{-1}^{1} (1-x)^{-\alpha-1} dx < \infty \quad \text{if} \quad \alpha < 0 ,$$

i.e. $p^{-1} \in L(-1,1)$ if $\alpha < 0$. Thus

$$p^{-1},\ w \text{ and } q \in L(-1,1) \quad \text{if} \quad -1 < \alpha < 0 \ . \qquad (3.5.7)$$

(Note that $q = \frac{1}{4}(1+\alpha+\beta)^2 w$ on (a,b).) Hence if

(i) if $-1 < \alpha < 0$, then $M_J[y]$ is <u>regular</u> at $x = +1$,

(ii) if $0 \le \alpha < 1$, then $M_J[y]$ is <u>limit-circle</u> at $x = +1$,

(iii) if $1 \le \alpha < \infty$, then $M_J[y]$ is <u>limit-point</u> at $x = +1$.

If we replace x with $-x$ (<u>i.e.</u> $x \to -1$) and α with β, we obtain the same classification for β; and this completes the proof for the right-definite case.

(b) Consider the left-definite case (<u>i.e.</u> in the space $H^2_{p,q}(-1,1)$). From (3.5.4),

$$y_1'(x) = (1-x)^{-\alpha-1}(1+x)^{-\beta-1} \sim 2^{-\beta-1}(1-x)^{-\alpha-1} \qquad (3.5.7)$$
$$(x \to +1)$$

and

$$y_2'(x) = 0 \ .$$

Similarly

$$p(x) \sim 2^{\beta+1}(1-x)^{\alpha+1} \quad (x \to +1) \qquad (3.5.8)$$

and

$$q(x) = \frac{1}{4}(1+\alpha+\beta)^2 w(x) \sim 2^{\beta-2}(1+\alpha+\beta)^2(1-x)^{\alpha} \qquad (3.5.9)$$
$$(x \to +1);$$

hence, if $\alpha \neq 0$, then (see (3.5.5)-(3.5.9)) for some $k \in R_+$

$$\int_{-1}^{1} \{p|y_1'|^2 + q|y_1|^2\}$$

$$= \int_{-1}^{1} \{(1-x)^{\alpha+1}(1+x)^{\beta+1}|y_1'(x)|^2 + \frac{(1+\alpha+\beta)^2}{4}(1-x)^{\alpha}(1+x)^{\beta}|y_1(x)|^2\}dx$$

$$\le k\int_{-1}^{1} \{\frac{(1-x)^{-\alpha-1}}{2^{\beta+1}} + \frac{(1+\alpha+\beta)}{2^{\beta+4}\alpha}(1-x)^{-\alpha}\}dx < \infty \quad \text{if} \quad \alpha < 0 \ ,$$

i.e.

$$y_1 \in H^2_{p,q}(-1,1) \quad \text{if} \quad \alpha < 0 \ . \tag{3.5.10}$$

Similarly

$$y_2 \in H^2_{p,q}(-1,1) \quad \text{if} \quad -1 < \alpha < \infty \ . \tag{3.5.11}$$

Also from (3.5.7), p^{-1}, q and $w \in L(-1,1)$ if $-1 < \alpha < 0$. Hence both (3.5.10) and (3.5.11) imply that

(i) if $-1 < \alpha < 0$, then $M_J[y]$ is regular at $x = +1$,

(ii) if $0 \le \alpha < \infty$, then $M_J[y]$ is limit-point at $x = +1$.

Replacing x with -x (i.e. $x \to -1$) gives the same classification in $H^2_{p,q}(-1,1)$ for β.

Put $\alpha = \beta = \nu - 1/2$, then (3.5.1) reduces to Gegenbauer's equation

$$-((1-x^2)^{\nu+1/2}y'(x))' + \nu^2(1-x^2)^{\nu-1/2}y(x) = \lambda(1-x^2)^{\nu-1/2}y(x) \tag{3.5.12}$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C},\ \nu > -1/2);$$

writing $M_G[y] = \lambda(1-x^2)^{\nu-1/2}y(x)$, we have:

Corollary 3.5.2

(a) For the right-definite case, with $L^2_w(-1,1)$ as the underlying space

(i) $M_G[y]$ is regular at $x = \pm 1$ if $-1/2 < \nu < 1/2$;

(ii) $M_G[y]$ is limit-circle at $x = \pm 1$ if $1/2 \le \nu < 3/2$;

(iii) $M_G[y]$ is limit-point at $x = \pm 1$ if $3/2 \le \nu < \infty$;

(b) however, for the left-definite case, with $H^2_{p,q}(-1,1)$ as the underlying space

(i) $M_G[y]$ is regular at $x = \pm 1$ if $-1/2 < \nu < 1/2$;

(ii) $M_G[y]$ is limit-point at $x = \pm 1$ if $1/2 \le \nu < \infty$.

If we put $\alpha = \beta = 0 = \nu - 1/2$, then (3.5.1) reduces to Legendre's equation

$$M_L[y] := -((1-x^2)y'(x))' + \frac{1}{4}y(x) = \lambda y(x) \qquad (3.5.11)$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C}).$$

Hence, from Theorem 3.5.1, we deduce the following:

Corollary 3.5.3

(a) $M_L[y]$ is limit-circle at $x = \pm 1$ in the right-definite case (i.e. in $L^2(-1,1)$); but

(b) $M_L[y]$ is limit-point at $x = \pm 1$ in the left-definite case (i.e. in $H^2_{p,q}(-1,1)$).

II We consider now Laguerre's equation

$$-(x^{\alpha+1}e^{-x}y'(x))' + \frac{\alpha+1}{2}x^{\alpha}e^{-x}y(x) = \lambda x^{\alpha}e^{-x}y(x) \qquad (3.5.12)$$

$$(x \in (0,\infty),\ \alpha > -1,\ \lambda \in \mathbb{C});$$

where λ in (3.2.14) is replaced by $\lambda - \frac{\alpha+1}{2}$, and

$$\left.\begin{aligned} p(x) &= x^{\alpha+1}e^{-x}, \\ q(x) &= \frac{\alpha+1}{2}w(x) = \frac{\alpha+1}{2}x^{\alpha}e^{-x} \quad (x \in (0,\infty),\ \alpha > -1) \end{aligned}\right\} \qquad (3.5.13)$$

Let $M[y](x) = \lambda x^{\alpha}e^{-x}y(x)$ $(x \in (0,\infty))$, then $M[y]$ has the following properties:

Theorem 3.5.4

(a) Consider the right-definite case (i.e. in $L_w^2(0,\infty)$), then $M[y]$ is

(i) regular at $x = 0$ if $-1 < \alpha < 0$;

(ii) limit-circle at $x = 0$ if $0 \leq \alpha < 1$;

(iii) limit-point at $x = 0$ if $1 \leq \alpha < \infty$;

(iv) limit-point at $x = +\infty$ for all $\alpha > -1$.

(b) For the left-definite case (i.e. in $H_{p,q}^2(0,\infty)$), $M[y]$ is

(i) regular at $x = 0$ if $-1 < \alpha < 0$;

(ii) limit-point at $x = 0$ if $0 \leq \alpha < \infty$;

(iii) limit-point at $x = +\infty$ for all $\alpha > -1$.

Proof: Using part (c) of Remark 2.1.1, let $\lambda = \frac{\alpha+1}{2}$; then

$$-(x^{\alpha+1}e^{-x}y'(x))' = 0 \qquad (x \in (0,\infty)),$$

$$y'(x) = k_1 x^{-\alpha-1}e^{x},$$

and

$$y(x) = k_1\int_1^x \frac{e^t dt}{t^{\alpha+1}} + k_2$$

$$= k_1 y_1(x) + k_2 y_2(x),$$

where $y_2(x) = 1$ and

$$y_1(x) = \int_1^x \frac{e^t dt}{t^{\alpha+1}} \tag{3.5.14}$$

and some $k_1, k_2 \in R$.

Now, if $\alpha > -1$, then $(x \in (0,\infty))$;

$$\int_1^x \frac{e^t dt}{t^{\alpha+1}} = \frac{e^t}{t^{\alpha+1}}\Big|_1^x + (\alpha+1)\int_1^x e^t\, t^{-\alpha-2} dt \quad , (\alpha > -1)$$

Hence

$$y_1(x) \geq \frac{e^x}{x^{\alpha+1}} - e \tag{3.5.15}$$

and

$$\int_1^\infty w|y_1|^2 \geq \int_1^\infty \frac{e^x dx}{x^{\alpha+2}} = \infty$$

i.e. $y_1 \notin L^2_w(1,\infty)$ and so $M[y]$ is limit-point at $x = \infty$, and this holds for all α.

Consider $x \to 0$, then, from (3.5.14), $x^{-\alpha-1}e^x \sim x^{-\alpha-1}$; and for some $k_0 \in R_+$

$$|y_1(x)| \leq k_0 \int_x^1 |t|^{-\alpha-1} dt,$$

i.e.

$$y_1(x) = \begin{cases} O(|x|^{-\alpha}) & (x \to 0,\ \alpha \neq 0) \\ O(\log x) & (x \to 0,\ \alpha = 0); \end{cases} \tag{3.5.16}$$

then if $\alpha \neq 0$ and some $k_1 \in R_+$

$$\int_0^\infty x^\alpha e^{-x} |x|^{-2\alpha} dx \leq k_1 \int_0^\infty x^{-\alpha} e^{-x} dx < \infty \quad \text{if} \quad \alpha < 1 .$$

Note that if $\alpha = 0$ then, for some $k_2 \in R_+$,

$$\int_0^\infty e^{-x} |\log x|^2 dx \leq k_2 \int_{-\infty}^\infty |u|^2 \exp[u - \exp[u]] du < \infty .$$

ence, in the neighbourhood of $x = 0$,

$$y_1 \in L_w^2(0,\infty) \quad \text{for} \quad -1 < \alpha < 1 ;$$

nd from (3.5.14), $y_2 \in L_w^2(0,\infty)$ if $\alpha > -1$; furthermore,

$$\int_0 p^{-1} = \int_0 x^{-\alpha-1} e^x dx < \infty \qquad \alpha < 0 , \tag{3.5.17}$$

and both w and $q \in L(0,\infty)$ at $x = 0$ if $\alpha > -1$. It follows then that at $x = 0$, $M[y]$ is regular if $-1 < \alpha < 0$, limit-circle if $0 \le \alpha < 1$, and limit-point if $1 \le \alpha < \infty$

(b) Now, from (3.5.14), $y_2'(x) = 0$ (pp $x \in (0,\infty)$) and $y_1'(x) = x^{-\alpha-1}e^x$ $(x \in (0,\infty))$. As in part (a), since

$$y_1'(x) = e^x x^{-\alpha-1} \tag{3.5.18}$$

it follows from this and (3.5.15) that $y_1 \notin H_{p,q}^2(1,\infty)$, and so $M[y]$ is <u>limit-point</u> at $x = \infty$ (in $H_{p,q}^2(1,\infty)$).

Also from (3.5.14) or (3.5.16),

$$y_1'(x) = O(|x|^{-\alpha-1}) \qquad (x \to 0); \tag{3.5.19}$$

hence, using this relation and (3.5.16), with some $k_5 \in R_+$,

$$\int_0^\infty \{p|y_1'|^2 + q|y_1|^2\} \le k_5 \int_0^\infty \{x^{\alpha+1}|x|^{-2\alpha-2} + \frac{\alpha+1}{2}x^{\alpha}|x|^{-2\alpha}\}e^{-x}dx \qquad (\alpha \ne 0)$$

$$= k_5 \int_0^\infty \{x^{-\alpha-1} + \frac{\alpha+1}{2}x^{-\alpha}\}e^{-x}dx$$

$$< \infty \quad \text{if} \quad \alpha < 0 ,$$

i.e.

$$\left.\begin{array}{ll} y_1 \in H^2_{p,q}(0,\infty) & \text{if} \quad \alpha < 0 \; ; \\ \text{also} \quad y_2 \in H^2_{p,q}(0,\infty) & \text{if} \quad \alpha > -1 \; . \end{array}\right\} \qquad (3.5.20)$$

We have seen from part (a) that p^{-1}, q and $w \in L(0,\infty)$ in the neighbourhood of $x = 0$ if $-1 < \alpha < 0$; thus we conclude that $M[y]$ is regular at $x = 0$ if $-1 < \alpha < 0$ and limit-point if $0 \le \alpha < \infty$. This completes the proof.

III Finally, we consider Hermite's equation

$$-(e^{-x^2} y'(x))' + e^{-x^2} y(x) = \lambda e^{-x^2} y(x) \qquad (3.5.21)$$

$$(x \in (-\infty,\infty),\ \lambda \in \mathbb{C});$$

where λ in (3.2.17) is replaced by $\lambda - 1$, and

$$p(x) = q(x) = w(x) = e^{-x^2} \quad (x \in (-\infty,\infty)). \qquad (3.5.22)$$

Let $M_H[y](x) = \lambda e^{-x^2} y(x)$ (where H stands for Hermite); we obtain the following result:

Theorem 3.5.5

$M_H[y]$ is limit-point at $x = \pm\infty$ in both right- and left-definite cases, i.e. in $L^2_w(-\infty,\infty)$ and $H^2_{p,q}(-\infty,\infty)$, respectively.

Proof: Consider the right-definite case; appealing to part (c) of Remark 2.1.1, we put $\lambda = 1$, then

$$(e^{-x^2} y'(x))' = 0 \quad (x \in (-\infty,\infty)), \qquad (3.5.23)$$

and

$$y'(x) = k_1 e^{x^2} \quad (k \in R).$$

nce

$$y(x) = k_1 \int_0^x e^{t^2}\,dt + k_2$$

$$= k_1 y_1(x) + k_2 y_2(x) ,$$

here

$$\left.\begin{aligned} y_1(x) &= \int_0^x e^{t^2}\,dt \\ \text{and} \quad y_2(x) &= 1 \quad (x \in (-\infty,\infty)) \end{aligned}\right\} \qquad (3.5.24)$$

here y_1 and y_2 are linearly independent solutions of (3.5.23). It is clear from this relation that $y_2 \in L^2_w(-\infty,\infty)$, but $y_1 \notin L^2_w(-\infty,\infty)$, and therefore proves the first part.

Also from (3.5.24), $y_1'(x) = e^{x^2}$ $(x \in (-\infty,\infty))$, and $y_2'(x) = 0$ (pp $x \in (-\infty,\infty)$); hence

$$\int_{-\infty}^{\infty} \{p|y_2'|^2 + q|y_2|^2\} = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} ,$$

i.e. $y_2 \in H^2_{p,q}(-\infty,\infty)$. On the other hand,

$$\int_0^{\infty} \{p|y_1'|^2 + q|y_1|^2\} = \int_0^{\infty} \{e^{x^2} + e^{-x^2}|y_1(x)|^2\}dx$$

$$= \infty ,$$

i.e. $y_1 \notin H^2_{p,q}(0,\infty)$, similarly $y_1 \notin H^2_{p,q}(-\infty,0)$. It follows from these that $M_H[y]$ is limit-point at $x = \pm\infty$; this completes the proof.

CHAPTER FOUR

DIFFERENTIAL OPERATORS

§4.0 Preliminary

This chapter is concerned with the generation of a self-adjoint operator T in an integrable-square Hilbert function space, from a boundary value problem associated with the differential equation

$$M[y] := -(py')' + qy = \lambda wy \quad \text{on} \quad [a,b) \ . \qquad (3.0.1)'$$

We assume in this chapter that a is a regular end-point, b is a singular (finite or infinite) end-point, and the coefficients p,q,w satisfying the conditions (3.1.2). In addition, if $-\infty < a < b < \infty$ and b is regular or singular, then we assume $p', q, w \in C[a,b)$, $p > 0$ on $[a,b)$, $p(b) = 0$, $q \geq 0$ and $w \geq 0$ on $[a,b]$, with

$$\int^{b} p^{-1} = \infty \ , \quad \int_a^b q > 0 \quad \text{and} \quad \int_a^b w > 0 \ . \qquad (4.0.1)$$

We consider first the right-definite case, and for this we require the necessary boundary conditions for determining the domain of T. The operator T in this case is determined directly by relating it to (3.1.1) (see (4.1.8) below). This, however, is not possible in the left-definite case, where $H^2_{p,q}[a,b)$ is the underlying Hilbert space (see definition 3.4.1). Instead, we use an idea of Everitt (1972) and Shotwell (1971) which involves establishing the existence of T (in $H^2_{p,q}[a,b)$ and hence in $L^2_w[a,b)$), as the inverse of a bounded self-adjoint operator A, defined on the whole of $H^2_{p,q}[a,b)$ and such that the range of A is not dense in $H^2_{p,q}[a,b)$.

To begin with, we establish the Green's formula for our case.

Let f and g be two complex-valued functions satisfying (3.0.1)', and for $i = 0,1$, let $f^{[i]}$ and $g^{[i]}$ be the quasi-derivatives defined in §3.1, with

$$f^{[i]}, g^{[i]} \in AC_{loc}[a,b) \; , \qquad (4.0.1)'$$

such that f, $w^{-1}M[f]$, g and $w^{-1}M[g] \in L^2_w[a,b)$. Then

$$\int_a^X \{M[f]\overline{g} - f\overline{M[g]}\} = [f,g](X) - [f,g](a) \qquad (X \in [a,b)) \qquad (4.0.2)$$

where $[f,g](\cdot)$ is a sesquilinear form on $[a,b)$ defined by

$$\begin{aligned} [f,g](x) &= f^{[0]}(x)\overline{g}^{[1]}(x) - f^{[1]}(x)\overline{g}^{[0]}(x) \qquad (x \in [a,b)) \\ &= f(x)(p\overline{g}')(x) - (pf')(x)\overline{g}(x) \\ &= (f.p\overline{g}' - pf'.\overline{g})(x) \qquad (x \in [a,b)). \end{aligned}$$

It follows that $[f,g](x) = \overline{-[g,f](x)}$ $(x \in [a,b))$.

Now since the integral in (4.0.2) has a limit value for $X \to b-$, we infer from (4.0.2) that $\lim_{X \to b-}[f,g](X)$ also exists. Define

$$[f,g](b) := \lim_{X \to b-}[f,g](X), \qquad (4.0.3)$$

then for all f and g satisfying (4.0.1) such that f, $w^{-1}M[f]$, g and $w^{-1}M[g] \in L^2_w[a,b)$ we have

$$\int_a^b \{M[f]\overline{g} - f\overline{M[g]}\} = [f,g](b) - [f,g](a) \; ; \qquad (4.0.4)$$

and this is known as <u>Green's formula</u>.

§4.1 The right-definite case

Consider now the differential equation (3.0.1)' above, i.e.

$$M[y] := -(py')' + qy = \lambda wy \quad \text{on} \quad [a,b) .$$

At the singular end-point b, M[y] is either limit-point or limit-circle (see §2.1). In these two cases we select appropriate boundary conditions and these then lead to the determination of the domains of self-adjoint operators generated by M[y] in the Hilbert function space $L_w^2[a,b)$. We begin with the limit-point case.

I The limit-point case

Let the linear manifold Δ of $L_w^2[a,b)$ be defined as follows:

$$\Delta = \{f \in L_w^2[a,b) : f, pf' \in AC_{loc}[a,b) \text{ and } w^{-1}M[f] \in L_w^2[a,b)\} . \tag{4.1.1}$$

Clearly Δ depends on the coefficients p, q and w. When $f,g \in \Delta$, then it is known, from Green's formula, that the limit

$$\lim_{b-}[f,g](\cdot) = \lim_{b-}\{f.p\overline{g}' - pf'.\overline{g}\} \tag{4.1.2}$$

exists and is finite.

Lemma 4.1.1

A necessary and sufficient condition for M[y] to be limit-point at b is that

$$\lim_{b-}[f,g](\cdot) = 0 \quad (f,g \in \Delta) . \tag{4.1.3}$$

Proof: See Everitt (1963) or Naimark (1967; §17.4) who prove the result for the unit weight function $w(x) = 1$ $(x \in [a,b))$; however, the extension to general weight function w presents no additional difficulty (see, for instance, Onyango-Otieno (1978; §2.1)).

At the end-point a, we impose a regular boundary condition of the type

$$\left.\begin{aligned} & f(a)\cos\alpha + (pf')(a)\sin\alpha = 0 \quad (\alpha \in (-\pi/2,\pi/2]) \\ & \text{i.e. (see (3.3.1))}, [f,\phi](a) = 0 \ . \end{aligned}\right\} \tag{4.1.4}$$

In the limit-point case, this boundary condition is sufficient to determine the domain of a self-adjoint operator.

Now to determine a self-adjoint operator for M[y] let Δ_p, a subspace of the Hilbert space $L_w^2[a,b)$, be defined by

$$\Delta_p = \{f \in \Delta : f(a)\cos\alpha + (pf')(a)\sin\alpha = 0 \text{ or, equivalently, } [f,\phi](a) = 0\} \ . \tag{4.1.5}$$

Consider the function $\Phi = \Phi(x,\lambda;f)$ defined as follows (see (2.1.10)):

$$\begin{aligned} \Phi(x,\lambda;f) &= \psi(x,\lambda)\int_a^x w(t)\phi(t,\lambda)f(t)dt + \phi(x,\lambda)\int_x^b w(t)\psi(t,\lambda)f(t)dt \\ &= -\int_a^b w(t)G(x,t,\lambda)f(t)dt \quad (f \in L_w^2[a,b)), \end{aligned} \tag{4.1.6}$$

i.e.

$$G(x,t,\lambda) = \begin{cases} -\psi(x,\lambda)\phi(t,\lambda) & (t \le x) \\ -\phi(x,\lambda)\psi(t,\lambda) & (x < t) \end{cases} \tag{4.1.7}$$

$(x \in [a,b),\ \lambda \in \mathbb{C} - R)$; then it can be shown that Φ has the following properties:

Theorem 4.1.2

(a) Φ satisfies the non-homogeneous equation

$$M[\Phi] = \lambda w\Phi + wf \quad \text{on} \quad [a,b);$$

(b) $[\Phi,\phi](a) = 0;$

(c) for $f \in L^2_w[a,b)$, $\Phi(\cdot,\lambda,f) \in L^2_w[a,b);$

(d) $w^{-1}M[\Phi] \in L^2_w[a,b).$

(Note that (b), (c) and (d) imply that $\Phi \in \Delta_p$.)

Proof: This is similar to the proof given in Titchmarsh (1962; §1.4, 2.6, 2.8) for the Liouville normal form of (3.0.1); so we shall omit it.

Now let T_p (where p stands for limit-point) denote a linear operator on the Hilbert space $L^2_w(a,b)$, with the domain

$$D(T_p) := \Delta_p$$

and

$$T_p f := w^{-1}M[f] \quad (f \in D(T_p)). \tag{4.1.8}$$

Then the following properties hold:

Theorem 4.1.3

(a) $D(T_p)$ is dense in $L^2_w[a,b)$, i.e.

$$\overline{D(T_p)} = L^2_w[a,b) \ ;$$

(b) $(T_p f,g)_w = (f,T_p g)_w \quad (f,g \in D(T_p));$

(a) and (b) together imply that T_p is a symmetric operator)

(c) T_p is self-adjoint.

Proof:

(a) Let $\Delta_{p,0}$ be the set of all functions $f \in \Delta_p$ with a compact support $[\xi,\eta] \subset [a,b)$ $(a \le \xi)$,

(this interval may be different for each function).

Let

$$T_{p,0} = T_p \Big|_{\Delta_{p,0}},$$

such that

$$D(T_{p,0}) = \Delta_{p,0}$$

and

$$T_{p,0}f = T_p f = w^{-1}M[f] \quad (f \in D(T_{p,0}));$$

then

$$T_{p,0} \subset T_p,$$

and

$$D(T_{p,0}) \subset D(T_p) \subset L^2_w[a,b).$$

We claim that $D(T_{p,0})$ is dense in $L^2_w[a,b)$; for this, let $h \in L^2_w[a,b)$ such that h is orthogonal to $D(T_{p,0})$, i.e.

$$(h,f)_w = 0 \quad (f \in D(T_{p,0})).$$

Also let $[m,n] = \square$ be a fixed, closed interval contained entirely within $[a,b)$; and let $D_{\square,0}(T_p) = \{f \in D(T_p) : f^{[i]}(m) = f^{[i]}(n) = 0, i = 0,1\}$. Then every $f \in D_{\square,0}(T_p)$ can be regarded as an element of $D(T_{p,0})$, hence h is orthogonal to $D_{\square,0}(T_p)$ i.e.

$$(h,f)_w = 0 \quad (f \in D_{\square,0}(T_p)).$$

However

$$\overline{D_{\square,0}(T_p)} = L^2_w[m,n]$$

(see Naimark §17.3, property III); consequently

$$h(x) = 0 \quad \text{a.e. in } [m,n];$$

and since the interval $[m,n] \subset [a,b)$ is arbitrary, it follows that $h(x) = 0$ a.e. in $[a,b)$. Hence $D(T_{p,0})$ is dense in $L^2_w[a,b)$ and this implies $D(T_p)$ is dense in $L^2_w[a,b)$.

(b) Let $f,g \in D(T_p)$; then

$$(T_p f,g)_w = \int_a^b wT_p f\bar{g}$$

$$= \int_a^b ww^{-1}M[f]\bar{g} \quad \text{(see (4.1.8))}$$

$$= \int_a^b M[f]\bar{g}$$

$$= \lim_{b-}[f,g](\cdot) - [f,g](a) + \int_a^b f\cdot\overline{M[g]} .$$

Since $M[y]$ is limit-point at b, we obtain from Lemma 4.1.1 that $\lim_{b-}[f,g](\cdot) = 0$. Also, since a is a regular end-point, it follows from (4.1.4) after some rearrangements that $[f,g](a) = 0$. Hence

$$(T_p f,g)_w = \int_a^b f\cdot\overline{M[g]}$$

$$= \int_a^b wf\overline{T_p g},$$

i.e.

$$(T_p f, g)_w = (f, T_p g)_w;$$

i.e. T_p is a symmetric operator.

(c) For this part, we appeal to a criterion for self-adjointness given in Theorem 2 and Definition (1) of Hellwig's book (1967; Chapter 10), viz. let A and A* be linear operators with D(A) and D(A*) as their respective domains in a Hilbert space H. Then A is self-adjoint, i.e. A = A* if and only if

(i) A is symmetric; and

(ii) $(A \pm iI)D(A) = H$;

i.e. the range of $(A \pm iI)$ is the whole of H (here A* is the adjoint of A, $i \in \mathbb{C} - \mathbb{R}$ and I is the identity operator).

From the property (a) it is clear that

$$(T_p \pm iI)D(T_p) \subseteq L^2_w[a,b).$$

Now let $f \in L^2_w[a,b)$; we need to exhibit an element $g \in D(T_p)$ such that

$$(T_p - iI)g = f.$$

Let

$$g = \Phi(\cdot, i; f)$$

where Φ is defined by equation (4.1.6) (with $\lambda = i$). Then from part (a) of Theorem 4.1.2

$$w^{-1}M[g] = ig + f \quad \text{on } [a,b)$$

or

$$T_p g = ig + f \quad \text{(see (4.1.8))}$$

i.e.

$$(T_p - iI)g = f.$$

Similarly for $T_p + iI$. Hence

$$L^2_w[a,b) \subseteq (T_p \pm iI)D(T_p);$$

and so

$$(T_p \pm iI)D(T_p) = L^2_w[a,b);$$

thus T_p is a self-adjoint (unbounded) differential operator in the space $L^2_w[a,b)$.

Remark 4.1.4

Since T_p is self-adjoint, all the eigenvalues - if any - of T_p are real; and if λ is an eigenvalue, then the eigenvectors of T_p are the solutions of the differential equation

$$M[y] = \lambda wy$$

which belong to $L^2_w[a,b)$, and which satisfy the boundary condition (4.1.4) at a.

II The limit-circle case

Let λ be any fixed point in $\mathbb{C} - \mathbb{R}$ (i.e. $\text{im}[\lambda] \neq 0$). Consider a particular point $m(\lambda)$ on the limit circle corresponding to λ. Then the corresponding solution

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

of

$$M[f] = \lambda wf \quad \text{on} \quad [a,b)$$

satisfies the condition

$$[\psi(\cdot,\lambda),\psi(\cdot,\lambda)](b) = 0 \quad \text{(see equation (2.1.9))}.$$

Now at a, we impose a regular boundary condition of the form (4.1.4), i.e.

$$[f,\phi](a) = 0,$$

and at the singular end-point b we impose the Titchmarsh-Naimark type of boundary conditions:

$$\lim_{x\to b-}[f,\psi(\cdot,\lambda)](x) = 0 \,. \tag{4.1.9}$$

Note that, unlike the limit-point case, here we need the two boundary conditions in order to determine the required self-adjoint operator.

Let Δ_c denote the class of functions in Δ defined as follows:

$$\Delta_c = \{f \in \Delta : [f,\phi(\cdot,\lambda)](a) = 0;\ \lim_{x\to b-}[f,\psi(\cdot,\lambda)](x) = 0\} \tag{4.1.10}$$

(here c stands for limit-circle), and consider the function Φ defined by equation (4.1.6). Then it can be shown, as in Theorem 4.1.2, that Φ has the following properties:

Theorem 4.1.5

(a) Φ is a solution of the equation

$$M[\Phi] = \lambda w\Phi + wf \text{ on } [a,b);$$

(b) $\Phi \in \Delta_c$ if $f \in L^2_w[a,b)$.

Now let T_c be a linear operator on the Hilbert space $L^2_w(a,b)$, with the domain

$$D(T_c) = \Delta_c,$$

and such that

$$T_c f := w^{-1}M[f] \quad (f \in D(T_c)); \tag{4.1.11}$$

then the following properties hold:

Theorem 4.1.6

(a) $D(T_c)$ is dense in $L^2_w(a,b)$;

(b) $(T_c f, g) = (f, T_c g) \quad (f,g \in D(T_c))$;

(c) T_c is a self-adjoint operator.

Proof: This is similar to that of Theorem 4.1.3, so we shall omit it. See also Remark 4.1.4.

§4.2 The left-definite case

Consider the equation (3.0.1)':

$$M[y] = -(py')' + qy = \lambda wy \quad \text{on} \quad [a,b) \quad (\lambda \in \mathbb{C})$$

and let $H^2_{p,q}[a,b)$ denote the integrable-square function space defined as in §3.4 by

$$H^2_{p,q}[a,b) = \{f : [a,b) \to \mathbb{C} : f \in AC_{loc}[a,b), \; q^{1/2}f \text{ and } p^{1/2}f' \in L^2[a,b)\};$$

then, as in the right-definite case (see §4.1), there is a similar theory of self-adjoint operators in $H^2_{p,q}[a,b)$ generated by $M[y]$; however, unlike the case considered in §4.1, it is not possible to give the explicit form of the differential operators in this case. We shall give some details for specific examples (viz. Legendre, Gegenbauer, Laguerre and Hermite) in later chapters. In this section, we consider the possibility of extending a method adopted in Everitt's paper (1972).

Now let the coefficients p, q and w of the differential equation

$$M[y] = \lambda wy \quad \text{on} \quad [a,b)$$

be real-valued on [a,b) such that

(i) $p^{-1} \in L_{loc}[a,b)$ and $p > 0$ on $[a,b)$; (4.2.1)

(ii) $q \in L_{loc}[a,b)$ and $q \geq 0$ on $[a,b)$; (4.2.2)

(iii) $w \in L_{loc}[a,b)$ and in our case $w > 0$ so that

$$0 < \int_a^b w \leq \infty \; ; \tag{4.2.3}$$

(iv) for some constant $k \in R_+$

$$w \leq kq \quad \text{on} \quad [a,b).$$

Note that in general w may not be of one sign on [a,b); in this case we write above inequalities with $|w|$ (see Everitt (1972; §2)).

Now with (4.2.1), (4.2.2) and (4.2.3) satisfied, it follows that the standard existence theorems may be applied to M[y] to obtain solutions defined on [a,b).

Guided by Everitt (1972; §2), we let θ and ϕ be the solutions of M[y] such that

$$\theta(a,\lambda) = 0, \; p(a)\theta'(a,\lambda) = -1 \tag{4.2.4}$$

$$\phi(a,\lambda) = 1, \; p(a)\phi'(a,\lambda) = 0; \tag{4.2.5}$$

then θ and ϕ are linearly independent solutions of M[y] on [a,b).

Now let f be a complex-valued function defined on $[a,b)$ such that $f' \in AC_{loc}[a,b)$ and

$$p(a)f'(a) = 0. \tag{4.2.6}$$

(Note that this is a special case of the boundary condition (4.1.4) in which $\alpha = \pi/2$; see also Everitt (1972; §2).) Also we recall from §3.4 that the space $H^2_{p,q}[a,b)$ is endowed with the scalar product

$$(f,g)_H = \int_a^b \{pf'\overline{g}' + qf\overline{g}\} \qquad (f,g \in H^2_{p,q}[a,b)) \tag{4.2.7}$$

and such that

$$\| f \|^2_H = \int_a^b \{p|f'|^2 + q|f|^2\} < \infty . \tag{4.2.8}$$

Let $\underline{B}$ denote a classical boundary value problem on $[a,b)$, which may be determined by requiring the solutions of the differential equation (3.0.1)' to satisfy the boundary condition (4.2.6) at a, and the "boundary condition" (4.2.8) at b (i.e. the solutions should belong to $H^2_{p,q}[a,b)$). Then, according to Everitt (1972), $\underline{B}$ may be characterised by a uniquely determined unbounded self-adjoint operator. We state this formally as follows:

Theorem 4.2.1

Let the real-valued coefficients p, q and w of the differential equation (3.0.1)', viz.

$$M[y] = -(py')' + qy = \lambda wy \quad \text{on} \quad [a,b)$$

satisfy the basic conditions (4.2.1), (4.2.2) and (4.2.3); and, in addition, let these coefficients satisfy

$$\int_a^b p^{-1} = \infty \quad \text{or} \quad \int_a^b q = \infty \tag{4.2.9}$$

and for some positive number k

$$w(x) \leq kq(x) \quad (x \in [a,b)) \tag{4.2.10}$$

(recall that in our case $w > 0$ on $[a,b)$); then there is a uniquely determined operator

$$S : D(S) \subset H^2_{p,q}[a,b) \to H^2_{p,q}[a,b)$$

with the following properties:

(i) the domain $D(S) \subset \{f \in H^2_{p,q}[a,b) : f' \in AC_{loc}[a,b)\}$;

(ii) S is a self-adjoint, unbounded operator in $H^2_{p,q}[a,b)$;

(iii) S^{-1} exists and is a bounded operator defined on the whole of $H^2_{p,q}[a,b)$;

(iv) for all $f \in H^2_{p,q}[a,b)$

$$M[S^{-1}f] = wf \quad \text{a.e. on } [a,b);$$

(v) S represents the boundary value problem B in the sense given above.

Proof: This is similar to that of Everitt (1972) in which $a = 0$, $b = \infty$ (see also the remarks preceding his proof), so we shall omit it.

Remark 4.2.2

(i) While this theorem holds for one singular end-point (finite or infinite), an examination of the results of Everitt (1972) and Atkinson-Everitt-Ong (1974; §7 and 9) shows that it is possible to extend the above results to the case in which both end-points a and b of the interval (a,b) are singular, provided that p, q and w satisfy the conditions (4.2.1), (4.2.2) and (4.2.3) on (a,b), and

$$\left.\begin{array}{l} \displaystyle\int_a^c p^{-1} = \infty \quad \underline{\text{and}} \quad \int_c^b p^{-1} = \infty \, , \\ \\ \underline{\text{or}} \\ \\ \displaystyle\int_a^c q = \infty \quad \underline{\text{and}} \quad \int_c^b q = \infty \quad ; \end{array}\right\} \qquad (4.2.11)$$

where c is a point in (a,b) at which the initial condition of the type (4.2.6) is satisfied, i.e.

$$p(c)f'(c) = 0 \, . \qquad (4.2.12)$$

(ii) In the remaining chapters we intend to obtain the operator S, by first constructing its inverse S^{-1} (see part (iii) of Theorem 4.2.1), making use of the solutions of the differential equations of the type (3.0.1).

CHAPTER FIVE

THE LEGENDRE EXPANSION

§5.0 Preliminary

One example of the symmetric differential equation (3.0.1) is Legendre's equation, which for our purposes is written as

$$M[y](x) = -((1-x^2)y'(x))' + \frac{1}{4}y(x) = \lambda y(x) \tag{5.0.1}$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C})$$

where $a = -1$, $b = 1$ and

$$p(x) = (1-x^2),\ q(x) = \frac{1}{4},\ w(x) = 1 \quad (x \in (-1,1)). \tag{5.0.2}$$

Written in this form, we can then study (5.0.1) both in the right- and left-definite cases, i.e. in the Hilbert spaces $L^2(-1,1)$ and $H^2_{p,q}(-1,1)$ respectively; where in the latter case we recall from (3.4.4) that

$$\int_{-1}^{1} \{(1-x^2)|f'(x)|^2 + \frac{1}{4}|f(x)|^2\}dx < \infty \quad (f \in H^2(-1,1)). \tag{5.0.3}$$

Our purpose in this chapter is to give a brief summary of the method of defining self-adjoint operators T and S in $L^2(-1,1)$ and $H^2_{p,q}(-1,1)$, respectively, which have the Legendre polynomials as eigenvectors. For the details involved in this analysis, see Everitt (1980). We shall extend this analysis of Everitt in the next chapter, to the case of the Gegenbauer differential equation which includes Legendre's equation (5.0.1) as a special case.

§5.1 The solutions of Legendre's equation

Again consider Legendre's equation (5.0.1). With the analysis of Titchmarsh (1962; §4.3-4.5) in mind, we write

$$s = \sqrt{\lambda} = \exp[\tfrac{1}{2}\ln(|\lambda|) + \tfrac{i}{2}\arg(\lambda)] \ ,$$

i.e. $s^2 = \lambda$, and determine the $\sqrt{(\cdot)}$ function by requiring

$$0 \le \arg s < \pi \quad \text{when} \quad 0 \le \arg \lambda < 2\pi \ . \tag{5.1.1}$$

Then two solutions of (5.0.1) may be determined by the following integrals of Dirichlet-Mehler type (see (1.5.4)):

$$Y(x,\lambda) = \int_{-\cos^{-1}x}^{\cos^{-1}x} \frac{\cos st\, dt}{(\cos t - x)^{1/2}} \quad (x \in (-1,1),\ \lambda \in \mathbb{C}) \tag{5.1.2}$$

$$Z(x,\lambda) = \int_{-(\pi/2+\sin^{-1}x)}^{\pi/2+\sin^{-1}x} \frac{\cos st\, dt}{(\cos\ t + x)^{1/2}} \tag{5.1.3}$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C}),$$

where in each case the positive square root is implied; and the inverse trigonometric functions $\cos^{-1}$, $\sin^{-1}$ are determined by requiring $\cos^{-1}x$ to decrease from π to 0, and $\sin^{-1}x$ to increase from $-\pi/2$ to $\pi/2$, as in both cases x increases from -1 to 1.

The proof that Y and Z satisfy (5.0.1) (see §6.1) involves writing (5.1.2) and (5.1.3) as well-defined contour integrals, after the manner of Titchmarsh (1962; §4.5).

Now, the initial values of Y and Z at $x = 0$ may be calculated on using Lemma 4.4 in Titchmarsh (1962; §4.4) to give, for all $\lambda \in \mathbb{C}$ (recall that $\lambda = s^2$),

$$Y(0,\lambda) = \frac{2^{1/2}\pi^{3/2}}{\Gamma(\frac{3}{4}+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}s)} = Z(0,\lambda) \qquad (5.1.4)$$

$$Y'(0,\lambda) = -\frac{2^{5/2}\pi^{3/2}}{\Gamma(\frac{1}{4}+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}s)} = -Z'(0,\lambda) \;. \qquad (5.1.5)$$

We note from these results that Y and Z are linearly independent, except when $Y(0,\lambda) = 0$ or $Y'(0,\lambda) = 0$, i.e. when

$$\lambda = (n+\tfrac{1}{2})^2 \;, \; n = 0,1,2,\ldots \;. \qquad (5.1.6)$$

Now following the existence theorem in Titchmarsh (1962; §1.5-1.6), we may form the solutions θ and ϕ of Legendre's equation (5.0.1) by

$$\theta(x,\lambda) = \frac{Y(x,\lambda)+Z(x,\lambda)}{2Y(0,\lambda)} \quad (x \in (-1,1),\ \lambda \in \mathbb{C}) \qquad (5.1.7)$$

and

$$\phi(x,\lambda) = \frac{Y(x,\lambda)-Z(x,\lambda)}{2Y'(0,\lambda)} \quad (x \in (-1,1),\ \lambda \in \mathbb{C}). \qquad (5.1.8)$$

Then θ and ϕ satisfy the following initial conditions

$$\left.\begin{aligned} \theta^{[0]}(0,\lambda) &= 1 \;, \quad \theta^{[1]}(0,\lambda) = 0 \\ \phi^{[0]}(0,\lambda) &= 0 \;, \quad \phi^{[1]}(0,\lambda) = 1 \;. \end{aligned}\right\} \qquad (5.1.9)$$

Furthermore, the Wronskian $p(0)W[\theta(\cdot,\lambda),\phi(\cdot,\lambda)](0) = 1$, i.e. θ and ϕ are linearly independent for all $\lambda \in \mathbb{C}$.

Since in the right-definite case the differential expression $M[y] = -((1-x^2)y'(x)) + \frac{1}{4}q(x)$ $(x \in (-1,1))$ is limit-circle at 1 and -1, a particular choice of the Titchmarsh-Weyl m-coefficients $m(\cdot)$ and $n(\cdot)$ (see Everitt (1980)) gives the L^2-solutions

$$\left.\begin{aligned} \psi(x,\lambda) &= \theta(x,\lambda) + m(\lambda)\phi(x,\lambda) \\ &\text{and} \\ \chi(x,\lambda) &= \theta(x,\lambda) + n(\lambda)\phi(x,\lambda) \end{aligned}\right\} (x \in (-1,1),\ \lambda \in \mathbb{C} - \mathbb{R}) \quad (5.1.10)$$

where $\psi(\cdot,\lambda) \in L^2(0,1)$, $\chi(\cdot,\lambda) \in L^2(-1,0)$, with

$$m(\lambda) = \frac{Y'(0,\lambda)}{Y(0,\lambda)} = -\frac{4\Gamma(\frac{3}{4}+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}s)}{\Gamma(\frac{1}{4}+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}s)} \quad (5.1.11)$$

and

$$n(\lambda) = \frac{Z'(0,\lambda)}{Z(0,\lambda)} = -m(\lambda) \quad \text{(see §6.4)}. \quad (5.1.12)$$

Putting these values of $m(\cdot)$ and $n(\cdot)$ in (5.1.10) gives

$$\left.\begin{aligned} \psi(x,\lambda) &= \frac{Y(x,\lambda)}{Y(0,\lambda)} \in L^2(0,1) \\ &\text{and} \\ \chi(x,\lambda) &= \frac{Z(x,\lambda)}{Z(0,\lambda)} \in L^2(-1,0)\ . \end{aligned}\right\} \quad (5.1.13)$$

In the left-definite case, however, with $H^2_{p,q}(-1,1)$ as the underlying space, the Titchmarsh-Weyl coefficients are unique since $M[y]$ is limit-point at 1 and -1. The associated m-coefficients $\tilde{m}(\cdot)$ and $\tilde{n}$ (we use the tilda notation to distinguish the left-definite case) turn out to be identical with $m(\cdot)$ and $n(\cdot)$ given above; furthermore, as in the right-definite case,

$$\left.\begin{aligned} \tilde{\psi}(x,\lambda) &= \frac{Y(x,\lambda)}{Y(0,\lambda)} \ (= \psi(x,\lambda)) \in H^2_{p,q}(0,1) \\ &\text{and} \\ \tilde{\chi}(x,\lambda) &= \frac{Z(x,\lambda)}{Z(0,\lambda)} \ (= \chi(x,\lambda)) \in H^2_{p,q}(-1,0)\ . \end{aligned}\right\} \quad (5.1.14)$$

Consider the Green's function given by

$$G(x,t,\lambda) = \begin{cases} -\dfrac{\psi(x,\lambda)\chi(t,\lambda)}{\{pW(\psi,\chi)\}(\lambda)} & (-1 < t < x < 1) \\ -\dfrac{\chi(x,\lambda)\psi(t,\lambda)}{\{pW(\psi,\chi)\}(\lambda)} & (-1 < x < t < 1) \end{cases} \tag{5.1.15}$$

where ψ and χ are given as above and

$$\{pW(\psi,\chi)\}(\lambda) = p(x)(\psi(x,\lambda)\chi'(x,\lambda)-\psi'(x,\lambda)\chi(x,\lambda)) \quad (x \in (-1,1))$$

$$= n(\lambda) - m(\lambda) \quad (\lambda \in \mathbb{C} - R) \quad \text{(see (3.3.5))}.$$

From the general theory of differential equations of the type (5.0.1), with two singular end-points (see Titchmarsh (1962; §2.18)), it is known that the eigenvalues of Legendre's equation (5.0.1) are given by the zeros and poles of $n(\cdot) - m(\cdot)$. In this case

$$n(\lambda) - m(\lambda) = -2m(\lambda) = \frac{8\Gamma(\frac{3}{4}+\frac{1}{2}s)\Gamma(\frac{3}{4}-\frac{1}{2}s)}{\Gamma(\frac{1}{4}+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}s)}$$

and a calculation shows that the eigenvalues are

$$\lambda_n = (n+\tfrac{1}{2})^2 \quad (n \in N_0). \tag{5.1.16}$$

Anticipating the definition of the operator T below, let $P\sigma(T)$ denote the set of eigenvalues (5.1.16), <u>i.e.</u>

$$P\sigma(T) = \{(n+\tfrac{1}{2})^2,\ n \in N_0\}\ . \tag{5.1.17}$$

Note that $m(\cdot)$ and $n(\cdot)$ are regular at $\lambda = 0$.

Following the analysis due to Titchmarsh (1962; §4.5), it may be shown that the eigenfunctions $\{\psi_n,\ n \in N_0\}$, corresponding to (5.1.17), are given by

$$\psi_n(x) = (n+\tfrac{1}{2})^{1/2}P_n(x) \quad (x \in [-1,1],\ n \in N_0) \tag{5.1.18}$$

where $\{P_n,\ n \in N_0\}$ are the Legendre polynomials (see §1.4).

Leaving the above classical study of Legendre's equation (5.0.1), we wish now to look at its associated differential operators both in the left- and right-definite cases.

§5.2 The Differential Operators T and S

We begin with the left-definite case.

I The operator S

Let the function $\Phi : (-1,1) \times (\mathbb{C} - \mathbb{R}) \times H^2_{p,q}(-1,1) \to \mathbb{C}$ be defined by (see (2.1.10))

$$\Phi(x,\lambda;f) = -\int_{-1}^{1} G(x,t,\lambda)f(t)dt \tag{5.2.1}$$

where G is given by (5.1.15); or

$$\Phi(x,0;f) = -\int_{-1}^{1} G(x,t,0)f(t)dt \tag{5.2.2}$$

(G is regular at $\lambda = 0$ because $n(\cdot)$ and $m(\cdot)$ are regular at $\lambda = 0$). Then Φ has the following properties (see Everitt (1980)):

(i) $M[\Phi(x,0;f)] = f(x) \quad (x \in (-1,1))$;

(ii) $\Phi(\cdot,0;f) \in C[-1,1]$; (5.2.3)

(iii) $\lim_{\pm 1} p\Phi'(\cdot,0,f)g = 0 \quad (f,g \in H^2_{p,q}(-1,1))$.

Now define a linear operator A on $H^2_{p,q}(-1,1)$ by

$$(Af)(x) = \Phi(x,0;f) \; ; \qquad (5.2.4)$$

then, using the properties (5.2.3), Everitt (1980) has shown that A is bounded, self-adjoint and has an inverse operator denoted by $S := A^{-1}$, and furthermore that S is an unbounded self-adjoint operator in $H^2_{p,q}(-1,1)$. We shall return to this later in §6.9 and 6.11, in the case of Gegenbauer's equation.

II The operator T in terms of Φ

In a similar way, if we define Φ in the right-definite case (replacing $H^2_{p,q}$ by L^2 above) by

$$\Phi(x,0;f) = \int_{-1}^{1} G(x,t,0)f(t) \qquad (f \in L^2(-1,1))$$

and define, as in (5.2.4), a linear operator B in $L^2(-1,1)$ by

$$(Bf)(x) = \Phi(x,0;f) \qquad (5.2.5)$$

it may be shown (see §6.12) as in the case of A that B is bounded, self-adjoint and its inverse operator $T := B^{-1}$ is a self-adjoint unbounded operator in $L^2(-1,1)$.

Note that both operators T and S have the same discrete spectrum (Everitt (1980)), i.e.

$$P\sigma(T) = P\sigma(S) = \{(n+\tfrac{1}{2})^2, \; n \in N_0\} \; .$$

This then implies the completeness of the Legendre polynomials $\{P_n, \; n \in N_0\}$ in $H^2_{p,q}(-1,1)$ and in $L^2(-1,1)$.

In the next section, we show that, unlike the operator S in the left-definite case, the operator T in the right-definite case can also be determined directly in terms of the differential expression $M[y]$ (see §4.1).

§5.3 The Operator T in terms of M[f]

Again, consider Legendre's differential expression $M[f]$, namely:

$$M[f](x) = -((1-x^2)f'(x))' + \frac{1}{4}f(x) \qquad (x \in (-1,1))$$

where $f : (-1,1) \to \mathbb{C}$, with f and $pf' \in AC_{loc}(-1,1)$. Then f and g, with these properties, satisfy the Green's formula (4.0.4) on $(-1,1)$.

As in (4.1.1), we define the linear manifold $\Delta \in L^2(-1,1)$ by $f \in \Delta$ if

$$\left.\begin{array}{ll} \text{(i)} & f : (-1,1) \to \mathbb{C} \text{ and } f \in L^2(-1,1) \\ \text{(ii)} & f \text{ and } pf' \in AC_{loc}(-1,1) \\ \text{(iii)} & M[f] \in L^2(-1,1) \,. \end{array}\right\} \tag{5.3.1}$$

Since $M[f]$ is limit-circle at 1 and -1, we require the boundary conditions of the type (4.1.9), viz.

$$\lim_{x\to 1} [f(x),\psi(x,\lambda)] = 0 \,, \quad \lim_{x\to -1} [f(x),\chi(x,\lambda)] = 0 \tag{5.3.2}$$

where $f \in \Delta$ and ψ and χ are given by (5.1.10).

Now let T_0 be a linear operator in $L^2(-1,1)$, with the domain $D(T_0)$, where

$$D(T_0) = \{f \in \Delta : \lim_{1}[f,\psi] = 0, \lim_{-1}[f,\chi] = 0\}, \tag{5.3.3}$$

and such that

$$T_0 f := M[f] \quad (f \in D(T_0)); \tag{5.3.4}$$

then the general theory (see Theorem 4.1.6) tells us that T_0 is a self-adjoint operator. Furthermore, its domain $D(T_0) \subset L^2(-1,1)$, and strictly so, for if $f \in D(T_0)$, then $pf' \in AC_{loc}(-1,1)$, and there are vectors in $L^2(-1,1)$ without this property. Hence T_0 is the required unbounded, self-adjoint differential operator, and such that (see Everitt (1980)) T_0 has a discrete spectrum $P\sigma(T_0) = \{(n+\frac{1}{2})^2, n \in N_0\}$. As in the preceding section, this implies the completeness of the Legendre polynomials $\{P_n(\cdot), n \in N_0\}$ in $L^2(-1,1)$; in this case (see §6.12), we may identify T_0 with the operator T in §5.2 by

$$Tf := T_0 f ,$$

i.e. $T = T_0$.

This method then gives an alternative way of determining the differential operator T. We reserve the comparison between the operators T and S for the next chapter.

§5.4 The half-range Legendre series

If we look at Legendre's equation (5.0.1) on the interval [0,1), then we can construct a self-adjoint operator T_1, as in §5.3 with the domain given as

$$D(T_1) = \{f \in L^2(0,1) : f \text{ and } pf' \in AC_{loc}[0,1), \quad M[f] \in L^2(0,1),\ f(0) = 0 \text{ and } \lim_1 [f,\psi] = 0\} \qquad (5.4.1)$$

and

$$T_1 f = M[f] \quad (f \in D(T_1)). \qquad (5.4.2)$$

It may be seen that T_1 has a simple and discrete spectrum at $\lambda_n = (2n+\frac{3}{2})^2$, with $n = 0,1,2,\ldots$, i.e. $P\sigma(T_1) = \{(2n+\frac{3}{2})^2, n \in N_0\}$, and the eigenfunctions $\{P_{2n+1}(\cdot), n \in N_0\}$.

Similarly we may construct T_2, in which the boundary condition is $(pf')(0) = 0$, but with the spectrum $\{(2n+\frac{1}{2})^2, n \in N_0\}$ and the eigen-functions $\{P_{2n}(\cdot), n \in N_0\}$. From this it follows that the collections of functions $\{P_{2n}(x) : x \in [0,1], n \in N_0\}$, $\{P_{2n+1}(x) : x \in [0,1], n \in N_0\}$ give separate orthogonal sets in $L^2(0,1)$, both of which are complete in this space. The two collections combined and extended to the interval $[-1,1]$ give an orthogonal set which is complete in $L^2(-1,1)$. This compares with the half-range Fourier sine and cosine series.

There are similar half-range Legendre expansions in the left-definite case.

CHAPTER SIX

THE GEGENBAUER EXPANSION

§6.0 Preliminary

Another example of the symmetric differential equation (3.0.1), for which the interval (a,b) is finite, is Gegenbauer's equation

$$-((1-x^2)^{\nu+1/2}y'(x))' + \nu^2(1-x^2)^{\nu-1/2}y(x) = \lambda(1-x^2)^{\nu-1/2}y(x) \quad (6.0.1)$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C},\ \nu > -1/2\ ,\ \nu \text{ is assumed real; see footnote(*)}).$$

Compared with (3.0.1), the coefficients p, q and w are

$$p(x) = (1-x^2)^{\nu+1/2};\ q(x) = \nu^2 w(x);\ w(x) = (1-x^2)^{\nu-1/2} \quad (6.0.2)$$

$$(x \in (-1,1)).$$

Note that $q \geq 0$ on $(-1,1)$; this then enables us to study (6.0.1) in both the right- and left-definite cases, i.e. in the integrable-square function spaces $L^2_w(-1,1)$ and $H^2_{p,q}(-1,1)$ $(:= H^2_{\nu,p,q}(-1,1))$, where we recall from (3.4.4) that

$$\int_{-1}^{1} \{(1-x^2)^{\nu+1/2}|f'(x)|^2 + \nu^2(1-x^2)^{\nu-1/2}|f(x)|^2\}dx < \infty \quad (6.0.3)$$

$$(f \in H^2_{p,q}(-1,1)).$$

We also note that if $\nu = 1/2$, then (6.0.1) reduces to Legendre's differential equation (5.0.1).

We look at the solutions of (6.0.1) in §6.1, and their asymptotic behaviour in §6.2. Sections 6.3, 6.4 and 6.5 are given over to the application of Titchmarsh's method to obtain the eigenvalues and eigenvectors in the regular, limit-circle and limit-point cases in the space $L^2_w(-1,1)$ (followed by a remark concerning the left-definite case).

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* The case $\nu = 0$ gives Tchebichef's differential equation, and it is appare[nt] from the methods developed in this chapter that this case, ie. $\nu = 0$, would not wo[rk]. A different approach is required if $\nu = 0$. in this chapter We assume $\nu \neq 0$.

Sections 6.6, 6.7, 6.8 and 6.12 are devoted to studying the associated differential operators in $L^2_w(-1,1)$, followed by a similar study in §6.9, 6.10 and 6.11 in $H^2_{p,q}(-1,1)$. We end this chapter with the comparison of the operators in $L^2_w(-1,1)$ and $H^2_{p,q}(-1,1)$.

§6.1 The solutions of Gegenbauer's equation

Again, consider equation (6.0.1), i.e.

$$-((1-x^2)^{\nu+1/2}y'(x))' + \nu^2(1-x^2)^{\nu-1/2}y(x) = \lambda(1-x^2)^{\nu-1/2}y(x) \quad (6.0.1)$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C},\ \nu > -1/2).$$

We write $s = \sqrt{\lambda} := \exp[\frac{1}{2}\log|\lambda| + \frac{1}{2} i \arg \lambda]$, i.e. $s^2 = \lambda$, and determine the $\sqrt{(\cdot)}$-function by requiring

$$0 \le \arg \sqrt{\lambda} < \pi \quad \text{when} \quad 0 \le \arg \lambda < 2\pi \; . \quad (6.1.1)$$

Our first result now follows:

Theorem 6.1.1

Let $\lambda \in \mathbb{C}$ be defined as above and let $-\pi/2 \le \arg z < 3\pi/2$; then, for all $x \in (-1,1)$, two solutions of (6.0.1) may be determined by the following integrals:

$$Y_\nu(x,\lambda) = \frac{\exp[(\nu-1/2)\pi i]}{2}\int_{C_1} \frac{\cos sz \, dz}{(\cos z - x)^\nu} \quad (6.1.2)$$

$$Z_\nu(x,\lambda) = \frac{\exp[(\nu-1/2)\pi i]}{2}\int_{C_2} \frac{\cos sz \, dz}{(\cos z + x)^\nu} \quad (6.1.3)$$

where C_1 is a closed contour enclosing the branch-points $\cos^{-1}x$ and $-\cos^{-1}x$, but excluding all other singularities. Similarly, C_2 encloses the branch-points $\pi/2 + \sin^{-1}x$ and $-\pi/2 - \sin^{-1}x$, but excludes all the others. Both inverse trigonometrical functions are determined by requiring $\cos^{-1}x$ to decrease from π to 0, and $\sin^{-1}x$ to increase from $-\pi/2$ to $\pi/2$, as in both cases x increases from -1 to 1; and both integrals hold for all ν.

Proof: Consider the integrals

$$U(x,\lambda) = \int_{C_1} \frac{\cos sz \, dz}{(\cos z - x)^{\nu}} \qquad (6.1.4)$$

and

$$V(x,\lambda) = \int_{C_2} \frac{\cos sz \, dz}{(\cos z + x)^{\nu}} \; ; \qquad (6.1.5)$$

the parameter ν is not necessarily an integer; hence the integrands in both cases assume many values. In (6.1.4) the integrand is made single-valued by cutting the z-plane from $-\cos^{-1}x$ to $\cos^{-1}x$; similarly for (6.1.5) the cut is from $-(\pi/2 + \sin^{-1}x)$ to $\pi/2 + \sin^{-1}x$. The contours C_1 and C_2 may now be taken as circles with centre the origin of the z-plane and of radius r and ρ respectively, where $\cos^{-1}x < r < \pi$ and $\pi/2 + \sin^{-1}x < \rho < \pi$.

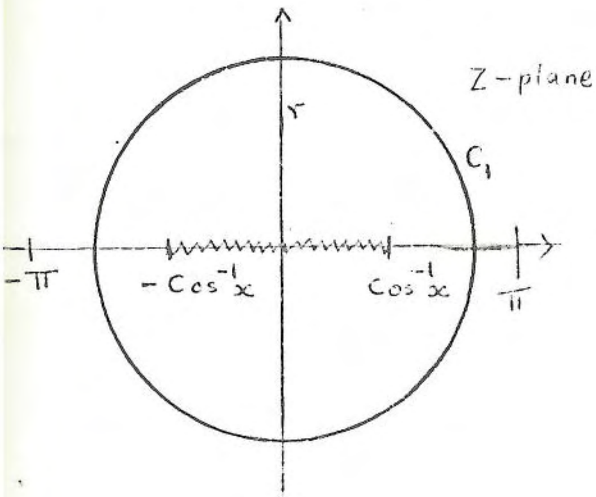

Figure 1

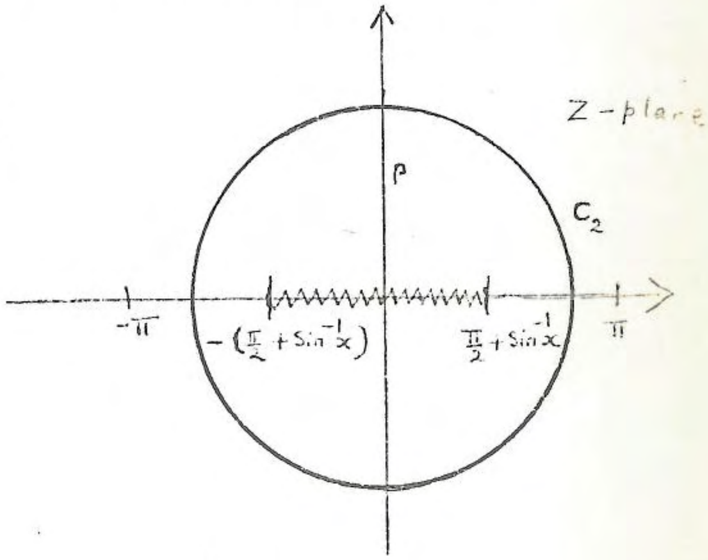

Figure 2

Now let $-\pi/2 \le \arg z < 3\pi/2$ denote the principal branch of the integrands in U and V and let $(\cos z \pm x)^{\nu}$ be real and positive when $\arg z = 0$. Then both integrands are now single-valued and regular within and on the closed contours C_1 and C_2 respectively. Thus the integrals (6.1.2) and (6.1.3) are well-defined.

To show that U (and hence Y_{ν}) is a solution of (6.0.1), we note from the above remarks that the integrand

$$f(x,z) = \frac{\cos sz}{(\cos z - x)^{\nu}} \qquad (x \in (-1,1))$$

is a continuous function of both variables and, for every z on C_1, $f(\cdot,z)$ is an analytic function of x in (-1,1). Hence from a standard result in Titchmarsh (1978; §2.83) $U(\cdot,\lambda)$ is an analytic function of x in (-1,1), and furthermore

$$U'(x,\lambda) = \int_{C_1} \frac{\nu \cos sz \, dz}{(\cos z - x)^{\nu+1}} \qquad (\lambda \in \mathbb{C},\ x \in (-1,1))$$

and

$$(1-x^2)^{\nu+1/2} U'(x,\lambda) = \int_{C_1} \frac{(1-x^2)^{\nu+1/2} \nu \cos sz \, dz}{(\cos z - x)^{\nu+1}} .$$

Applying the same procedure to this gives

$$((1-x^2)^{\nu+1/2} U'(x,\lambda))'$$

$$= (1-x^2)^{\nu-1/2} \int_{C_1} \frac{(-(2x\nu^2+\nu x)\cos z + \nu^2 x^2 + \nu^2 + \nu)\cos sz \, dz}{(\cos z - x)^{\nu+2}} ;$$

hence

$$-((1-x^2)^{\nu+1/2}U'(x,\lambda))' + \nu^2(1-x^2)^{\nu-1/2}U(x,\lambda)$$

$$= (1-x^2)^{\nu-1/2}\int_{C_1} \frac{(-\nu^2\sin^2 z + \nu x\cos z - \nu)\cos sz\,dz}{(\cos z - x)^{\nu+2}} . \qquad (6.1.6)$$

On the other hand, since the integrand $\cos sz\,(\cos z - x)^{-\nu}$ in (6.1.4) is analytic and single-valued at every point of C_1, we may integrate (6.1.4) by parts (Carathéodory (1954; §229)). Then

$$U(x,\lambda) = \int_{C_1} \frac{\cos sz\,dz}{(\cos z - x)^{\nu}}$$

$$= -\frac{1}{s}\int_{C_1} \frac{\nu\sin z\sin sz\,dz}{(\cos z - x)^{\nu+1}}$$

(note that the integrated term vanishes because C_1 is a simple closed contour and so the end-points coincide)

$$= \frac{1}{s^2}\int_{C_1} \frac{(-\nu^2\sin^2 z + \nu x\cos z - \nu)\cos sz\,dz}{(\cos z - x)^{\nu+2}} ;$$

hence

$$\lambda(1-x^2)^{\nu-1/2}U(x,\lambda) = (1-x^2)^{\nu-1/2}\int_{C_1} \frac{(-\nu^2\sin^2 z + \nu x\cos z - \nu)\cos sz\,dz}{(\cos z - x)^{\nu+2}} . \qquad (6.1.7)$$

Both (6.1.6) and (6.1.7) imply that $U(\cdot,\lambda)$ and hence $Y_\nu(\cdot,\lambda)$ is a solution of (6.0.1). A similar result holds if we replace x by -x in the foregoing analysis. Also from (6.1.2) and (6.1.3) and the properties of inverse trigonometrical functions (note that $\pi/2 + \sin^{-1}x = \cos^{-1}(-x)$ for $-\pi/2 \le \sin^{-1}x \le \pi/2$) it follows that

$$Y_\nu(-x,\lambda) = Z_\nu(x,\lambda) ; \qquad (6.1.8)$$

then $Z_\nu(\cdot,\lambda)$ is also a solution of (6.0.1). This completes the proof.

The next result concerns the initial values of Y_ν and Z_ν at 0.

Theorem 6.1.2

Let Y_ν and Z_ν be defined as before; then at $x = 0$

$$Y_\nu(0,\lambda) = \frac{2^\nu \pi \sin \pi\nu \, \Gamma(1-\nu)}{\Gamma(1-\nu/2+s/2)\Gamma(1-\nu/2-s/2)} = Z_\nu(0,\lambda) \qquad (6.1.9)$$

for all ν; (recall that $s^2 = \lambda$). Also

$$Y_\nu'(0,\lambda) = -\frac{2^{\nu+1} \pi \sin \pi\nu \, \Gamma(1-\nu)}{\Gamma(1/2-\nu/2+s/2)\Gamma(1/2-\nu/2-s/2)} = -Z_\nu'(0,\lambda) \qquad (6.1.10)$$

(all ν as above).

Proof: We take a similar line as the result in Titchmarsh (1962; §4.4); recall that $-\pi/2 \le \arg z < 3\pi/2$, then at $x = 0$

$$Y_\nu(0,\lambda) = \frac{\exp[(\nu-1/2)\pi i]}{2}\int_{C_1} \frac{\cos sz\, dz}{(\cos z)^\nu} \quad (\text{all } \nu) \qquad (6.1.11)$$

$$= \exp[(\nu-1/2)\pi i](1-\exp[-2\nu\pi i])\int_0^{\pi/2} (\cos z)^{-\nu}.\cos sz\, dz \quad (\nu < 1)$$

(and on using a formula in Magnus et al.*
(1966; §1.1))

$$= \frac{2^\nu \pi \sin \pi\nu \, \Gamma(1-\nu)}{\Gamma(1-\nu/2+s/2)\Gamma(1-\nu/2-s/2)} \quad (\nu < 1). \qquad (6.1.12)$$

Suppose $\nu \ge 1$ (and recall that the integrand in Y_ν is analytic within and on the contour C_1); then

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* Reference is made to Magnus, Oberhettinger and Soni (1966) in the literature as Magnus et al. (1966).

$$\int_{C_1} \frac{\cos sz\,dz}{(\cos z)^{\nu}} = \int_{C_1} \frac{\cos (s-1) z.\cos z - \sin (s-1)z.\sin z\,dz}{(\cos z)^{\nu}}$$

$$= \int_{C_1} \{\frac{\cos (s-1) z}{(\cos z)^{\nu-1}} - \frac{\sin(s-1)z.\sin z}{(\cos z)^{\nu}}\}dz \ ;$$

each of the terms in {---} is also analytic and single valued within and on C_1. Hence we can integrate each term (Titchmarsh (1978; §1.71)), i.e.

$$\int_{C_1} \frac{\cos sz\,dz}{(\cos z)^{\nu}} = \int_{C_1} \frac{\cos(s-1)z\ dz}{(\cos z)^{\nu-1}} + \int_{C_1} \frac{-\sin(s-1)x.\sin z\,dz}{(\cos z)^{\nu}} .$$

We integrate the second term on the right hand side by parts (see proof of Theorem 6.1.1); then

$$\int_{C_1} \frac{\cos sz\,dz}{(\cos z)^{\nu}} = \int_{C_1} \frac{\cos(s-1)\ z\,dz}{(\cos z)^{\nu-1}} + \frac{s-1}{\nu-1}\int_{C_1} \frac{\cos(s-1)\ z\,dz}{(\cos z)^{\nu-1}}$$

$$= \frac{\nu+s-2}{\nu-1}\int_{C_1} \frac{\cos(s-1)\ z\,dz}{(\cos z)^{\nu-1}}$$

and in general

$$\int_{C_1} \frac{\cos sz\,dz}{(\cos z)^{\nu}} = \frac{(\nu+s-2)(\nu+s-4)\ \ldots\ (\nu+s-2n+2)}{(\nu-1)(\nu-2)\ \ldots\ (\nu-n+1)}\int_{C_1} \frac{\cos(s-n+1)\ z\,dz}{(\cos z)^{\nu-n+1}}$$

$$= 2(1-\exp[-2\pi i\nu])\frac{(\nu+s-2)(\nu+s-4)\ldots(\nu+s-2n+2)}{(\nu-1)(\nu-2)\ldots(\nu-n+1)}\int_0^{\pi/2} \frac{\cos(s-n+1)\ z\,dz}{(\cos z)^{\nu-n+1}} .$$

Hence, as in (6.1.12) above,

$$Y_{\nu}(0,\lambda) = \frac{2^{\nu-n}\pi \sin\pi\nu\,(\nu+s-2)(\nu+s-4)\ \ldots\ (\nu+s-2n+2)\Gamma(n-\nu)}{(\nu-1)(\nu-2)\ldots(\nu-n+1)\Gamma(1-\nu/2+s/2)\Gamma(n-\nu/2-s/2)} \quad (6.1.13)$$

$$(\quad \nu < n).$$

Note that the integral (6.1.11) for $Y_\nu(0,\lambda)$ is defined for any ν, while the function $\Gamma(n-\nu)$ is defined for all ν except $\nu = n, n+1, n+2, \ldots$ $(n \in N_0)$. Since $n \in N_0$ is arbitrary, we see that (6.1.13) is the analytic continuation of (6.1.12) into the whole real line R; moreover, repeated application of the relation $\Gamma(n+1) = n\Gamma(n)$ gives

$$\frac{2^{\nu-n}\pi \sin\pi\nu\,(\nu+s-2)\ldots(\nu+s-2n+2)\Gamma(n-\nu)}{(\nu-1)(\nu-2)\ldots(\nu-n+1)\Gamma(1-\nu/2+s/2)\Gamma(n-\nu/2-s/2)}$$

$$= \frac{2^{\nu}\pi\sin\pi\nu\,(1-\nu/2-s/2)\ldots(n-\nu/2-s/2)\Gamma(n-\nu)}{(1-\nu)(2-\nu)\ldots(n-\nu-1)\Gamma(1-\nu/2+s/2)\Gamma(1+n-\nu/2-s/2)}$$

$$= \frac{2^{\nu}\pi\sin\pi\nu\,\Gamma(1-\nu)}{\Gamma(1-\nu/2+s/2)\Gamma(1-\nu/2-s/2)} .$$

Hence

$$Y_\nu(0,\lambda) = \frac{2^{\nu}\pi\sin\pi\nu\,\Gamma(1-\nu)}{\Gamma(1-\nu/2+s/2)\Gamma(1-\nu/2-s/2)} ,$$

for all ν by analytic continuation.

A similar argument obtains for $Y'_\nu(0,\lambda)$. Note from (6.1.8) that $Y'_\nu(-x,\lambda) = -Z'_\nu(x,\lambda)$, which gives $Y'_\nu(0,\lambda) = -Z'_\nu(0,\lambda)$. Also we note from these results that Y_ν and Z_ν are linearly independent, except when $Y_\nu(0,\lambda) = 0$ or $Y'_\nu(0,\lambda) = 0$, i.e. when

$$\lambda = (n+\nu)^2 \quad (n \in N_0).$$

§6.2 Asymptotic behaviour of the solutions near -1 and 1

Due to the symmetry of the interval (-1,1), it suffices to consider $x \to 1$.

Theorem 6.2.1

Let $Y_\nu(\cdot,\lambda)$ and $Z_\nu(\cdot,\lambda)$ be the solutions of Gegenbauer's equation, defined by (6.1.2) and (6.1.3) respectively ($\lambda \in \mathbb{C}$); then

(i) $$\lim_{x\to 1} Y_\nu(x,\lambda) = \frac{(-1)^\nu 2^\nu \pi.\sin\pi\nu.\cos\pi s.\Gamma(1-2\nu)}{\Gamma(1-\nu+s)\Gamma(1-\nu-s)} \qquad (6.2.1)$$

and

$$\lim_{x\to 1} Y'_\nu(x,\lambda) = \frac{(-1)^{\nu+1} 2^{\nu+1} \pi.\sin\pi\nu.\cos\pi s.\Gamma(-1-2\nu)}{\Gamma(-\nu+s)\Gamma(-\nu-s)} \;; \qquad (6.2.2)$$

both (6.2.1) and (6.2.2) hold, by analytic continuation, for all ν;

(ii) as $x \to 1$ ($\lambda \in \mathbb{C}$, $\nu > -1/2$),

$$Z_\nu(x,\lambda) = \begin{cases} 2\sqrt{2}\cos s\pi.\ln(\frac{1}{1-x}) + O(1) & (\nu = 1/2) \\ \frac{\sqrt{2}\cos s\pi\,\Gamma(1/2)\,\Gamma(\nu-1/2)}{\Gamma(\nu)}.(1-x)^{1/2-\nu} + O(1) & \\ & (\nu \neq 1/2) \end{cases} \qquad (6.2.3)$$

and

$$Z'_\nu(x,\lambda) = \frac{\sqrt{2}\cos s\pi\,\Gamma(1/2)\,\Gamma(\nu+1/2)}{\Gamma(\nu+1)} (1-x)^{-1/2-\nu} + O(1). \qquad (6.2.4)$$

Proof: Consider the integral (6.1.2), namely

$$Y_\nu(x,\lambda) = \frac{\exp[(\nu-1/2)\pi i]}{2}\int_{C_1} \frac{\cos sz\,dz}{(\cos z - x)^\nu} \qquad (x \in (-1,1);$$

then as $x \to 1$, the branch-cut tends to the origin of the z-plane (see Figure 1), i.e.

$$\lim_{x\to 1} Y_\nu(x,\lambda) = \frac{\exp[(\nu-1/2)\pi i]}{2}\int_{C_1} \frac{\cos sz\,dz}{(\cos z - 1)^\nu}$$

$$= \frac{(-1)^\nu \exp[(\nu-1/2)\pi i]}{2^{\nu+1}}\int_{C_1} \sin^{-2\nu}(z/2)\,.\cos sz\,dz$$

$$= \frac{(-1)^{\nu}\exp[(\nu-1/2)\pi i](1-\exp[-2\pi\nu i])}{2^{\nu}}\int_0^{\pi}\sin^{-2\nu}(z/2)\,.\cos sz\,dz$$
$$(\nu < 1/2)$$

$$= \frac{(-1)^{\nu}2^{\nu}\pi.\sin\pi\nu.\cos\pi s.\Gamma(1-2\nu)}{\Gamma(1-\nu+s)\Gamma(1-\nu-s)} \quad (\nu < 1/2). \qquad (6.2.5)$$

As in the case of Theorem 6.1.2, it may be shown that (6.2.5) holds, by analytic continuation, for all ν; i.e.

$$\lim_{x\to 1} Y_{\nu}(x,\lambda) = \frac{(-1)^{\nu+1}2^{\nu}\pi\sin\pi\nu\cos\pi s\,\Gamma(1-2\nu)}{\Gamma(1-\nu+s)\Gamma(1-\nu-s)} \quad (\nu \text{ and } \lambda \in \mathbb{C})$$

or, equivalently,

$$Y_{\nu}(x,\lambda) = O(1) \quad (x \to 1).$$

A similar argument holds for $Y'_{\nu}(x,\lambda)$ in (6.2.2) as required.

(ii) Consider the integral (6.1.3), viz.:

$$Z_{\nu}(x,\lambda) = \frac{\exp[(\nu-1/2)\pi i]}{2}\int_{C_2}\frac{\cos sz\,dz}{(\cos z + x)^{\nu}}; \qquad (6.1.3)$$

here the method of part (i) fails because (see Figure 2) the branch-cut extends to $-\pi$ and π as $x \to 1$. On the other hand, keeping $|x| < 1$, we can replace the contour C_2 by any other simple closed contour (Cauchy's theorem), in particular by a circle of radius π.

Put $z = \pi\exp[\theta i]$, $-\pi/2 \le \theta \le 3\pi/2$; then

$$Z_{\nu}(x,\lambda) = \frac{\exp[(\nu-1/2)\pi i]}{2}\int_{|z|=\pi}\frac{\cos sz\,dz}{(\cos z + x)^{\nu}}$$

$$= \frac{\pi\exp[\nu\pi i]}{2}\int_{-\pi/2}^{3\pi/2}\frac{\cos(s\pi\exp[\theta i]).\exp[\theta i]\,d\theta}{(\cos(\pi\exp[\theta i]) + x)^{\nu}}$$

$$= \frac{\pi \exp[\nu\pi i]}{2}\left\{\int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2}\right\}\frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]\,d\theta}{(\cos(\pi \exp[\theta i]) + x)^{\nu}} . \qquad (6.2.6)$$

Consider the first integral in (6.2.6) in the neighbourhood of $\theta = 0$ (which is the only singularity as $x \to 1$); then, for a fixed δ, $0 < \delta < \pi/2$,

$$\int_{-\pi/2}^{\pi/2} \frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]\,d\theta}{(\cos(\pi \exp[\theta i]) + x)^{\nu}}$$

$$= \left\{\int_{-\pi/2}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi/2}\right\}\frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]\,d\theta}{(\cos(\pi \exp[\theta i]) + x)^{\nu}} . \qquad (6.2.7)$$

Now in the neighbourhood of $\theta = 0$,

$$\cos(\pi \exp[\theta i]) = \cos(\pi \cos\theta + i\pi \sin\theta)$$

$$= \cos(\pi\cos\theta).\cosh(\pi\sin\theta) - i\sin(\pi\cos\theta).\sinh(\pi\sin\theta)$$

$$\sim \cos(\pi\cos\theta).(1 + \frac{\pi^2\theta^2}{2}) - i\sin(\pi\cos\theta).(\pi\theta)$$

$$\sim \cos(\pi - \frac{\pi\theta^2}{2}).(1 + \frac{\pi^2\theta^2}{2}) - i\sin(\pi - \frac{\pi\theta^2}{2}).(\pi\theta)$$

$$= -\cos(\frac{\pi\theta^2}{2}).(1 + \frac{\pi^2\theta^2}{2}) - i\sin(\frac{\pi\theta^2}{2}).(\pi\theta)$$

$$\sim -(1 - \frac{\pi^2\theta^4}{8}).(1 + \frac{\pi^2\theta^2}{2}) - i\,\frac{\pi^2\theta^3}{2} ,$$

<u>i.e.</u>

$$\cos(\pi \exp[\theta i]) = -1 - \frac{\pi^2\theta^2}{2} + O(|\theta|^3) . \qquad (6.2.8)$$

Similarly

$$\cos(s\pi \exp[\theta i]) = \cos(s\pi \cos\theta).\cosh(s\pi \sin\theta)$$

$$- i \sin(s\pi \cos\theta).\sinh(s\pi \sin\theta)$$

$$= \cos s\pi . (1+\frac{s^2\pi^2\theta^2}{2}) - \sin s\pi . \frac{s\pi\theta^2}{2}$$

$$- i \sin s\pi . s\pi\theta + O(|\theta|^3)$$

and in the neighbourhood of $\theta = 0$

$$\frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]}{(\cos(\pi \exp[\theta i]) + x)^{\nu}} \sim \frac{\cos s\pi}{(-1 -(\pi^2\theta^2/2)+x)^{\nu}} \tag{6.2.9}$$

for

$$\lim_{\theta\to 0}\{\frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]/(\cos(\pi \exp[\theta i])+x)^{\nu}}{\cos s\pi/(-1-(\pi^2\theta^2/2)+x)^{\nu}}\}$$

$$= \frac{\cos s\pi/(-1+x)^{\nu}}{\cos s\pi/(-1+x)^{\nu}}$$

$$= 1 ;$$

hence, for some $k \in R_+$,

$$\int_{-\delta}^{\delta} |\frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]}{(\cos(\pi \exp[\theta i]) + x)^{\nu}}| d\theta \le k\int_{-\delta}^{\delta} |\frac{\cos s\pi}{(-1-(\pi^2\theta^2/2)+x)^{\nu}}| d\theta . \tag{6.2.10}$$

Also

$$\int_{-\delta}^{\delta} \frac{\cos s\pi \, d\theta}{(-1-(\pi^2\theta^2/2)+x)^{\nu}} = \exp[-\nu\pi i] \cos s\pi \int_{-\delta}^{\delta} \frac{d\theta}{(1-x+(\pi^2\theta^2/2))^{\nu}} ; \tag{6.2.11}$$

put $(1-x) = a^2$, where $a > 0$, because $-1 < x < 1$, and put $t = \pi\theta/\sqrt{2}$, then

$$\int_{-\delta}^{\delta} \frac{d\theta}{(1-x+(\pi^2\theta^2/2))^{\nu}} = \frac{2\sqrt{2}}{\pi}\int_{0}^{\pi\delta/\sqrt{2}} \frac{dt}{(a^2+t^2)^{\nu}} .$$

Note that, if $\nu = 1/2$, then

$$\frac{2\sqrt{2}}{\pi}\int_0^{\pi\delta/\sqrt{2}} \frac{dt}{(a^2+t^2)^{1/2}} = \frac{2\sqrt{2}}{\pi} \ln (a+\sqrt{a^2+t^2})\Big|_0^{\delta\pi/\sqrt{2}}$$

$$= -\frac{2\sqrt{2}}{\pi} \ln (2a) + O(1) \quad (x \to 1)$$

$$= \frac{2\sqrt{2}}{\pi} \ln (\frac{1}{1-x}) + O(1) \quad (x \to 1)$$

i.e.

$$\int_{-\delta}^{\delta} \frac{\cos s\pi \, d\theta}{(-1-(\pi^2\theta^2/2)+x)^{1/2}} = 2\sqrt{2} \cos \pi . \exp[-(1/2)\pi i] \ln (\frac{1}{1-x}) + O(1) \quad (6.2.12)$$

$$(x \to 1).$$

However, if $\nu \neq 1/2$, then

$$\frac{2\sqrt{2}}{\pi}\int_0^{\pi\delta/\sqrt{2}} \frac{dt}{(a^2+t^2)^{\nu}} = \frac{2\sqrt{2}}{\pi}\int_0^{\infty} \frac{dt}{(a^2+t^2)^{\nu}} + O(1)$$

$$= \frac{2\sqrt{2}}{\pi} a^{1-2\nu} \int_0^{\infty} \frac{dr}{(1+r^2)^{\nu}} + O(1)$$

(on putting $t = ar$)

$$= \frac{2\sqrt{2}\Gamma(1/2).\Gamma(\nu-1/2)}{\pi 2\Gamma(\nu)} a^{1-2\nu} + O(1)$$

(see Magnus et al (1966; §1.1)); and this holds for $\nu > 1/2$, and so by analytic continuation for all $\nu > -1/2$ (with $\nu \neq 1/2$ as above). Hence from (6.2.11)

$$\int_{-\delta}^{\delta} \frac{\cos s\pi \, d\theta}{(-1-(\pi^2\theta^2/2)+x)^{\nu}} = \frac{\sqrt{2} \cos s\pi . \exp[-\nu\pi i].\Gamma(\nu-1/2)}{\sqrt{\pi}\Gamma(\nu)} . (1-x)^{1/2-\nu}$$

$$+ O(1) \quad (x \to 1). \quad (6.2.13)$$

For the case of the integral

$$\int_{-\pi/2}^{-\delta} \frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]d\theta}{(\cos(\pi \exp[\theta i])+x)^{\nu}},$$

with x → 1, the integrand does not have any singularities, and so (by continuity) remains bounded, i.e.

$$\int_{-\pi/2}^{-\delta} \frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]d\theta}{(\cos(\pi \exp[\theta i])+x)^{\nu}} = 0(1) \quad (x \to 1); \tag{6.2.14}$$

similarly

$$\int_{\delta}^{\pi/2} \frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]d\theta}{(\cos(\pi \exp[\theta i])+x)^{\nu}} = 0(1) \quad (x \to 1). \tag{6.2.15}$$

It follows from (6.2.7), (6.2.10)-(6.2.15) that, as x → 1,

$$\int_{-\pi/2}^{\pi/2} \frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]d\theta}{(\cos(\pi \exp[\theta i])+x)^{\nu}}$$

$$= \begin{cases} \dfrac{2\sqrt{2}\cos s\pi.\exp[-\pi i/2]}{\pi}.\ln(\dfrac{1}{1-x}) + 0(1), & \nu = 1/2 \\[2ex] \dfrac{\sqrt{2}\cos s\pi.\exp[-\nu\pi i].\Gamma(\nu-1/2)}{\sqrt{\pi}\Gamma(\nu)}.(1-x)^{1/2-\nu} + 0(1), & \\ & \nu \neq 1/2 . \end{cases} \tag{6.2.16}$$

Consider now the second integral of (6.2.6), then, on putting $\phi = \pi - \theta$, we have

$$\int_{\pi/2}^{3\pi/2} \frac{\cos(s\pi \exp[\theta i]).\exp[\theta i]d\theta}{(\cos(\pi \exp[\theta i])+x)^{\nu}}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos(s\pi \exp[-\phi i]).\exp[-\phi i]d\phi}{(\cos(\pi \exp[-\phi i])+x)^{\nu}}$$

which is equivalent to the first integral of (6.2.6). A similar analysis then gives the result (6.2.16).

It follows from (6.2.6) and (6.2.16) that, as $x \to 1$,

$$Z_\nu(x,\lambda) = \begin{cases} 2\sqrt{2}\cos s\pi . \ln(\frac{1}{1-x}) + O(1) & (\nu = 1/2) \\ \frac{\sqrt{2}\cos s\pi\, \Gamma(1/2).\Gamma(\nu-1/2)}{\Gamma(\nu)} . (1-x)^{1/2-\nu} + O(1) & (\nu \neq 1/2) \end{cases}$$

which is the required result.

In the case of $Z_\nu'(\cdot,\lambda)$, we note that

$$Z_\nu'(x,\lambda) = -\frac{\nu \exp[(\nu-1/2)\pi i]}{2} \int_{C_2} \frac{\cos sz\, dz}{(\cos z + x)^{\nu+1}} . \tag{6.2.17}$$

Following similar arguments as in the case of $Z_\nu(\cdot,\lambda)$, with ν replaced by $\nu + 1$, we obtain the result (6.2.4).

Similar results obtain, by symmetry, as $x \to -1$. This completes the proof.

Now, following the existence theorem in Titchmarsh (1962; §1.5-1.6), we may form the solutions θ_ν and ϕ_ν of Gegenbauer's equation (6.0.1) by

$$\theta_\nu(x,\lambda) = \frac{Y_\nu(x,\lambda) + Z_\nu(x,\lambda)}{2Y(0,\lambda)} \quad (x \in (-1,1),\ \lambda \in \mathbb{C}), \tag{6.2.18}$$

$$\phi_\nu(x,\lambda) = \frac{Y_\nu(x,\lambda) - Z_\nu(x,\lambda)}{2Y'(0,\lambda)} \quad (x \in (-1,1),\ \lambda \in \mathbb{C}), \tag{6.2.19}$$

such that

$$\left.\begin{array}{l} \theta_\nu(0,\lambda) = 1 , \quad \theta_\nu'(0,\lambda) = 0 \\ \phi_\nu(0,\lambda) = 0 , \quad \phi_\nu'(0,\lambda) = 1 ; \end{array}\right\} \tag{6.2.20}$$

and the Wronskian

$$(\theta_\nu(0,\lambda)\phi'_\nu(0,\lambda) - \theta'_\nu(0,\lambda)\phi_\nu(0,\lambda)) = 1 ,$$

i.e. θ_ν and ϕ_ν are linearly independent for all $\lambda \in \mathbb{C}$.

The asymptotic forms of θ_ν and ϕ_ν follow from those of Y_ν and Z_ν (Theorem 6.2.1), i.e. if $\nu \neq 1/2$, then, as $x \to 1$,

$$\theta_\nu(x,\lambda) = \frac{\sqrt{2}\cos s\pi.\Gamma(1/2)\Gamma(\nu-1/2).\Gamma(1-\nu/2+s/2).\Gamma(1-\nu/2-s/2)}{2^{1+\nu}\Gamma(\nu)\pi.\sin\pi\nu.\Gamma(1-\nu).(1-x)^{\nu-1/2}}$$

$$+ O(1)$$

$$= \frac{2^{-1/2-\nu}.\cos s\pi.\Gamma(1/2).\Gamma(\nu-1/2).(1-x)^{1/2-\nu}}{\Gamma(\nu/2+s/2).\sin(\nu/2+s/2)\pi.\Gamma(\nu/2-s/2).\sin(\nu/2-s/2)\pi}$$

$$+O(1)$$

(on using the relation $\Gamma(z).\Gamma(1-z) = \pi/\sin\pi z$). Similarly, if $\nu = 1/2$, then

$$\theta_{1/2}(x,\lambda) = \frac{2\sqrt{\pi}}{\Gamma(1/4+s/2).\Gamma(1/4-s/2)}.\ln(\frac{1}{1-x}) + O(1) \quad (x \to 1);$$

thus, as $x \to 1$,

$$\theta_\nu(x,\lambda) = \begin{cases} \dfrac{2\sqrt{\pi}}{\Gamma(1/4+s/2).\Gamma(1/4-s/2)}.\ln(\frac{1}{1-x}) + O(1) , & (\nu = 1/2) , \\ \dfrac{2^{-1/2-\nu}\cos\pi s.\Gamma(1/2)\Gamma(\nu-1/2).(1-x)^{1/2-\nu}}{\Gamma(\nu/2+s/2).\sin(\nu/2+s/2)\pi.\Gamma(\nu/2-s/2).\sin(\nu/2-s/2)\pi} \\ + O(1), & (\nu \neq 1/2) , \end{cases} \quad (6.2.21)$$

(care is to be taken at the critical points $s = \pm(\nu+2m)$ ($m \in N_0$) by taking the limit of the right-hand side as $s \to \pm(\nu+2m)$). In a similar way we obtain, as $x \to 1$,

$$\theta'_\nu(x,\lambda) = \frac{2^{-1/2-\nu}\cos\pi s.\Gamma(1/2).\Gamma(\nu+1/2).(1-x)^{-1/2-\nu}}{\Gamma(\nu/2+s/2).\sin(\nu/2+s/2)\pi.\Gamma(\nu/2-s/2).\sin(\nu/2-s/2)\pi}$$

$$+ O(1) , \quad (6.2.22)$$

$$\phi_\nu(x,\lambda) = \begin{cases} \dfrac{-\pi^{1/2}\ln(1/(1-x))}{\Gamma(3/4 - s/2).\Gamma(3/4 + s/2)} + O(1) \quad (\nu = 1/2) \\ \dfrac{-2^{-3/2-\nu}\cos\pi s.\Gamma(\nu-1/2).(1-x)^{1/2-\nu}}{\Gamma(1/2+\nu/2+s/2).\cos(\nu/2+s/2)\pi.\Gamma(1/2+\nu/2-s/2).\cos(\nu/2-s/2)\pi} \\ + O(1) \quad (\nu \neq 1/2); \end{cases} \tag{6.2.23}$$

and

$$\phi_\nu'(x,\lambda) = \frac{-2^{-3/2-\nu}\cos\pi s.\Gamma(\nu+1/2).(1-x)^{-1/2-\nu}}{\Gamma(1/2+\nu/2+s/2)\cos(\nu/2+s/2)\pi\Gamma(1/2+\nu/2-s/2)\cos(\nu/2-s/2)\pi} + O(1)\ . \tag{6.2.24}$$

Similar results may be obtained as $x \to -1$.

Remark 6.2.2

On the basis of these asymptotic results, the limit-point, limit-circle classification established in Corollary 3.5.2 for Gegenbauer's differential equation (6.0.1) may be verified and is seen to hold for all $\lambda \in \mathbb{C}$, in both right- and left-definite cases.

§6.3 The associated eigenvalues and eigenfunctions in $L_w^2(-1,1)$: the regular case $(-1/2 < \nu < 1/2)$

It follows from the definition of θ_ν and ϕ_ν (see (6.2.18), (6.2.19) and (6.2.20)) that, for $-1/2 < \nu < 1/2$, any two solutions ψ_ν and χ_ν are of the form

$$\psi_\nu(x,\lambda) = \theta_\nu(x,\lambda) + \ell_0\phi_\nu(x,\lambda) \tag{6.3.1}$$

and

$$\chi_\nu(x,\lambda) = \theta_\nu(x,\lambda) + k_0\phi_\nu(x,\lambda) \tag{6.3.2}$$

and such that both ψ_ν and χ_ν satisfy the regular boundary conditions of

$$(py')(\pm 1) = 0 \tag{6.3.3}$$

(see (4.1.4) with $\alpha = \pi/2$).

The coefficients $\ell_0 = \ell_0(\lambda)$ and $k_0 = k_0(\lambda)$ are meromorphic functions of λ in $\mathbb{C}$, and may be determined by using (6.3.3) as follows: let $(p\psi_\nu'(\cdot,\lambda))(1) = 0$, then

$$(p\theta_\nu'(\cdot,\lambda))(1) + \ell_0(p\phi_\nu'(\cdot,\lambda))(1) = 0$$

i.e.

$$\ell_0 = -\frac{(p\theta_\nu'(\cdot,\lambda))(1)}{(p\phi_\nu'(\cdot,\lambda))(1)}$$

$$= -\frac{(pY_\nu'(\cdot,\lambda)+pZ_\nu'(\cdot,\lambda))(1)}{(pY_\nu'(\cdot,\lambda)-pZ_\nu'(\cdot,\lambda))(1)}\cdot\frac{Y_\nu'(0,\lambda)}{Y_\nu(0,\lambda)} .$$

Appealing to the asymptotic results of Y_ν' and Z_ν' (see Theorem 6.2.1) gives

$$\ell_0 = -\frac{0+K}{0-K}\cdot\frac{Y_\nu'(0,\lambda)}{Y_\nu(0,\lambda)}$$

where $K = \sqrt{2}\cos s\pi.\Gamma(1/2).\Gamma(\nu+1/2)/\Gamma(\nu+1)$ (see (6.2.4), and recall that $p(x) = (1-x^2)^{\nu+1/2}$ and $\nu > -1/2$). Hence

$$\ell_0 = \frac{Y_\nu'(0,\lambda)}{Y_\nu(0,\lambda)} . \tag{6.3.4}$$

Similarly, on putting $(p\chi_\nu'(\cdot,\lambda))(-1) = 0$, we may obtain

$$k_0 = -\frac{Y_\nu'(0,\lambda)}{Y_\nu(0,\lambda)} . \tag{6.3.5}$$

The functions ψ_ν and χ_ν now reduce to

$$\psi_\nu(x,\lambda) = \theta_\nu(x,\lambda) + \frac{Y'_\nu(0,\lambda)}{Y_\nu(0,\lambda)}\cdot\phi_\nu(x,\lambda)$$

$$= \frac{Y_\nu(x,\lambda) + Z_\nu(x,\lambda)}{2Y_\nu(0,\lambda)} + \frac{Y'_\nu(0,\lambda)}{Y_\nu(0,\lambda)}\cdot\frac{Y_\nu(x,\lambda) - Z_\nu(x,\lambda)}{2Y'_\nu(0,\lambda)} ,$$

i.e.

$$\psi_\nu(x,\lambda) = \frac{Y_\nu(x,\lambda)}{Y_\nu(0,\lambda)} \quad (x \in (-1,1),\ \lambda \in \mathbb{C} - R). \tag{6.3.6}$$

Similarly,

$$\chi_\nu(x,\lambda) = \frac{Z_\nu(x,\lambda)}{Z_\nu(0,\lambda)} \quad (x \in (-1,1),\ \lambda \in \mathbb{C} - R). \tag{6.3.7}$$

Besides these, ψ_ν and χ_ν also satisfy the following properties:

(i) $\psi_\nu(\cdot,\lambda) \in L^2_w(-1,1) \quad (\lambda \in \mathbb{C} - R)$;

(ii) $\lim_{x\to 1} \psi_\nu(x,\lambda) = Y_\nu(1,\lambda)/Y_\nu(0,\lambda)$ (see (6.2.1) and (6.1.9));

(iii) $\psi_\nu(x,\lambda) = O(|1+x|^{1/2-\nu})$ $(x \to -1)$, and $\psi_\nu(\cdot,\lambda)$ is bounded in the neighbourhood of -1, provided $\nu < 1/2$, and hence for $-1/2 < \nu < 1/2$;

(iv) $\psi'_\nu(x,\lambda) = O(|1+x|^{-1/2-\nu})$ $(x \to -1)$, and $\psi'_\nu(\cdot,\lambda)$ is unbounded in the neighbourhood of -1 $(\lambda \in \mathbb{C} - R,\ -1/2 < \nu < \infty)$

Similarly

(i) $\chi_\nu(\cdot,\lambda) \in L^2_w(-1,1) \quad (\lambda \in \mathbb{C} - R)$;

(ii) $\lim_{x\to -1}\chi_\nu(x,\lambda) = Z_\nu(1,\lambda)/Z_\nu(0,\lambda)$ (as above);

(iii) $\chi_\nu(x,\lambda) = O(|1-x|^{1/2-\nu})$ $(x \to 1)$, and $\chi_\nu(\cdot,\lambda)$ is bounded in the neighbourhood of 1 for $-1/2 < \nu < 1/2$; but

(iv) $\chi_\nu'(x,\lambda) = O(|1-x|^{-1/2-\nu})$ $(x \to 1)$, and $\chi'_\nu(\cdot,\lambda)$ is unbounded in the neighbourhood of 1 $(\lambda \in \mathbb{C} - R,\ -1/2 < \nu < \infty)$.

Consider now the Wronskian of ψ_ν and χ_ν in the form

$$\omega(\lambda) = p(x)W(\psi_\nu(x,\lambda),\chi_\nu(x,\lambda)) ;$$

then

$$\omega(\lambda) = p(0)\{Y_\nu(\cdot,\lambda)Z_\nu'(\cdot,\lambda) - Y_\nu'(\cdot,\lambda)Z_\nu(\cdot,\lambda)\}(0)$$

$$= -\frac{2Y_\nu'(0,\lambda)}{Y_\nu(0,\lambda)} .$$

The Green's function G (see (3.3.4)) now becomes

$$G(x,t,\lambda) = \begin{cases} \dfrac{-\psi_\nu(x,\lambda)\chi_\nu(t,\lambda)}{\{pW(\psi_\nu,\chi_\nu)\}(\lambda)} & (-1 < t < x < 1) \\[2ex] \dfrac{-\chi_\nu(x,\lambda)\psi_\nu(t,\lambda)}{\{pW(\psi_\nu,\chi_\nu)\}(\lambda)} & (-1 < x < t < 1) , \end{cases} \tag{6.3.8}$$

i.e.

$$G(x,t,\lambda) = \begin{cases} \dfrac{Y_\nu(x,\lambda)Z_\nu(t,\lambda)}{2Y_\nu(0,\lambda)Y_\nu'(0,\lambda)} & (-1 < t < x < 1) \\[2ex] \dfrac{Z_\nu(x,\lambda)Y_\nu(t,\lambda)}{2Y_\nu(0,\lambda)Y_\nu'(0,\lambda)} & (-1 < x < t < 1) . \end{cases}$$

Put $\omega_0(\lambda) = 2Y_\nu(0,\lambda)Y_\nu'(0,\lambda)$, then

$$\omega_0(\lambda) = \frac{-2^{2+2\nu}\pi^2\{\sin\pi\nu\,\Gamma(1-\nu)\}^2}{\Gamma(1-\nu/2+s/2)\Gamma(1-\nu/2-s/2)\Gamma(1/2-\nu/2+s/2)\Gamma(1/2-\nu/2-s/2)}$$

$$= \frac{-4\pi^3}{[\Gamma(\nu)]^2\Gamma(1-\nu+s)\Gamma(1-\nu-s)} \tag{6.3.9}$$

(on using Legendre's duplicating formula), or

$$\omega_0(\lambda) = \frac{-2\pi(\cos\pi s-\cos\pi\nu).\Gamma(\nu-s).\Gamma(\nu+s)}{[\Gamma(\nu)]^2} , \tag{6.3.10}$$

and this has poles at

$$s = \pm(n+\nu) \quad (n \in N_0).$$

It follows from §3.3 that in this case the associated eigenvalues are given by

$$\lambda_n = (n+\nu)^2 \quad (n \in N_0, -1/2 < \nu < 1/2). \tag{6.3.11}$$

Anticipating the definition of the operator T_r below, we put

$$P\sigma(T_r) = \{(n+\nu)^2,\ n \in N_0,\ -1/2 < \nu < 1/2\}\ . \tag{6.3.12}$$

Let $\Phi : (-1,1) \times (\mathbb{C} - P\sigma(T_r)) \times L^2_w(-1,1) \to \mathbb{C}$ be the resolvent function given by (3.3.6), <u>i.e.</u>

$$\begin{aligned}\Phi(x,\lambda;f) &= \frac{\psi_\nu(x,\lambda)}{\omega(\lambda)}\int_{-1}^{x} w(t)\chi_\nu(t,\lambda)f(t)dt \\ &+ \frac{\chi_\nu(x,\lambda)}{\omega(\lambda)}\int_{x}^{1} w(t)\psi_\nu(t,\lambda)f(t)dt \\ &= \frac{Y_\nu(x,\lambda)}{\omega_0(\lambda)}\int_{-1}^{x} w(t)Z_\nu(t,\lambda)f(t) \\ &+ \frac{Z_\nu(x,\lambda)}{\omega_0(\lambda)}\int_{x}^{1} w(t)Y_\nu(t,\lambda)f(t)dt\ .\end{aligned} \tag{6.3.13}$$

At $\lambda = \lambda_n$,

$$Y_\nu(x,\lambda_n) = \frac{\exp[(\nu-1/2)\pi i]}{2}\int_{C_1} \frac{\cos(n+\nu)\,z\,dz}{(\cos z - x)^\nu}\ ,$$

and if we now define $C_n^\nu(x)$ (see (1.5.13)) as

$$C_n^\nu(x) = \frac{\exp[(\nu-1/2)\pi i]}{2^{\nu+1}\pi}\int_{C} \frac{\cos(n+\nu)\,z\,dz}{(\cos z - x)^\nu}$$

$$(x \in (-1,1),\ -\pi/2 \le \arg z < 3\pi/2,\ \nu \text{ arbitrary}),$$

then

$$Y_\nu(x,\lambda_n) = 2^\nu \pi C_n^\nu(x) = (-1)^n 2^\nu \pi C_n^\nu(-x)$$

$$= (-1)^n Z_\nu(x,\lambda_n) \ .$$

If $s = n + \nu + \epsilon$, with $\epsilon \to 0$, then (6.3.9) now becomes

$$\omega_0(\lambda_n) = -4\pi^3/[\Gamma(\nu)]^2\Gamma(1+n+\epsilon)\Gamma(1-n-2\nu-\epsilon)$$

$$= -4\pi^2\Gamma(n+2\nu+\epsilon).\sin(n+2\nu+\epsilon)\pi/[\Gamma(\nu)]^2\Gamma(1+n+\epsilon)$$

(on using the relation $\Gamma(1-Z)\Gamma(Z) = \pi/\sin\pi Z$)

$$= \frac{-4\pi^2}{[\Gamma(\nu)]^2}\cdot\frac{\Gamma(n+2\nu+\epsilon)}{\Gamma(n+1+\epsilon)}\{\sin(n+2\nu)\pi\cos\pi\epsilon + \sin\pi\epsilon\cos(n+2\nu)\pi\}$$

$$\sim \frac{-4\pi^3}{[\Gamma(\nu)]^2}\cdot(-1)^n\,\frac{\Gamma(n+2\nu)}{n!}\cdot\epsilon \quad \text{(if } 2\nu \text{ is an integer)}$$

$$\sim \frac{-4\pi^3}{[\Gamma(\nu)]^2}\cdot(-1)^n\,\frac{\Gamma(n+2\nu)}{n!}\cdot\frac{\lambda-(n+\nu)^2}{2(n+\nu)} \qquad (\lambda \to \lambda_n).$$

Hence the function $\Phi(x,\cdot;f)$ above has a simple pole at λ_n, with the residue

$$\text{Res}[\Phi(x,\cdot;f);\lambda_n] = \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)\Phi(x,\lambda;f)$$

$$= \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)\int_{-1}^{1}\frac{(-1)^n w(t).Y_\nu(x,\lambda_n).Y_\nu(t,\lambda_n).f(t)dt}{\omega_0(\lambda)}$$

$$= \int_{-1}^{1}\frac{(-1)^n 2^\nu \pi C_n^\nu(x)\ 2^\nu \pi\, w(t)\ C_n^\nu(t)\ f(t)dt}{2\pi^3(-1)^n[\Gamma(\nu)]^{-2}.\Gamma(n+2\nu).(n!)^{-1}.(n+\nu)^{-1}}$$

$$= \frac{2^{2\nu-1}[\Gamma(\nu)]^2\ n!\ (n+\nu)}{\pi\Gamma(n+2\nu)}\ C_n^\nu(x)\int_{-1}^{1} w(t)C_n^\nu(t)f(t)dt \ .$$

Thus, corresponding to the eigenvalues λ_n in (6.3.12) are the eigenfunctions $\{\phi_{n,\nu}(\cdot),\ n \in N_0,\ -1/2 < \nu < 1/2\}$, where

$$\phi_{n,\nu}(x) = \left\{\frac{2^{2\nu-1} n!(n+\nu)[\Gamma(\nu)]^2}{\pi\Gamma(n+2\nu)}\right\}^{1/2} C_n^{\nu}(x) \qquad (6.3.14)$$

with $\{C_n^{\nu}(\cdot),\ n \in N_0\}$ being the Gegenbauer polynomials (see §1.4).

Remark 6.3.1

We recall from Corollary 3.5.2 that the equation (6.0.1) is also regular at ±1 in the left-definite case (*i.e.* $H^2_{p,q}(-1,1)$) if $-1/2 < \nu < 1/2$. Thus, we can replace the space $L^2_w(-1,1)$ with $H^2_{p,q}(-1,1)$ and obtain the same results.

An immediate consequence of these results are the following properties:

Theorem 6.3.2

(i) *The set* $\{\phi_{n,\nu}(\cdot),\ n \in N_0,\ -1/2 < \nu < 1/2\}$, *defined by* (6.3.14), *is orthonormal in* $L^2_w(-1,1)$;

(ii) *the set* $\{(n+\nu)^{-1}\phi_{n,\nu}(\cdot),\ n \in N_0,\ -1/2 < \nu < 1/2\}$ *is orthonormal in* $H^2_{p,q}(-1,1)$.

Proof:

(i) The orthogonality part is clear; also

$$\int_{-1}^{1} \frac{2^{2\nu-1} n!(n+\nu)[\Gamma(\nu)]^2 (1-x^2)^{\nu-1/2}}{\pi\Gamma(n+2\nu)} [C_n^{\nu}(x)]^2 dx$$

$$= \frac{2^{2\nu-1} n!(n+\nu)[\Gamma(\nu)]^2}{\pi\Gamma(n+2\nu)} \int_{-1}^{1} (1-x^2)^{\nu-1/2} [C_n^{\nu}(x)]^2 dx$$

$$= \frac{2^{2\nu-1} n!(n+\nu)[\Gamma(\nu)]^2}{\pi\Gamma(n+2\nu)} \cdot \frac{2^{1-2\nu}\pi\Gamma(n+2\nu)}{n!(n+\nu)[\Gamma(\nu)]^2}$$

$= 1$, because (see Magnus et al. (1966; §5.3))

$$\int_{-1}^{1} (1-x^2)^{\nu-1/2}[C_n^{\nu}(x)]^2 dx = \frac{2^{1-2\nu}\pi\Gamma(n+2\nu)}{n!(n+\nu)[\Gamma(\nu)]^2} . \tag{6.3.15}$$

(ii) For this part, it is sufficient to show that

$$\int_{-1}^{1} \{p|(n+\nu)^{-1}\phi'_{n,\nu}(\cdot)|^2 + q|(n+\nu)^{-1}\phi_{n,\nu}(\cdot)|^2\} = 1 . \tag{6.3.16}$$

Since $(d/dx)C_n^{\nu}(x) = 2\nu C_{n-1}^{\nu+1}(x)$ (see Magnus et al. (1966; §5.3)), the left-hand side of (6.3.16) now becomes

$$\int_{-1}^{1} \{\frac{(1-x^2)^{\nu+1/2}2^{2\nu+1}n!(n+\nu)[\Gamma(\nu)]^2\nu^2}{\pi(n+\nu)^2\Gamma(n+2\nu)}[C_{n-1}^{\nu+1}(x)]^2$$

$$+\frac{\nu^2(1-x^2)^{\nu-1/2}n!2^{2\nu-1}(n+\nu)}{\pi(n+\nu)^2\Gamma(n+2\nu)}[\Gamma(\nu)]^2[C_n^{\nu}(x)]\}dx$$

$$= \frac{2^{2\nu-1}n![\Gamma(\nu)]^2\nu^2}{\pi\Gamma(n+2\nu).(n+\nu)}\int_{-1}^{1} \{4(1-x^2)^{\nu+1/2}[C_{n-1}^{\nu+1}(x)]^2 + (1-x^2)^{\nu-1/2}[C_n^{\nu}(x)]^2\}dx .$$

$$= \frac{2^{2\nu-1}n![\Gamma(\nu)]^2\nu^2}{\pi\Gamma(n+2\nu).(n+\nu)}\{\frac{2^{1-2\nu}\pi\Gamma(n+2\nu+1)}{(n-1)!(n+\nu)[\Gamma(\nu+1)]^2} + \frac{2^{1-2\nu}\pi\Gamma(n+2\nu)}{n!(n+\nu)[\Gamma(\nu)]^2}\}$$

(on using (6.3.7) above)

$$= \frac{2^{2\nu-1}n![\Gamma(\nu)]^2\nu^2}{\pi\Gamma(n+2\nu).(n+\nu)}\{\frac{2^{1-2\nu}\pi(n+2\nu)\Gamma(n+2\nu)}{(n-1)!(n+\nu)\nu^2[\Gamma(\nu)]^2} + \frac{2^{1-2\nu}\pi\Gamma(n+2\nu)}{n!(n+\nu)[\Gamma(\nu)]^2}\}$$

$= 1$.

This completes the proof.

Remarks 6.3.3

These two properties of Theorem 6.3.2 are in fact satisfied for all $\nu > -1/2$.

§6.4 The associated eigenvalues and eigenfunctions in $L^2_w(-1,1)$: the limit-circle case ($1/2 \le \nu < 3/2$)

Again consider Gegenbauer's differential equation (6.0.1) on $(-1,1)$, i.e.

$$-((1-x^2)^{\nu+1/2}y'(x\quad))' + \nu^2(1-x^2)^{\nu-1/2}y(x) = \lambda(1-x^2)^{\nu-1/2}y(x) \qquad (6.0.1)$$
$$(x \in (-1,1),\ \lambda \in \mathbb{C},\ \nu > -1/2).$$

Let θ_ν and ϕ_ν be the solutions of this equation defined by (6.2.18) and (6.2.19) respectively, then the general theory in Titchmarsh (1962; Chapter 2) gives the existence of the Titchmarsh-Weyl m-coefficient (see §2.1 above), and determines a particular solution ψ_ν of (6.0.1) in the form

$$\psi_\nu(x,\lambda) = \theta_\nu(x,\lambda) + m_c(\lambda)\phi_\nu(x,\lambda) \qquad (6.4.1)$$
$$(x \in (-1,1),\ \lambda \in \mathbb{C} - R)$$

(the subscript c denotes limit-circle).

Since the differential equation (6.0.1) is limit-circle at ±1, the m-coefficient is not unique; and in order to determine the differential operator associated with the Gegenbauer polynomials, we need to make a suitable choice from the family of m-coefficients belonging to the end-point 1. This is done by following the limit process which determines $m_c(\cdot)$ from the ℓ-functions; for the general theory, see Titchmarsh (1962; §2.1).

With the solutions θ_ν and ϕ_ν determined from (6.2.18) and (6.2.19) and with $\lambda \in \mathbb{C} - \mathbb{R}$ and $1/2 \le \nu < 3/2$, let $\psi_{X,\nu}$ be a solution of (6.0.1) - here $X \in (0,1)$ - given by $\psi_{X,\nu}(x,\lambda) = \theta_\nu(x,\lambda) + \ell(\lambda,X,\beta)\phi_\nu(x,\lambda)$ $(x \in (-1,1))$. The function ℓ is chosen so that $\psi_{X,\nu}$ satisfies the following boundary conditions at X (see also (4.1.4))

$$\psi_{X,\nu}(X,\lambda)\cos\beta + p(X)\psi'_{X,\nu}(X,\lambda)\sin\beta = 0\ , \tag{6.4.2}$$

for some $\beta \in (-\pi/2,\pi/2]$; thus, for all $\lambda \in \mathbb{C} - \mathbb{R}$,

$$\ell(\lambda,X,\beta) = -\frac{\theta_\nu(X,\lambda)\cos\beta + p(X)\theta'_\nu(X,\lambda)\sin\beta}{\phi_\nu(X,\lambda)\cos\beta + p(X)\phi'_\nu(X,\lambda)\sin\beta}\ .$$

Now let $X \to 1-$ and choose β as a function of X so that ℓ tends to a limit $m_c(\cdot)$, where $m_c(\cdot) : \mathbb{C} \to \mathbb{C}$ and is regular on $\mathbb{C} - \mathbb{R}$. In this the Gegenbauer case, with $1/2 \le \nu < 3/2$, this may be done by writing ℓ in the form

$$\ell(\lambda,X,\beta) = -\frac{\theta_\nu(X,\lambda)(1-X)^{\nu-1/2} + p(X)\theta'_\nu(X,\lambda)(1-X)^{\nu-1/2}\tan\beta(X)}{\phi_\nu(X,\lambda)(1-X)^{\nu-1/2} + p(X)\phi'_\nu(X,\lambda)(1-X)^{\nu-1/2}\tan\beta(X)} \tag{6.4.3}$$

and then choose $\beta(X)$ $(X \in (0,1))$, so that for some $\gamma \in (-\pi/2,\pi/2]$, $\tan\beta(X).(1-X)^{\nu-1/2} \to \tan\gamma$ as $X \to 1-$. (Note that the limits of all the remaining terms in (6.4.3) may be calculated from the asymptotic formulae (6.2.21) to (6.2.24).) In our case, guided by Titchmarsh (1962; §4.5), we take $\gamma = 0$ and so choose $\beta(X) = 0$ $(X \in (0,1))$; then

$$\begin{aligned} m_c(\lambda) &= \lim_{X\to 1} \ell(\lambda,X,0) = \lim_{X\to 1}\{-\theta_\nu(X,\lambda)/\phi_\nu(X,\lambda)\} \\ &= \lim_{X\to 1}\left\{\frac{-Y_\nu(X,\lambda) + Z_\nu(X,\lambda)}{2Y_\nu(0,\lambda)}\cdot\frac{2Y'_\nu(0,\lambda)}{Y_\nu(X,\lambda) - Z_\nu(X,\lambda)}\right\} \\ &= \frac{Y'_\nu(0,\lambda)}{Y_\nu(0,\lambda)} \end{aligned}$$

$$= \frac{-2\Gamma(1 - \nu/2 + s/2).\Gamma(1 - \nu/2 - s/2)}{\Gamma(1/2 - \nu/2 + s/2).\Gamma(1/2 - \nu/2 - s/2)} \qquad (6.4.4)$$

(where we recall (see (6.1.1)) that $s^2 = \lambda$).

It follows from (6.4.4) and (6.4.1) that the solution $\psi_\nu(\cdot,\lambda)$ has the properties

(i) $\psi_\nu(x,\lambda) = Y_\nu(x,\lambda)/Y_\nu(0,\lambda) \quad (x \in (-1,1),\ \lambda \in \mathbb{C} - R)$;

(ii) $\psi_\nu(\cdot,\lambda) \in L^2_w(-1,1) \quad (\lambda \in \mathbb{C} - R,\ 1/2 \le \nu < 3/2)$;

(iii) $\lim_{x \to 1} \psi_\nu(x,\lambda) = Y_\nu(1,\lambda)/Y_\nu(0,\lambda)$

$$= \frac{(-1)^\nu \cos s\pi.\Gamma(1-2\nu).\Gamma(1 - \nu/2 + s/2).\Gamma(1 - \nu/2 - s/2)}{\Gamma(1-\nu+s).\Gamma(1-\nu-s).\Gamma(1-\nu)}$$

(see (6.1.9) and (6.2.1)); similarly

(iv) $\lim_{x \to 1} \psi'_\nu(x,\lambda) = (-1)^{\nu+1} \dfrac{2\cos \pi s.\Gamma(-1-2\nu).\Gamma(1 - \nu/2 + s/2).\Gamma(1 - \nu/2 - s/2)}{\Gamma(1-\nu).\Gamma(-\nu+s).\Gamma(-\nu-s)}$;

(v) $\psi_\nu(\cdot,\lambda)$ and $\psi'_\nu(\cdot,\lambda)$ are unbounded in the neighbourhood of -1 $(\lambda \in \mathbb{C} - R,\ 1/2 \le \nu < \infty)$.

A similar analysis holds for the singular end-point -1; there is a solution $\chi_\nu(\cdot,\lambda)$ of (6.0.1) which has the form

$$\chi_\nu(x,\lambda) = \theta_\nu(x,\lambda) + n_c(\lambda)\phi_\nu(x,\lambda) \qquad (6.4.5)$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C} - R)$$

where

$$n_c(\lambda) = Z'_\nu(0,\lambda)/Z_\nu(0,\lambda) = -m_c(\lambda) \quad (\lambda \in \mathbb{C} - R), \qquad (6.4.6)$$

such that $\chi_\nu(\cdot,\lambda)$ has the properties

(i) $\chi_\nu(x,\lambda) = Z_\nu(x,\lambda)/Z_\nu(0,\lambda) \quad (x \in (-1,1),\ \lambda \in \mathbb{C} - R)$;

(ii) $\chi_\nu(\cdot,\lambda) \in L^2_w(-1,1) \quad (\lambda \in \mathbb{C} - R,\ 1/2 \le \nu < 3/2)$;

(iii) $\lim_{x\to -1}\chi_\nu(x,\lambda) = \dfrac{(-1)^\nu \cos s\pi.\Gamma(1-2\nu).\Gamma(1-\nu/2+s/2).\Gamma(1-\nu/2-s/2)}{\Gamma(1-\nu+s)\Gamma(1-\nu-s)\Gamma(1-\nu)}$;

(iv) $\lim_{x\to -1}\chi_\nu'(x,\lambda) = \dfrac{(-1)^\nu 2\cos \pi s.\Gamma(-1-2\nu).\Gamma(1-\nu/2+s/2).\Gamma(1-\nu/2-s/2)}{\Gamma(1-\nu)\Gamma(-\nu+s)\Gamma(-\nu-s)}$;

(v) $\chi_\nu(\cdot,\lambda)$ and $\chi_\nu'(\cdot,\lambda)$ are unbounded in the neighbourhood of 1 $(\lambda \in \mathbb{C} - R,\ 1/2 \le \nu < \infty)$.

The Green's function for this choice of $m_c(\cdot)$ and $n_c(\cdot)$ is given by

$$G(x,t;\lambda) = \begin{cases} -\dfrac{\psi_\nu(x,\lambda)\chi_\nu(t,\lambda)}{\{pW(\psi_\nu,\chi_\nu)\}(\lambda)} & (-1 < t < x < 1) \\ -\dfrac{\chi_\nu(x,\lambda)\psi_\nu(t,\lambda)}{\{pW(\psi_\nu,\chi_\nu)\}(\lambda)} & (-1 < x < t < 1) \end{cases} \tag{6.4.7}$$

where, from the form of ψ_ν and χ_ν above,

$$\{pW(\psi_\nu,\chi_\nu)\}(\lambda) = p(x)\big(\psi_\nu(x,\lambda)\chi_\nu'(x,\lambda) - \psi_\nu'(x,\lambda)\chi_\nu(x,\lambda)\big) \qquad (x \in (-1,1))$$

$$= n_c(\lambda) - m_c(\lambda) \qquad (\lambda \in \mathbb{C} - R). \tag{6.4.8}$$

From the general theory of differential equations of the type (6.0.1) with two singular end-points, it is known (Titchmarsh (1962; §2.18)) that the eigenvalues of the equation, with the particular choice of ψ_ν and χ_ν above, are given by the zeros and poles of $n_c(\lambda) - m_c(\lambda)$. In this case, from (6.4.6),

$$n_c(\lambda) - m_c(\lambda) = -2m_c(\lambda)$$

$$= \frac{4\Gamma(1-\nu/2+s/2)\Gamma(1-\nu/2-s/2)}{\Gamma(1/2-\nu/2+s/2)\Gamma(1/2-\nu/2-s/2)} \tag{6.4.9}$$

and from this we obtain, as in §6.3, the associated eigenvalues as

$$\lambda_n = (n+\nu)^2 \qquad (n \in N_0,\ 1/2 \le \nu < 3/2). \tag{6.4.9}$$

By a similar analysis to that of §6.3 above, we may obtain the corresponding eigenfunctions $\{\phi_{n,\nu}(\cdot),\ n \in N_0,\ 1/2 \le \nu < 3/2\}$, where

$$\phi_{n,\nu}(x,\lambda) = \{\frac{2^{2\nu-1} n!(n+\nu)[\Gamma(\nu)]^2}{\pi\Gamma(n+2\nu)}\}^{1/2} C_n^{\nu}(x) \qquad (6.4.10)$$

and again $C_n^{\nu}(\cdot)$ denotes the Gegenbauer polynomial . See also Theorem 6.3.2 and Remarks 6.3.1 and 6.3.3.

§6.5 The Associated Eigenvalues and Eigenfunctions in $L_w^2(-1,1)$: The Limit-point case $(3/2 \le \nu < \infty)$

Again consider Gegenbauer's differential equation (6.0.1), i.e.

$$-((1-x^2)^{\nu+1/2}y'(x))' + \nu^2(1-x^2)^{\nu-1/2}y(x) = \lambda(1-x^2)^{\nu-1/2}y(x) \qquad (6.0.1)$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C},\ \nu > -1/2).$$

This equation is limit-point at ± 1 in $L_w^2(-1,1)$ if $3/2 \le \nu < \infty$. Let θ_ν and ϕ_ν be the solutions of (6.0.1) defined by (6.2.18) and (6.2.19) respectively. Then the general theory of Titchmarsh and Weyl in §2.1 now gives the existence of unique m-coefficients $m_p(\cdot)$ and $n_p(\cdot)$ (where the subscript p denotes limit-point), both analytic mappings of $\mathbb{C} \to \mathbb{C}$, such that, for all $x \in (-1,1)$ and $\lambda \in \mathbb{C} - R$,

$$\left.\begin{aligned}\psi_\nu(x,\lambda) &:= \theta_\nu(x,\lambda) + m_p(\lambda)\phi_\nu(x,\lambda) \in L_w^2(0,1)\\ \chi_\nu(x,\lambda) &:= \theta_\nu(x,\lambda) + n_p(\lambda)\phi_\nu(x,\lambda) \in L_w^2(-1,0)\ .\end{aligned}\right\} \qquad (6.5.1)$$

Then ψ_ν and χ_ν are multiples of Y_ν and Z_ν respectively.

To calculate $m_p(\cdot)$ and $n_p(\cdot)$, put

$$\psi_\nu(x,\lambda) = \theta_\nu(x,\lambda) + m_p(\lambda)\phi_\nu(x,\lambda) = K(\lambda)Y_\nu(x,\lambda)$$

and

$$\psi_\nu'(x,\lambda) = \theta_\nu'(x,\lambda) + m_p(\lambda)\phi_\nu'(x,\lambda) = K(\lambda)Y_\nu'(x,\lambda) .$$

Then at $x = 0$ (see (6.2.20))

$$1 = K(\lambda)Y_\nu(0,\lambda)$$

and

$$m_p(\lambda) = K(\lambda)Y_\nu'(0,\lambda) ,$$

i.e.

$$m_p(\lambda) = \frac{Y_\nu'(0,\lambda)}{Y_\nu(0,\lambda)}$$

$$= \frac{-2\Gamma(1-\nu/2+s/2)\Gamma(1-\nu/2-s/2)}{\Gamma(1/2-\nu/2+s/2)\Gamma(1/2-\nu/2-s/2)} . \qquad (6.5.2)$$

Similarly

$$n_p(\lambda) = \frac{Z_\nu'(0,\lambda)}{Z_\nu(0,\lambda)} = \frac{-Y_\nu'(0,\lambda)}{Y_\nu(0,\lambda)} ,$$

i.e.

$$m_p(\lambda) = -n_p(\lambda) . \qquad (6.5.3)$$

From (6.5.2), we find that the solution $\psi_\nu(\cdot,\lambda)$ has the properties (see (6.5.1))

(i) $\psi_\nu(x,\lambda) = Y_\nu(x,\lambda)/Y_\nu(0,\lambda)$ $(x \in (-1,1),\ \lambda \in \mathbb{C} - R)$;

(ii) $\psi_\nu(\cdot,\lambda) \in L^2_w(0,1)$ $(\lambda \in \mathbb{C} - R,\ 3/2 \le \nu < \infty)$;

(iii) ψ_ν and ψ_ν' are unbounded in the neighbourhood of -1 $(3/2 \le \nu < \infty,\ \lambda \in \mathbb{C} - R)$.

Similarly, it can be shown that

(i) $\chi_\nu(x,\lambda) = Z_\nu(x,\lambda)/Z_\nu(0,\lambda)$ $(x \in (-1,1),\ \lambda \in \mathbb{C} - R)$;

(ii) $\chi_\nu(\cdot,\lambda) \in L^2_w(-1,0)$ $(\lambda \in \mathbb{C} - R,\ 3/2 \le \nu < \infty)$;

(iii) χ_ν and χ'_ν are unbounded in the neighbourhood of 1

$(3/2 \le \nu < \infty,\ \lambda \in \mathbb{C} - \mathbb{R})$.

The other limiting properties of ψ_ν and χ_ν in §6.4 for $1/2 \le \nu < 3/2$ also hold for this case (i.e. for $3/2 \le \nu < \infty$).

As before (see (6.3.8)), the Green's function for these values of $m_p(\cdot)$ and $n_p(\cdot)$ is given by

$$G(x,t;\lambda) = \begin{cases} -\dfrac{\psi_\nu(x,\lambda)\chi_\nu(t,\lambda)}{\{pW(\psi_\nu,\chi_\nu)\}(\lambda)} & (-1 < t < x < 1) \\[2ex] -\dfrac{\chi_\nu(x,\lambda)\psi_\nu(t,\lambda)}{\{pW(\psi_\nu,\chi_\nu)\}(\lambda)} & (-1 < x < t < 1) \end{cases} \tag{6.5.4}$$

where $\{pW(\psi_\nu,\chi_\nu)\}(\lambda) = n_p(\lambda) - m_p(\lambda)$ (see (6.4.8)).

Following §6.3, the associated eigenvalues in this the limit-point case are also given by the zeros and poles of $n_p(\lambda) - m_p(\lambda)$; from which we obtain (see (6.4.9))

$$\lambda_n = (n+\nu)^2 \quad (n \in \mathbb{N}_0,\ 3/2 \le \nu < \infty) . \tag{6.5.5}$$

A similar analysis to that of §6.3 now gives the corresponding set of eigenfunctions $\{\phi_{n,\nu}(\cdot),\ n \in \mathbb{N}_0,\ 3/2 \le \nu < \infty\}$ where, as before,

$$\phi_{n,\nu}(x,\lambda) = \left\{\frac{2^{2\nu-1} n!(n+\nu)[\Gamma(\nu)]^2}{\pi\Gamma(n+2\nu)}\right\}^{1/2} C_n^\nu(x) . \tag{6.5.6}$$

Remark 6.5.1

(i) Again we recall from Corollary 3.5.2 that Gegenbauer's differential equation (6.0.1) is also limit-point at ±1 in the left-definite case (i.e. in $H^2_{p,q}(-1,1)$) if $1/2 \le \nu < \infty$. Thus, by replacing the space $L^2_w(-1,1)$ with $H^2_{p,q}(-1,1)$ in the analysis above (see Everitt (1974)), we obtain similar results, viz. the eigenvalues

$$\lambda_n = (n+\nu)^2 \quad (n \in N_0,\ 1/2 \le \nu < \infty). \tag{6.5.9}$$

and the eigenfunctions $\{\phi_{n,\nu}(\cdot),\ n \in N_0,\ 1/2 \le \nu < \infty\}$, with $\phi_{n,\nu}$ given by (6.5.6).

(ii) The orthonormality properties in Theorem 6.3.2 also hold in this case.

We now leave the classical study of Gegenbauer's differential equation (6.0.1) and turn to the study of the associated differential operators in all the three cases (i.e. regular, limit-point, limit-circle) in the right-definite case, and the two cases (i.e. regular and limit-point) in the left-definite case.

§6.6 The operator T_r in $L^2_w(-1,1)$: the regular case $(-1/2 < \nu < 1/2)$

Again consider Gegenbauer's equation

$$M[y](x) := -((1-x^2)^{\nu+1/2}y'(x))' + \nu^2(1-x^2)^{\nu-1/2}y(x) = \lambda(1-x^2)^{\nu-1/2}y(x)$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C}). \tag{6.0.1}$$

Following (4.1.1), let Δ denote a linear manifold of $L^2_w(-1,1)$, defined by

$$\Delta := \{f : (-1,1) \to \mathbb{C} : f, pf' \in AC_{loc}(-1,1)$$
$$\text{and } f, W^{-1}M[f] \in L^2_w(-1,1)\} \tag{6.6.1}$$

where $p(x) = (1-x^2)^{\nu+1/2}$ and $w(x) = (1-x^2)^{\nu-1/2}$ are the coefficients in (6.0.1).

Since $M[y]$ is regular at ± 1 for $-1/2 < \nu < 1/2$, we can impose a regular boundary condition of the type (4.1.4) at ± 1, viz.

$$(pf')(\pm 1) = 0\ . \tag{6.6.2}$$

With this in mind, we now define a linear operator T_r and its domain $D(T_r)$ by

$$T_r f = w^{-1}M[f] \quad (f \in D(T_r)), \tag{6.6.3}$$

where

$$D(T_r) := \{f \in \Delta : (pf')(1) = 0 = (pf')(-1)\}; \tag{6.6.4}$$

then T_r has the following properties:

Theorem 6.6.1

(i) $D(T_r)$ is dense in $L^2_w(-1,1)$;

(ii) T_r is a symmetric operator in $L^2_w(-1,1)$;

(iii) T_r is self-adjoint.

Proof: This follows from Theorem 4.1.3 with the interval $[a,b)$ replaced by $(-1,1)$. So we shall omit it.

Thus T_r is the required self-adjoint (unbounded) differential operator.

We comment on the spectrum of T_r: Suppose λ is an eigenvalue of T_r, i.e. for some eigenvector $f \neq 0$, $T_r f := \lambda f$; then $W^{-1}M[f] = \lambda f$ and $M[f] = \lambda w f$. Hence, with $-1/2 < \nu < 1/2$ in mind, it follows that f is a non-trivial solution of the Gegenbauer equation (6.0.1) in $L^2_w(-1,1)$, and such that $(pf')(\pm 1) = 0$; from the properties of the solutions Y_ν and Z_ν given in §6.1 and §6.2, this can only happen if Y_ν and Z_ν are linearly dependent, i.e. $\lambda \in P\sigma(T_r)$ (see (6.3.12)), and f is then linearly dependent on the corresponding Gegenbauer polynomials from the set $\{C^\nu_n(\cdot),\ n \in N_0,\ -1/2 < \nu < 1/2\}$. Conversely, every point in the set $P\sigma(T_r)$ is an eigenvalue with the corresponding eigenvector in $\{C^\nu_n(\cdot),\ n \in N_0,\ -1/2 < \nu < 1/2\}$.

For any $\mu \in R - P\sigma(T_r)$ it may be shown as in the proof of part (c) of Theorem 4.1.3 that $(T_r - \mu I)D(T_r) = L_w^2(-1,1)$, i.e. μ is in the resolvent set of T_r (see Akhiezer and Glazman (1963; §43)). Hence the spectrum $\sigma(T_r)$ of T_r is discrete, i.e. $\sigma(T_r) = P\sigma(T_r)$.

An alternative proof of the discreteness of T_r may be obtained from the analysis given by Chaudhuri — Everitt (1968); in fact the discreteness of T_r and the fact that $P\sigma(T_r)$ is given by (6.3.12) follows from the fact that the Titchmarsh-Weyl m-coefficients are meromorphic, and their singularities in $\mathbb{C}$ are confined to the real axis.

The spectral theory for self-adjoint operators in Hilbert space (Akhiezer and Glazman (1963; Chapter VI)) now implies that the polynomials $\{C_n^\nu(\cdot),\ n \in N_0,\ -1/2 < \nu < 1/2\}$ are complete in $L_w^2(-1,1)$.

§6.7 The operator T_c in $L_w^2(-1,1)$: the limit-circle case $(1/2 \le \nu < 3/2)$

Here the differential expression $M[y]$ is limit-circle at ± 1. To determine a self-adjoint differential operator T_c in this case, and with the linear manifold Δ defined by (6.6.1), we require the following boundary conditions (see (4.1.9)):

$$\left.\begin{array}{ll} \text{(i)} & \lim_{x\to 1}[f(x),\psi_\nu(x,\lambda)] = 0 \\ \text{(ii)} & \lim_{x\to -1}[f(x),\chi_\nu(x,\lambda)] = 0 \end{array}\right\} (f \in \Delta,\ \lambda \in \mathbb{C} - P\sigma(T_c)). \qquad (6.7.1)$$

As in §4.0, these limits exist and are finite in view of the Green's formula (4.0.4). For a further discussion of (6.7.1) as the correct general form of boundary conditions in the limit-circle case, see Titchmarsh (1962; §2.7) and Naimark (1967; §1 8). Note that ψ_ν and χ_ν by themselves satisfy these boundary conditions.

With this in mind, we define a linear operator T_c and its domain

$D(T_c)$ by

$$T_c f := w^{-1}M[f] \quad (f \in D(T_c)) \tag{6.7.2}$$

where

$$D(T_c) := \{f \in \Delta : \lim_{x\to 1}[f(x),\psi_\nu(x,\lambda)] = 0;$$
$$\lim_{x\to -1}[f(x),\chi_\nu(x,\lambda)] = 0\} ; \tag{6.7.3}$$

then T_c has the following properties:

Theorem 6.7.1

(i) $D(T_c)$ is dense in $L^2_w(-1,1)$;
(ii) T_c is a symmetric operator in $L^2_w(-1,1)$;
(iii) T_c is a self-adjoint operator.

Proof: This follows from Theorem 4.1.6 on replacing the interval $[a,b)$ with $(-1,1)$; so we shall omit it.

Hence T_c is a self-adjoint (unbounded) differential operator. As in §6.6, it may be shown that the spectrum of T_c is simple and discrete and consists of the set

$$P\sigma(T_c) = \{(n+\nu)^2,\ n \in N_0,\ 1/2 \le \nu < 3/2\} , \tag{6.7.4}$$

and the corresponding eigenvectors are the Gegenbauer polynomials

$$\{C_n^\nu(\cdot),\ n \in N_0,\ 1/2 \le \nu < 3/2\} \tag{6.7.5}$$

which are complete in $L^2_w(-1,1)$.

We consider now a number of different but equivalent descriptions of $D(T_c)$, and in particular to find an alternative description of the boundary conditions (6.7.1). We take in this case $1/2 < \nu < 3/2$; the case $\nu = 1/2$, i.e. the Legendre's case, has been considered by Everitt (1980; §3) (see also Akhiezer and Glazman (1963; Appendix II, §9, Example II)).

We begin with the following result:

Theorem 6.7.2

Let $f \in D(T_c)$ *and note from Theorem* 6.2.1 *that* $Y_\nu(x,\lambda) = O(1)$ $(x \to 1)$ *and* $Z_\nu(x,\lambda) = O(|1-x|^{1/2-\nu})$ $(x \to 1)$; *then* $\lim_{x\to\pm 1} f(x)$ *exist and are finite, and if* $f(\pm 1) := \lim_{x\to\pm 1} f(x)$, *then* $f \in C[-1,1]$.

Proof: Let $f \in D(T_c)$, then from (2.1.14) (see also Titchmarsh (1962; Lemma 2.9)),

$$f(x) = \lambda\Phi(x,\lambda;f) - \Phi(x,\lambda;w^{-1}M[f]) \tag{6.7.6}$$

$$(x \in (-1,1),\ \lambda \in \mathbb{C} - P\sigma(T_c)),$$

where the function $\Phi : (-1,1) \times \left(\mathbb{C} - P\sigma(T_c)\right) \times L^2_w(-1,1) \to \mathbb{C}$ is defined by

$$\Phi(x,\lambda;f) = \frac{\psi_\nu(x,\lambda)}{n_c(\lambda) - m_c(\lambda)}\int_{-1}^{x} w(t)\chi_\nu(t,\lambda)f(t)dt$$

$$+ \frac{\chi_\nu(x,\lambda)}{n_c(\lambda) - m_c(\lambda)}\int_{x}^{1} w(t)\psi_\nu(t,\lambda)f(t)dt \tag{6.7.7}$$

and note that Φ, and hence f (see (6.7.6)), satisfies the boundary conditions (6.7.1).

Suppose $g \in L^2_w(-1,1)$ and $\lambda \in \mathbb{C} - P\sigma(T_c)$, then

$$(n_c(\lambda)-m_c(\lambda))\Phi(x,\lambda;g) = \psi_\nu(x,\lambda)\int_{-1}^{x} w(t)\chi_\nu(t,\lambda)g(t)dt$$

$$+ \chi_\nu(x,\lambda)\int_{x}^{1} w(t)\psi_\nu(t,\lambda)g(t)dt$$

and

$$|\chi_\nu(x,\lambda)\int_{x}^{1} w(t)\psi_\nu(t,\lambda)g(t)dt|$$

$$\leq \frac{|Z_\nu(x,\lambda)|}{|Z_\nu(0,\lambda)|}\left\{\int_{x}^{1} w(t)\frac{|Y_\nu(t,\lambda)|^2 dt}{|Y_\nu(t,\lambda)|^2}\int_{x}^{1} w(t)|g(t)|^2 dt\right\}^{1/2}$$

(on using the Cauchy-Schwarz inequality on the left-hand side, and the properties of ψ_ν and χ_ν (see §6.4))

$$= O((1-x)^{1/2-\nu}.(1-x)^{1/4+\nu/2}) \quad (x \to 1)$$

$$= O((1-x)^{3/4-\nu/2}) \quad (x \to 1)$$

$$= o(1) \quad (x \to 1) \tag{6.7.8}$$

(because $\nu < 3/2$).

Also

$$\lim_{x\to 1} \psi_\nu(x,\lambda)\int_{-1}^{x} w(t)\chi_\nu(t,\lambda)g(t)dt$$

$$= \frac{K(\lambda)}{|Y_\nu(0,\lambda)|}\int_{-1}^{1} w(t)\chi_\nu(t,\lambda)g(t)dt$$

(where $K(\lambda) = \lim_{x\to 1} Y_\nu(x,\lambda)$ is a constant given by (6.2.1), and this is finite because χ_ν and $g \in L_w^2(-1,1)$). There is a similar result if we take $x \to -1$.

Thus it now follows from (6.7.6) that if $f \in D(T_c)$ then $\lim_{x\to\pm 1} f(x)$

both exist and are finite; furthermore, if $f(\pm 1)$ is defined by these limits, then $f \in C[-1,1]$ for all $f \in D(T_c)$; hence the proof.

Now suppose $f \in \Delta$ and $\lim\limits_{\pm 1} f$ exist and are finite, then

$$\lim_{x \to \pm 1} p(x)f'(x) = 0 \ , \qquad (6.7.9)$$

for suppose, without loss of generality, that f is real-valued on $(-1,1)$; we have $w^{-1}M[f] \in L^2_w(0,1)$, i.e. $M[f] \in L(0,1)$ and $(pf')' \in L(0,1)$, hence for some $K \in R$

$$\lim_{1} pf' = (pf')(0) + \int_0^1 (pf')' = K \ . \qquad (6.7.10)$$

Now if $K \neq 0$, then we take $K > 0$ and obtain

$$f(x) = f(0) + \int_0^x p^{-1}(pf')$$

$$= f(0) + K\int_0^x p^{-1} \quad \text{(for x close to 1)}$$

$$\geq f(0) + \tfrac{1}{2}K\int_0^x p^{-1} \ ,$$

i.e. $\lim\limits_{x \to 1} f(x) = \infty$, a contradiction; hence $K = 0$ and $\lim\limits_{x \to 1} p(x)f'(x) = 0$. Similarly $\lim\limits_{x \to -1} p(x)f'(x) = 0$.

Now suppose $f \in \Delta$ and $\lim\limits_{\pm 1} pf' = 0$; then

$$|f(x)| \leq |f(0)| + \int_0^x |f'| = |f(0)| + \int_0^x p^{-1}|pf'|$$

$$\leq |f(0)| + K_0(1-x)^{1/2-\nu} \quad (x \in [0,1))$$

for some $k_0 \in R_+$. This last result together with (6.7.9) and the known properties of ψ_ν in §6.4 (recall that $[f,\psi_\nu] := p(f\psi_\nu' - f'\psi_\nu)$) proves that $\lim_{x\to 1} [f(x),\psi_\nu(x,\lambda)] = 0$ for any $\lambda \in \mathbb{C} - P\sigma(T_c)$; similarly $\lim_{x\to -1}[f(x),\chi_\nu(x,\lambda)] = 0$; hence $f \in D(T_c)$.

If now $f \in D(T_c)$, then from the above results $\lim_{\pm 1} pf'.\bar{f} = 0$; hence from

$$\int_{-X}^{X} \{p|f'|^2 + q|f|^2\} = [pf',\bar{f}]_{-X}^{X} + \int_{-X}^{X} M[f].f \qquad (6.7.11)$$

it follows that $p^{1/2}f' \in L^2(-1,1)$. Conversely, let $f \in \Delta$ and $p^{1/2}f' \in L^2(-1,1)$, then $M[f] \in L(0,1)$, and as in (6.7.10) $\lim_{\pm 1} pf' = k$ (say); if $k \neq 0$, then for x close to 1 it follows that for some $k_0 \in R_+$

$$\int^{x} p|f'|^2 \geq k_0 \int^{x} p^{-1}$$

and $p^{1/2}f' \notin L^2(0,1)$; this is a contradiction so $k = 0$; hence as above $f \in D(T_c)$.

Taking all these results together it follows that the domain $D(T_c)$ can be described in any one of the following five equivalent forms:

Theorem 6.7.3

Let $f \in \Delta$, then $f \in D(T_c)$ if either

(a) $\lim_{1} [f,\psi_\nu] = \lim_{-1} [f,\chi_\nu] = 0$; or

(b) $\lim_{\pm 1} f$ exist and are finite; or

(c) $\lim_{\pm 1} pf' = 0$; or

(d) $p^{1/2}f' \in L^2(-1,1)$; or

(e) $\lim_{1} [f,1] = \lim_{-1} [f,1] = 0$,

where in (e) the notation 1 is used to represent the function taking the value 1 on $(-1,1)$.

We can prove a little more; let $f \in D(T_c)$, then from (6.7.11) above we have

$$\int_{-1}^{1} \{p|f'|^2 + q|f|^2\} = \int_{-1}^{1} \{p|f'|^2 + \nu^2 w|f|^2\}$$

$$= \int_{-1}^{1} w^{-1} M[f].wf$$

$$= (T_c f, f)_w \quad (f \in D(T_c)),$$

i.e. $M[f]$ satisfies the so-called Dirichlet's formula on $D(T_c)$ (but not necessarily on Δ). Hence

$$(T_c f, f)_w \geq \nu^2 (f,f)_w \quad (f \in D(T_c),\ 1/2 \leq \nu < 3/2), \qquad (6.7.12)$$

with equality if and only if f is a constant over $(-1,1)$. This is a special case of a general inequality for self-adjoint operators which are bounded below; in fact the first eigenvalue λ_0 of T_c is $\lambda_0 = \nu^2$ $(1/2 \leq \nu < 3/2)$. For a further discussion on such inequalities, see Bradley-Everitt (1973) and Amos-Everitt (1978).

§6.8 The operator T_p in $L^2_w(-1,1)$: the limit-point case $(3/2 \leq \nu < \infty)$

Here, with $3/2 \leq \nu < \infty$, the differential expression

$$M[y] := -((1-x^2)^{\nu+1/2} y'(x))' + \nu^2 (1-x^2)^{\nu-1/2} y(x)$$
$$(x \in (-1,1))$$

is limit point at ± 1; hence the boundary conditions of the type (4.1.9) are not needed but are replaced by the integral conditions as in (4.1.1), (see also (4.1.3)).

Again let the linear manifold Δ be defined by (6.6.1), viz.

$$\Delta = \{f : (-1,1) \to \mathbb{C} : f, pf' \in AC_{loc}(-1,1)$$
$$\text{and } f, w^{-1}M[f] \in L^2_w(-1,1)\}$$

d let T_p be a linear operator defined by

$$T_p f := w^{-1}M[f] \quad (f \in D(T_p)) \tag{6.8.1}$$

iere $D(T_p) := \Delta$ is the domain of T_p; then T_p has the following
:operties:

heorem 6.8.1

(i) $D(T_p)$ is dense in $L^2_w(-1,1)$;
(ii) T_p is a symmetric operator in $L^2_w(-1,1)$;
iii) T_p is self-adjoint.

'roof: This follows from Theorem 4.1.3, with the end-points $a = -1$, $) = 1$, both being singular. So we shall omit it.

Thus, as in §6.6, T_p is the required self-adjoint (unbounded) differential operator, with a simple discrete spectrum

$$P\sigma(T_p) = \{(n+\nu)^2,\ n \in N_0,\ 3/2 \le \nu < \infty\}, \tag{6.8.2}$$

the corresponding eigenvectors are the Gegenbauer polynomials

$$\{C^{\nu}_n(\cdot),\ n \in N_0,\ 3/2 \le \nu < \infty\}, \tag{6.8.3}$$

which are complete in $L^2_w(-1,1)$.

Here we leave the right-definite case and turn to the study of the differential operators associated with Gegenbauer's differential equation in the left-definite case.

§6.9 The Properties of Φ

We recall from Corollary 3.5.2 that in the left-definite case, i.e. $H^2_{p,q}(-1,1)$, the differential expression

$$M[y] := -((1-x^2)^{\nu+1/2}y'(x))' + \nu^2(1-x^2)^{\nu-1/2}y(x) \qquad (x \in (-1,1))$$

is regular at ±1 if $-1/2 < \nu < 1/2$ and limit-point if $1/2 \le \nu < \infty$.

Here we use (3.4.8) to construct the resolvent function $\tilde{\Phi}$ as in the right-definite case; in fact we can identify $\tilde{\Phi}$ with Φ of (6.3.13)

$$\tilde{\Phi}(x,\lambda;f) = \Phi(x,\lambda;f) \tag{6.9.1}$$

but now defined for $x \in (-1,1)$, $\lambda \in \mathbb{C} - \{(n+\nu)^2,\ n \in N_0\}$ and all $f \in H^2_{p,q}(-1,1)$.

Also for either regular or limit-point case (see §6.3 or §6.5)

$$\omega(\lambda) = -\ 2Y'_\nu(0,\lambda)/Y_\nu(0,\lambda)$$

$$= \frac{4\Gamma(1-\nu/2+s/2)\Gamma(1-\nu/2-s/2)}{\Gamma(1/2-\nu/2+s/2)\Gamma(1/2-\nu/2-s/2)},$$

and at $\lambda = 0$,

$$\omega(0) = 4[\Gamma(1-\nu/2)]^2/[\Gamma(1/2-\nu/2)]^2 ;$$

hence, except for $\nu = 2n+1$ $(n \in N_0)$, Φ is regular at $\lambda = 0$. Put $\Psi(x,f) = \Phi(x,0;f)$, then (6.3.13) becomes

$$\Psi(x,f) = \frac{\psi_\nu(x,0)}{\omega(0)}\int_{-1}^{x} w(t)\chi_\nu(t,0)f(t)dt + \frac{\chi_\nu(x,0)}{\omega(0)}\int_{x}^{1} w(t)\psi_\nu(t,0)f(t)dt . \tag{6.9.2}$$

We prove now some properties of Ψ.

Lemma 6.9.1

Let $f \in H^2_{p,q}(-1,1)$ and Ψ be defined as above, then

$$M[\Psi(x,f)] = w(x)f(x) \quad (x \in (-1,1)).$$

Proof: This follows from part (a) of Theorem 4.1.2, with $\lambda = 0$, and the interval $[a,b)$ replaced with $(-1,1)$. So we shall omit it.

Theorem 6.9.2

Let $f \in H^2_{p,q}(-1,1)$ and let Ψ be defined by (6.9.2); and if $1/2 \le \nu < \infty$, then

(i) $\lim_{\pm 1} p\Psi'(\cdot,f)g = 0 \quad (f,g \in H^2_{p,q}(-1,1))$;

(ii) $\Psi(\cdot,f) \in L^2_w(-1,1) \quad (f \in H^2_{p,q}(-1,1))$.

Proof: We note from the properties of ψ_ν and χ_ν in §6.5, and the asymptotic results in Theorem 6.2.1, with $\nu \neq \frac{1}{2}$ (see Everitt [1980] for $\nu = \frac{1}{2}$), that

$$\left.\begin{aligned} &\psi_\nu(x,0) = O(1), \ \psi_\nu'(x,0) = O(1) \quad (x \to 1); \\ &\chi_\nu(x,0) = O((1-x)^{1/2-\nu}) \quad (x \to 1); \text{ and} \\ &\chi_\nu'(x,0) = O((1-x)^{-1/2-\nu}) \quad (x \to 1). \end{aligned}\right\} \quad (6.9.3)$$

Now for some $K \in R_+$, with $f \in H^2_{p,q}(-1,1)$,

$$\left|\int_{-1}^{x} w(t)\chi_\nu(t,0)f(t)dt\right| \le K \int_{-1}^{x} \left|(1-t)^{\nu-1/2}(1-t)^{1/2-\nu}f(t)\right|dt$$

$$= K \int_{-1}^{x} |f(t)|dt$$

$$\leq K\{\int_{-1}^{x} w^{-1}(t)dt\}^{1/2}\|f\|_H$$

(on using Cauchy-Schwarz inequality)

$$= K\{\int_{-1}^{x} (1-t)^{1/2-\nu}dt\}^{1/2}\|f\|_H;$$

hence

$$\int_{-1}^{x} w(t)\chi_\nu(t,0)f(t)dt = O((1-x)^{3/4-\nu/2}) \quad (x \to 1). \tag{6.9.4}$$

Also, for some $k_1 \in R_+$,

$$|\int_{x}^{1} w(t)\psi_\nu(t,0)f(t)dt| \leq k_1|\int_{x}^{1}(1-t)^{\nu-1/2}f(t)dt|$$

$$\leq k_1\{\int_{x}^{1}(1-t)^{\nu-1/2}dt\int_{x}^{1}(1-t)^{\nu-1/2}|f(t)|^2dt\}^{1/2}$$

(Cauchy-Schwarz inequality)

$$\leq k_1\{\int_{x}^{1}(1-t)^{\nu-1/2}dt\}^{1/2}\|f\|_H;$$

then

$$\int_{x}^{1} w(t)\psi_\nu(t,0)f(t)dt = O((1-x)^{\nu/2+1/4}) \quad (x \to 1). \tag{6.9.5}$$

Now let $g \in H^2_{p,q}(-1,1)$; then

$$|g(x)| \leq |g(0)| + |\int_{0}^{x} g'(t)dt|$$

$$\le |g(0)| + \{\int_0^x p^{-1}(t)dt \int_0^x p(t)|g'(t)|^2 dt\}^{1/2}$$

$$\le |g(0)| + \{\int_0^x (1-t)^{-1/2-\nu} dt\}^{1/2} \| g \|_H ;$$

then

$$g(x) = O((1-x)^{1/4-\nu/2}) \quad (x \to 1); \tag{6.9.6}$$

and, on using (6.9.3)-(6.9.6), we obtain

$$\frac{p\psi_\nu' g}{\omega(0)} \int_{-1}^{x} \omega\chi_\nu f = O((1-x)^{\nu+1/2}(1-x)^{1/4-\nu/2}(1-x)^{3/4-\nu/2})$$

$$= O((1-x)^{3/2}) \quad (x \to 1) \tag{6.9.7}$$

and

$$\frac{p\chi_\nu' g}{\omega(0)} \int_{x}^{1} \omega\psi_\nu f = O((1-x)^{\nu+1/2}(1-x)^{-1/2-\nu}(1-x)^{1/4-\nu/2}(1-x)^{\nu/2+1/4})$$

$$= O((1-x)^{1/2}) \quad (x \to 1). \tag{6.9.8}$$

Putting these together, we obtain

$$p(x)\Psi'(x,f)g(x) = O((1-x)^{1/2}) + O((1-x)^{3/2})$$

$$= o(1) \quad \text{as} \quad x \to 1 , \tag{6.9.9}$$

i.e.

$$\lim_{x\to 1} p(x)\Psi'(x,f)g(x) = 0 .$$

A similar result holds if $x \to -1$.

(ii) From (6.9.3)-(6.9.5)

$$\frac{\psi_\nu(x,0)}{\omega(0)}\int_{-1}^{x} w(t)\chi_\nu(t,0)f(t)dt = O((1-x)^{3/4-\nu/2}) \quad (x \to 1)$$

and

$$\frac{\chi_\nu(x,0)}{\omega(0)}\int_{x}^{1} w(t)\psi_\nu(t,0)f(t)dt = O((1-x)^{1/2-\nu}(1-x)^{\nu/2+1/4})$$

$$= O((1-x)^{3/4-\nu/2}) \quad (x \to 1),$$

i.e.

$$\Psi(x,f) = O((1-x)^{3/4-\nu/2}) \quad (x \to 1); \tag{6.9.10}$$

and for some constant $k > 0$

$$\int_{-1}^{1} w|\Psi(\cdot,f)|^2 \le k\int_{-1}^{1} (1-x)^{\nu-1/2}(1-x)^{3/2-\nu}dx < \infty ,$$

i.e.

$$\Psi(\cdot,f) \in L^2_w(-1,1) ;$$

this completes the proof.

Corollary 6.9.3

Let $f \in H^2_{p,q}(-1,1)$ and let Ψ be defined by (6.9.2), with $-1/2 < \nu < 1/2$; then

(i) $(p\Psi'(\cdot,f)g)(\pm 1) = 0 \quad (f,g \in H^2_{p,q}(-1,1)$;

(ii) $\Psi(\cdot,f) \in C[-1,1]$.

Proof: Part (i) follows from (6.9.9) and part (ii) from (6.9.10).

Theorem 6.9.4

Let $f \in H^2_{p,q}(-1,1)$ and let $\Psi(\cdot,f)$ be defined by (6.9.2), then $\Psi(\cdot,f) \in H^2_{p,q}(-1,1)$, (see also Theorem 7.5.4 (iii)).

Proof: Here we appeal to Lemma 6.9.1:

$$\int_{-X}^{X} wf\overline{\Psi}(\cdot,f) = \int_{-X}^{X} M[\Psi(\cdot,f)]\overline{\Psi}(\cdot,f) \qquad (f \in H^2_{p,q}(-1,1))$$

$$= \int_{-X}^{X} \{-(p\Psi'(\cdot,f))' + q\Psi(\cdot,f)\}\overline{\Psi}(\cdot,f)$$

$$= -[p\Psi'(\cdot,f)\overline{\Psi}(\cdot,f)]_{-X}^{X} + $$

$$+ \int_{-X}^{X} \{p|\Psi'(\cdot,f)|^2 + q|\Psi(\cdot,f)|^2\} \ ;$$

the integrated terms vanish by either Theorem 6.9.2 if $1/2 \le \nu < \infty$, or Corollary 6.9.3 if $-1/2 < \nu < 1/2$; hence

$$\int_{-1}^{1} wf\overline{\Psi}(\cdot,f) = \int_{-1}^{1} \{p|\Psi'(\cdot,f)|^2 + q|\Psi(\cdot,f)|^2\}$$

$$= ||\Psi(\cdot,f)||_H^2 \ .$$

On using Cauchy-Schwarz inequality on the left-hand side, we obtain

$$||\Psi(\cdot,f)||_H^2 \le \left|\int_{-1}^{1} wf\overline{\Psi}(\cdot,f)\right|$$

$$\leq \{\int_{-1}^{1} w|f|^2 \int_{-1}^{1} w|\Psi(\cdot,f)|^2\}^{1/2} \qquad (6.9.11)$$

$$< \infty$$

(because $f \in H^2_{p,q}(-1,1)$ and $\Psi(\cdot,f) \in L^2_w(-1,1)$ if $1/2 \leq \nu < \infty$, or $-1/2 < \nu < 1/2$); hence $\Psi(\cdot,f) \in L^2_w(-1,1)$.

§6.10 The Operator S_r in $H^2_{p,q}(-1,1)$: The Regular Case ($-1/2 < \nu < 1/2$, $\nu \neq 0$)

Consider now a linear operator A_r on the space $H^2_{p,q}(-1,1)$ defined by

$$(A_r f)(x) := \Psi(x,f) \quad (x \in (-1,1),\ f \in H^2_{p,q}(-1,1)) \qquad (6.10.1)$$

then A_r has the following properties:

Theorem 6.10.1

(i) A_r is a bounded linear operator on $H^2_{p,q}(-1,1)$;

(ii) the operator A_r is symmetric, and hence self-adjoint;

(iii) A_r has an inverse operator A_r^{-1}.

Proof:

(i) We note from (6.9.11) and (6.0.2) that

$$\|\Psi(\cdot,f)\|^2_H \leq \{\int_{-1}^{1} w|f|^2 \int_{-1}^{1} w|\Psi(\cdot,f)|^2\}^{1/2}$$

$$= \frac{1}{\nu^2}\{\int_{-1}^{1} q|f|^2 \int_{-1}^{1} q|\Psi(\cdot,f)|^2\}^{1/2} \qquad (\nu \neq 0\text{ , See foot note, pp 97})$$

$$\leq \frac{1}{\nu^2}\{\int_{-1}^{1}[p|f'|^2+q|f|^2]\int_{-1}^{1}[p|\Psi'(\cdot,f)|^2+q|\Psi(\cdot,f)|^2]\}^{1/2}$$

$$= \frac{1}{\nu^2}\,||f||_H\,||\Psi(\cdot,f)||_H$$

i.e.

$$||A_r f||_H = ||\Psi(\cdot,f)||_H \leq \frac{1}{\nu^2}\,||f||_H \qquad (\nu \neq 0);$$

hence the proof. Note that boundedness of A_r in this, the regular case, may also be deduced from part (ii) of Corollary 6.9.3.

(ii) Let $f,g \in H^2_{p,q}(-1,1)$; then

$$(A_r f,g)_H = (\Psi(\cdot,f),g)_H$$

$$= \int_{-1}^{1}\{p\Psi'(\cdot,f)\overline{g}' + q\Psi(\cdot,f)\overline{g}\}$$

$$= [p\Psi'(\cdot,f)\overline{g}]_{-1}^{1} + \int_{-1}^{1}\{-(p\Psi'(\cdot,f))' + q\Psi(\cdot,f)\}\overline{g}$$

$$= \int_{-1}^{1}\{-(p\Psi'(\cdot,f))' + q\Psi(\cdot,f)\}\overline{g} \quad \text{(Corollary 6.9.3)}$$

$$= \int_{-1}^{1} M[\Psi(\cdot,f)]\overline{g}$$

$$= \int_{-1}^{1} wf\overline{g} \quad \text{(Lemma 6.9.1)}$$

$$= \int_{-1}^{1} fM[\Psi(\cdot,\overline{g})]$$

$$= \int_{-1}^{1} \{pf'\overline{\Psi}'(\cdot,g) + qf\overline{\Psi}(\cdot,g)\}$$

(on reversing the argument)

$$= (f,\Psi(\cdot,g))_H ,$$

i.e.

$$(A_r f,g)_H = (f,A_r g)_H ;$$

hence A_r is symmetric and hence self-adjoint.

(iii) Suppose $A_r f = 0$ ($f \in H^2_{p,q}(-1,1)$); then $\Psi(\cdot,f) = 0$ and $0 = M[\Psi(\cdot,f)] = wf$ on $(-1,1)$. Since $w > 0$ on $(-1,1)$, it follows that $f = 0$; thus A_r has an inverse operator A_r^{-1}.

The existence of the inverse A_r^{-1} leads us to define the following operator:

$$S_r : D(S_r) \subset H^2_{p,q}(-1,1) \to H^2_{p,q}(-1,1)$$

by

$$\left.\begin{aligned} & D(S_r) := \{A_r f : f \in H^2_{p,q}(-1,1)\} \\ \text{and} \quad & \\ & S_r f := A_r^{-1} f \quad (f \in D(S_r)). \end{aligned}\right\} \qquad (6.10.2)$$

Then S_r is self-adjoint, bounded or unbounded (see Akhiezer and Glazman (1963; §41, Corollary to Theorem 1)) in $H^2_{p,q}(-1,1)$. Using the argument of Everitt (1980; §4), it may be shown that S_r is unbounded. Thus S_r is an unbounded self-adjoint operator in $H^2_{p,q}(-1,1)$.

Suppose λ is an eigenvalue of S_r, i.e. for some eigenvector $f \neq 0$, $S_r f = \lambda f$; then $A_r f = \lambda^{-1} f$ (note that $\lambda \neq 0$ because $S_r^{-1} := A_r$ exists). Hence $\Psi(\cdot,f) = \lambda^{-1} f$ on $(-1,1)$, and from Lemma 6.9.1,

$M[\Psi(\cdot,f)] = M[\lambda^{-1}f] = wf$ on $(-1,1)$, i.e. $M[f] = \lambda wf$ on $(-1,1)$. Hence f is a non-trivial solution of Gegenbauer's equation (6.0.1) in $H^2_{p,q}(-1,1)$: furthermore, since this is a regular case, $(pf')(\pm 1) = 0$. From the properties of the solutions Y_ν and Z_ν given in §6.1 and §6.2, this can only happen if Y_ν and Z_ν are linearly dependent, i.e. $\lambda \in P\sigma(S_r)$, where

$$P\sigma(S_r) = \{(n+\nu)^2,\ n \in N_0,\ -1/2 < \nu < 1/2\} \qquad (6.10.3)$$

(see Remark 6.3.1), and f is then linearly dependent on the corresponding Gegenbauer polynomials from the set $\{C_n^\nu(\cdot),\ n \in N_0,\ -1/2 < \nu < 1/2\}$. Conversely every point in the set $P\sigma(S_r)$ is an eigenvalue with the corresponding eigenvector in $\{C_n^\nu(\cdot),\ n \in N_0,\ -1/2 < \nu < 1/2\}$.

As in §6.6, it can be shown that the spectrum $\sigma(S_r) = P\sigma(S_r)$. The spectral theorem for self-adjoint operators in a Hilbert space (Akhiezer and Glazman (1963; Chapter VI)) now implies that the Gegenbauer polynomials form a complete orthogonal set in $H^2_{p,q}(-1,1)$, and hence in $L^2_w(-1,1)$, because the set $H^2_{p,q}(-1,1)$ is dense in $L^2_w(-1,1)$.

§6.11 The operator S_p in $H^2_{p,q}(-1,1)$: the limit-point case $(1/2 \le \nu < \infty)$

Here we consider a linear operator A_p on the space $H^2_{p,q}(-1,1)$ given by

$$(A_p f)(x) = \Psi(x,f) \qquad (x \in (-1,1),\ f \in H^2_{p,q}(-1,1)) \qquad (6.11.1)$$

then, on using Lemma 6.9.1, Theorems 6.9.2 and 6.9.4 and the analysis of §6.10 above, it may be shown that A_p has the following properties:

Theorem 6.11.1

(i) A_p is a bounded linear operator in $H^2_{p,q}(-1,1)$;

(ii) A_p is a symmetric, and hence self-adjoint operator in $H^2_{p,q}(-1,1)$;

(iii) A_p has an inverse operator A_p^{-1}.

Also, if we define the operator $S_p : D(S_p) \subset H^2_{p,q}(-1,1) \to H^2_{p,q}(-1,1)$ by

$$\left.\begin{aligned} &D(S_p) := \{A_p f : f \in H^2_{p,q}(-1,1)\} \\ \text{and} \\ &S_p f := A_p^{-1} f \quad (f \in D(S_p)) \end{aligned}\right\} \tag{6.11.2}$$

then, as in §6.10, it can be shown that S_p is a self-adjoint unbounded operator, having a simple discrete spectrum

$$P\sigma(S_p) = \{(n+\nu)^2,\ n \in N_0,\ 1/2 \le \nu < \infty\} \tag{6.11.3}$$

with the Gegenbauer polynomials $\{C_n^{\nu}(\cdot),\ n \in N_0,\ 1/2 \le \nu < \infty\}$ as the corresponding eigenvectors; and, furthermore, these polynomials are complete in $H^2_{p,q}(-1,1)$, and hence in $L^2_w(-1,1)$.

§6.12 The operator T: alternative definition

It is of interest to note that we could have defined any of the operators $T = T_r, T_c, T_p$ in the same way as S_r and S_p above. We take as an example $T = T_p$ (the limit-point case).

Let $f \in L^2_w(-1,1)$, and let the resolvent function in this case be given by

$$\Psi(x,f) := \Phi(x,0;f) = -\int_{-1}^{1} w(t)G(x,t,0)f(t)\,dt \tag{6.12.1}$$

(where $G(x,t,0)$ is given by (6.5.4)); and define a linear operator B_p in $L_w^2(-1,1)$ by

$$(B_p f)(x) := \Psi(x,f) \quad (x \in (-1,1),\ f \in L_w^2(-1,1)). \qquad (6.12.2)$$

Then, with arguments similar to those of §6.10 (and hence §6.11 in the case of A_p), we may prove that B_p is a bounded symmetric operator on $L_w^2(-1,1)$ into $L_w^2(-1,1)$, that the inverse B_p^{-1} exists and the operator T_p, as defined in §6.8, satisfies

$$T_p = B_p^{-1} .$$

This gives, therefore, an alternative way of determining the operator T_p in the right-definite case. Both T_r and T_c may also be determined in a similar way.

§6.13 Remarks on the operators T and S

It is of some interest to compare the operators T and S, where T denotes the operator T_r, T_c or T_p in the right-definite case; and S denotes S_r or S_p in the left-definite case.

T is an unbounded self-adjoint, differential operator in $L_w^2(-1,1)$ with a simple, discrete spectrum $\{(n+\nu)^2,\ n \in N_0\}$ and corresponding eigenvectors $\{C_n^\nu(\cdot),\ n \in N_0\}$.

S is an unbounded, self-adjoint operator in $H_{p,q}^2(-1,1)$ with the same simple, discrete spectrum and eigenvectors; however, we hesitate to call S a differential operator for the reasons given below.

The operator T, and its domain D(T), is defined directly in terms of the Gegenbauer differential expression $M[y] = -(py')' + qy$, with the coefficients p and q given by (6.0.2); also in the limit-circle case, we are able to give alternative and simplified descriptions of D(T) (see Theorem 6.7.3).

For the operator S, the situation is different; we defined S as the inverse A^{-1} of a bounded symmetric operator in $H^2_{p,q}(-1,1)$. While we can say something about the elements of D(S), it does not seem possible to characterise the operator S directly in terms of M[y]. The definition of S as S := A^{-1} depends upon a general theorem in Hilbert space theory (Akhiezer and Glazman (1963; §41)), which provides for the existence of S, but does not give a constructive definition in general. Thus S appears as a differential operator only in an indirect sense in comparison with T.

Note that in §6.12, and in the sense that $H^2_{p,q}(-1,1) \subset L^2_w(-1,1)$, we have A = B on $H^2_{p,q}(-1,1)$. However, the inverse A^{-1} has to be determined in $H^2_{p,q}(-1,1)$ and B^{-1} in $L^2_w(-1,1)$ so that there is no identification of S with T, even on D(S).

Thus in the right-definite case we are able to give an explicit characterisation of the inverse B^{-1} = T, a characterisation which is not possible in the left-definite case.

CHAPTER SEVEN

THE LAGUERRE EXPANSION

§7.0 Preliminary

One example of the symmetric differential equation (3.0.1), in which the interval (a,b) is unbounded, is Laguerre's equation given by (1.6.5), but which for our purposes may be written as

$$-(x^{\alpha+1}e^{-x}y'(x))' + \frac{\alpha+1}{2}x^{\alpha}e^{-x}y(x) = \lambda x^{\alpha}e^{-x}y(x) \qquad (7.0.1)$$

$$(x \in (0,\infty),\ \lambda \in \mathbb{C},\ \alpha > -1);$$

where $a = 0$, $b = \infty$ and the coefficients

$$p(x) = x^{\alpha+1}e^{-x}\ ,\ q(x) = \frac{\alpha+1}{2}w(x)\ ,\ w(x) = x^{\alpha}e^{-x}\ . \qquad (7.0.2)$$

First we consider the solutions of (7.0.1) in §7.1, and in §7.2; we study their asymptotic behaviour near the end-points 0 and ∞, and then adopt in §7.3 Titchmarsh's method (1962; §4.16) to obtain the Laguerre polynomials as the eigenvectors generated by (7.0.1). Finally, we consider in §7.4 and §7.5 the associated self-adjoint operators T_{α} in the right-definite case (i.e. in $L^2_w(0,\infty)$), and S_{α} in the left-definite case (i.e. in $H^2_{p,q}(0,\infty) := H^2_{\alpha,p,q}(0,\infty)$) respectively.

§7.1 The solutions of Laguerre's equation

The main result in this section is the following:

Theorem 7.1.1

Let $0 \leq \arg(-z) < 2\pi$ ($\lambda \in \mathbb{C}$), and let $\alpha > -1$ and $x \in (0,\infty)$ be given as above. Then a solution of (7.0.1) is of the form

$$Y_\alpha(x,\lambda) = \frac{-\Gamma((1-\alpha)/2+\lambda).x^{-\alpha}}{2\pi i}\int_\infty^{(0+)} \frac{\exp[-z].(z+x)^{\lambda+(\alpha-1)/2}}{(-z)^{\lambda-(\alpha-1)/2}}\,dz \qquad (7.1.1)(a)$$

$$(\mathrm{re}(\lambda-(\alpha+1)/2)\neq -n,\ n\in N_+)$$

$$= \frac{x^{-\alpha}}{\Gamma((\alpha+1)/2-\lambda)}\int_0^\infty \frac{\exp[-z].(z+x)^{\lambda+(\alpha-1)/2}}{z^{\lambda-(\alpha-1)/2}}\,dz \qquad (7.1.1)(b)$$

(otherwise).

Here, the contour of integration C, from $+\infty$ round 0 to $+\infty$, excludes the point $z = -x$ (note that the solution $Y_\alpha(\cdot,\lambda)$ remains defined as $x \to 0$, provided $\alpha > 0$).

A second solution of (7.0.1) takes the following form

$$Z_\alpha(x,\lambda) = 1 + \sum_{k=1}^{\infty} \frac{((\alpha+1)/2-\lambda)_k}{(\alpha+1)_k}.\frac{x^k}{k!} \qquad (7.1.2)$$

$$(\alpha > -1,\ x \in (0,\infty)).$$

Proof: First we note that $z = 0$ is a branch-point of order $\lambda - (\alpha-1)/2$. To render the integrand single-valued, we cut the z-plane from 0 to $+\infty$, with $0 \le \arg(-z) < 2\pi$. Then we take the value of $\arg(z+x)$ which tends to zero as $z \to 0$ by a path lying inside the contour C. Also, let

$$(-z)^{\lambda-(\alpha-1)/2} := \exp[(\lambda-(\alpha-1)/2).\log(-z)]$$

$$= \exp[(\lambda-(\alpha-1)/2).\log|-z|].\exp[(\lambda-(\alpha-1)/2)i\arg(-z)]$$

and

$$(z+x)^{\lambda+(\alpha-1)/2} := \exp[(\lambda+(\alpha-1)/2).\log|z+x|].\exp[(\lambda+(\alpha-1)/2)i\arg(z+x)].$$

Thus under these circumstances the integrand is analytic in x and z.

To show the convergence of the integral (7.1.1)(b), let $z = t + is$, $x + z = x + t + is$, and $\lambda = \mu + i\nu$. Then we have $\arg(-z) = \tan^{-1}(-s/-t)$

$$\lim_{x\to\infty} \left\{\frac{\int_a^x e^t t^{-\alpha-2} dt}{e^x x^{-\alpha-2}}\right\} = \lim_{x\to\infty} \left\{\frac{e^x x^{-\alpha-2}}{e^x x^{-\alpha-2}(1-(\alpha+2)x^{-1})}\right\}$$

$$= 1 ;$$

hence, for some constant $k_1 \in \mathbb{R}_+$,

$$\left|\int_a^x e^t t^{-\alpha-1} dt\right| \le e^x|x|^{-\alpha-1} + k_1 e^x|x|^{-\alpha-2} ,$$

i.e.

$$g(x) = e^{x/2} x^{-(\alpha+1)/2}(1+O(|x|^{-1/2})) \quad (x \to \infty). \tag{7.5.16}$$

Now, from (7.5.2), we have

$$p\Psi' g = \frac{p\psi_0' g}{\omega(0)}\int_0^x w\phi_0 f + \frac{p\phi_0' g}{\omega(0)}\int_x^\infty w\psi_0 g ; \tag{7.5.17}$$

then, from (7.5.4), (7.5.15) and part (i) of Lemma 7.5.2 (see (7.5.6) and (7.5.7)),

$$\frac{p\psi_0' g}{\omega(0)}\int_0^x w\psi_0 g = O(|x|^{\alpha+1} e^{-x}|x|^{-\alpha-1}|x|^{-\alpha/2}|x|^{(\alpha+1)/2})$$

$$= O(e^{-x}|x|^{1/2}) \quad (x \to 0)$$

$$= o(1) \quad (x \to 0) \tag{7.5.18}$$

and

$$\frac{p\phi_0' g}{\omega(0)}\int_x^\infty w\psi_0 g = O(|x|^{\alpha+1} e^{-x}|x|^{-\alpha/2}|x|^{-\alpha/2} e^{-x/2})$$

$$= O(|x| e^{-3x/2}) \quad (x \to 0)$$

$$= o(1) \quad (x \to 0). \tag{7.5.19}$$

Putting these together, we have, from (7.5.17),

$$p(x)\psi'(x,f)g(x) = o(1) \quad (x \to 0)$$

and this proves part (i) above.

(ii) For this part, we use (7.5.5), (7.5.16) and part (ii) of Lemma 7.5.2 (i.e. (7.5.8) and (7.5.9)) to obtain

$$\frac{p\psi_0'g}{\omega(0)}\int_0^x w\phi_0 f = O(|x|^{\alpha+1}e^{-x}|x|^{-(\alpha+3)/2}|x|^{-(\alpha+1)/2}e^{x/2}|x|^{-1/2}e^{x/2})$$

$$= O(|x|^{-3/2}) \quad (x \to \infty)$$

$$= o(1) \quad (x \to \infty) \tag{7.5.20}$$

and

$$\frac{p\phi_0'g}{\omega(0)}\int_x^\infty w\psi_0 f = O(|x|^{\alpha+1}e^{-x}|x|^{-(\alpha+1)/2}e^{x}|x|^{-(\alpha+1)/2}e^{x/2}|x|^{-1/2}e^{-x/2})$$

$$= O(|x|^{-1/2}) \quad (x \to \infty)$$

$$= o(1) \quad (x \to \infty). \tag{7.5.21}$$

Again, putting these together, we have, from (7.5.17),

$$p(x)\Psi'(x,f)g(x) = o(1) \quad (x \to \infty),$$

i.e.

$$\lim_{x\to\infty} p(x)\Psi'(x,f)g(x) = 0$$

and this completes the proof.

The results of Lemmas 7.5.1 and 7.5.2 may also be used to establish the following properties of Ψ:

Theorem 7.5.4

Let $f \in H^2_{p,q}(0,\infty)$; then

(i) $\Psi(\cdot,f) \in L^2_w(0,\infty)$; and

(ii) $\Psi(\cdot,f) \in H^2_{p,q}(0,\infty)$

Proof:

(i) From (7.5.4), (7.5.6) and (7.5.7), we have

$$\frac{\psi_0(x,0)}{\omega(0)}\int_0^x w(t)\phi_0(t,0)f(t)dt = O(|x|^{-\alpha}|x|^{(\alpha+1)/2})$$

$$= O(|x|^{(1-\alpha)/2}) \quad (x \to 0)$$

and

$$\frac{\phi_0(x,0)}{\omega(0)}\int_x^\infty w(t)\psi_0(t,0)f(t)dt = O(|x|^{-\alpha/2}e^{-x/2}) \quad (x \to 0);$$

hence

$$\Psi(x,f) = O(|x|^{1/2-\alpha/2}) + O(|x|^{-\alpha/2}e^{-x/2}) \quad (x \to 0).$$

Thus, for some $k_1,k_2,k_3 \in R_+$,

$$\int_0 w|\Psi(\cdot,f)|^2 \le \int_0 x^\alpha e^{-x}(k_1|x|^{1-\alpha}+k_2|x|^{-\alpha}e^{-x}+k_3|x|^{1/2-\alpha}e^{-x/2})dx$$

$$< \infty . \qquad (7.5.22)$$

On the other hand, if $x \to \infty$, then, from (7.5.5), (7.5.8) and (7.5.9), we have

$$\frac{\psi_0(x,0)}{\omega(0)}\int_0^x w(t)\phi_0(t,0)f(t)dt = O(|x|^{-(\alpha+1)/2}e^{x/2}|x|^{-1/2})$$

$$= O(|x|^{-(\alpha/2)-1}e^{x/2}) \quad (x \to \infty)$$

and

$$\frac{\phi_0(x,0)}{\omega(0)}\int_x^\infty w(t)\psi_0(t,0)dt = O(|x|^{-(\alpha+1)/2}e^{x}|x|^{-1/2}e^{-x/2})$$

$$= O(|x|^{-(\alpha/2)-1}e^{x/2}) \quad (x \to \infty),$$

i.e.

$$\Psi(x,0) = O(|x|^{-(\alpha/2)-1}e^{x/2}) \quad (x \to \infty).$$

Hence, in the neighbourhod of $x = \infty$ and some $k_0 \in R_+$,

$$\int^\infty w(x)|\Psi(x,f)|^2dx \leq k_0\int^\infty x^\alpha e^{-x}|x|^{-\alpha-2}e^{x}dx$$

$$= k_0\int^\infty x^{-2}dx < \infty . \tag{7.5.23}$$

Thus, from (7.5.22) and (7.5.23), we conclude that $\Psi(\cdot,f) \in L^2_w(0,\infty)$. (Note that this result may also be deduced from part (c) of Theorem 4.1.2.)

(ii) For this part, we appeal to Lemma 7.5.1 as follows:
Let $f \in H^2_{p,q}(0,\infty)$; then, for $0 < a < b < \infty$,

$$\int_a^b wf\overline{\Psi}(\cdot,f) = \int_a^b M[\Psi(\cdot,f)]\overline{\Psi}(\cdot,f) \quad \text{(Lemma 7.5.1)}$$

$$= \int_a^b \{-(p\Psi'(\cdot,f))' + q\Psi(\cdot,f)\}\overline{\Psi}(\cdot,f)$$

$$= [-p\Psi'(\cdot,f)\overline{\Psi}(\cdot,f)]_a^b + \int_a^b \{p|\Psi'(\cdot,f)|^2 + q|\Psi(\cdot,f)|^2\} .$$

We now let $a \to 0$ and $b \to \infty$; and use Theorem 7.5.3 to give

$$\int_0^\infty wf\overline{\Psi}(\cdot,f) = \int_0^\infty \{p|\Psi'(\cdot,f)|^2 + q|\Psi(\cdot,f)|^2\}$$

$$= \|\Psi(\cdot,f)\|_H^2 ;$$

also Cauchy-Schwarz's inequality, applied to the left-hand side, gives

$$\|\Psi(\cdot,f)\|_H^2 = |\int_0^\infty wf\Psi(\cdot,f)|$$

$$\leq \{\int_0^\infty w|f|^2 \int_0^\infty w|\Psi(\cdot,f)|^2\}^{1/2} \tag{7.5.24}$$

$$< \infty$$

(because $f \in H^2_{p,q}(0,\infty) \subset L^2_w(0,\infty)$, and $\Psi(\cdot,f) \in L^2_w(0,\infty)$; see part (i) above); hence $\Psi(\cdot,f) \in H^2_{p,q}(0,\infty)$; this completes the proof.

§7.6 The operator S_α in $H^2_{p,q}(0,\infty)$: the left-definite case

First we recall that the space $H^2_{p,q}(0,\infty)$ is defined by

$$H^2_{p,q}(0,\infty) = \{f : (0,\infty) \to \mathbb{C} : f \in AC_{loc}(0,\infty),\ q^{1/2}f \in L^2(0,\infty) \text{ and } p^{1/2}f' \in L^2(0,\infty)\} \tag{7.6.1}$$

(where $p(x) = x^{\alpha+1}e^{-x}$, $q(x) = (\alpha+1)/2.x^\alpha e^{-x}$ $(\alpha > -1)$) with the norm

$$\| f \|_H = \{\int_0^\infty [p|f'|^2 + q|f|^2]\}^{1/2} .$$

Now, with the above properties of the function $\Psi(\cdot,f)$ (see §7.5) in mind, we define a linear operator A_α on $H^2_{p,q}(0,\infty)$ by

$$(A_\alpha f)(x) := \Psi(x,f) \qquad (x \in (0,\infty),\ f \in H^2_{p,q}(0,\infty)). \tag{7.6.2}$$

Then A_α has the following properties:

Theorem 7.6.i

(i) A_α is a bounded linear operator on $H^2_{p,q}(0,\infty)$;

(ii) the operator A_α is symmetric, and hence self-adjoint;

(iii) A_α has an inverse operator A_α^{-1}.

Proof:

(i) Note from (7.5.24) that

$$\| \Psi(\cdot,f) \|^2_H \le \{\int_0^\infty w|f|^2 \int_0^\infty w|\Psi(\cdot,f)|^2\}^{1/2}$$

$$= \frac{2}{\alpha+1}\{\int_0^\infty \frac{\alpha+1}{2} w|f|^2 \int_0^\infty \frac{\alpha+1}{2} w|\Psi(\cdot,f)|^2\}^{1/2} \qquad (\alpha > -1)$$

$$= \frac{2}{\alpha+1}\{\int_0^\infty q|f|^2 \int_0^\infty q|\Psi(\cdot,f)|^2\}^{1/2}$$

$$\le \frac{2}{\alpha+1}\{\int_0^\infty [p|f'|^2 + q|f|^2] \int_0^\infty [p|\Psi'(\cdot,f)|^2 + q|\Psi(\cdot,f)|^2]\}^{1/2},$$

i.e.

$$\|\Psi(\cdot,f)\|_H^2 \le \frac{2}{\alpha+1}\|f\|_H \|\Psi(\cdot,f)\|_H$$

or

$$\|A_\alpha f\|_H = \|\Psi(\cdot,f)\|_H \le \frac{2}{\alpha+1}\|f\|_H$$

and this implies that A_α is bounded in $H^2_{p,q}(0,\infty)$.

(ii) Let $f,g \in H^2_{p,q}(0,\infty)$; then

$$(A_\alpha f,g)_H = (\Psi(\cdot,f),g)_H$$

$$= \lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b \{p\Psi'(\cdot,f)\overline{g}' + q\Psi(\cdot,f)\overline{g}\}$$

(where $0 < a < b < \infty$)

$$= \lim_{\substack{a\to 0\\ b\to\infty}} [p\Psi'(\cdot,f)\overline{g}]_a^b + \lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b \{-(p\Psi'(\cdot,f))' + q\Psi(\cdot,f)\}\overline{g}$$

$$= \int_0^\infty M[\Psi(\cdot,f)]\overline{g} \quad \text{(Theorem 7.5.3)}$$

$$= \int_0^\infty wf\overline{g} \quad \text{(Lemma 7.5.1)}$$

$$= \int_0^\infty f\cdot\overline{M[\Psi(\cdot,g)]}$$

$$= \int_0^\infty \{pf'\overline{\Psi}'(\cdot,g) + qf\overline{\Psi}(\cdot,g)\}$$

(on reversing the argument)

$$= (f,\Psi(\cdot,g))_H ,$$

<u>i.e.</u>

$$(A_\alpha f,g)_H = (f,A_\alpha g)_H \ ,$$

i.e. A_α is symmetric in $H^2_{p,q}(0,\infty)$, and hence self-adjoint (see part (i)).

(iii) Let $A_\alpha f = 0$ $(f \in H^2_{p,q}(0,\infty))$; then $\Psi(\cdot,f) = 0$, and

$$0 = M[\Psi(\cdot,f)] = wf \quad \text{on} \quad (0,\infty) \ .$$

Since $w > 0$ on $(0,\infty)$, it follows that $f = 0$. Thus A_α has an inverse operator A_α^{-1}.

The existence of A_α^{-1} leads us to define the following operator $S_\alpha : D(S_\alpha) \subset H^2_{p,q}(0,\infty) \to H^2_{p,q}(0,\infty)$ by

$$\left.\begin{aligned} &D(S_\alpha) := \{A_\alpha f : f \in H^2_{p,q}(0,\infty)\}, \text{ and} \\ &S_\alpha f := A_\alpha^{-1} f \quad (f \in D(S_\alpha)); \end{aligned}\right\} \tag{7.6.3}$$

then S_α is self-adjoint, bounded or unbounded in $H^2_{p,q}(0,\infty)$ (see Akhiezer and Glazman (1963; §41, Corollary to Theorem 1)). Using the arguments of Everitt (1980; §4), it may be shown that S_α is unbounded, i.e. S_α is a self-adjoint unbounded operator. Also, a similar analysis to that of §6.10 shows that S_α has a simple discrete spectrum

$$P\sigma(S_\alpha) = \{n + \frac{\alpha+1}{2},\ n \in N_0,\ \alpha > -1\} \tag{7.6.4}$$

and the corresponding eigenvectors being the Laguerre polynomials $\{L_n^{(\alpha)}(x),\ n \in N_0,\ \alpha > -1\}$; and furthermore these polynomials are complete in $H^2_{p,q}(0,\infty)$ and hence in $L^2_w(0,\infty)$ (see §6.6 and 6.10; also Everitt (1980) in the case of Legendre polynomials).

Remark 7.6.2

(i) The results of §6.12 concerning the alternative definition of T also hold in the case of T_α;

(ii) the comparisons given in §6.13 may also be extended to the case of T_α in $L^2_w(0,\infty)$ and S_α in $H^2_{p,q}(0,\infty)$.

CHAPTER EIGHT

THE HERMITE EXPANSION

§8.0 Preliminary

Another example of the general symmetric equation (3.0.1), in which the interval (a,b) is unbounded, is Hermite's equation given by (1.7.5), but which in our case we write (with 2n replaced by $\lambda - 1$) as

$$-(e^{-x^2} y'(x))' + e^{-x^2} y(x) = \lambda e^{-x^2} y(x) \qquad (8.0.1)$$

$$(x \in (-\infty,\infty),\ \lambda \in \mathbb{C}).$$

In comparison with (3.0.1), here $a = -\infty$, $b = +\infty$, and the coefficients p, q and w are

$$p(x) = q(x) = w(x) = e^{-x^2} . \qquad (8.0.2)$$

The work in this chapter follows a similar pattern to that of Chapter Seven. We consider the solutions of (8.0.1) in §8.1, followed by their asymptotic behaviour near $-\infty$ and $+\infty$ in §8.2. We then apply Titchmarsh's method (1962; §4.2) in §8.3 to obtain the Hermite polynomials as eigenvectors generated by (8.0.1). Finally, we discuss in §8.4 and §8.5 the associated self-adjoint operators T in $L^2_w(-\infty,\infty)$ (*i.e.* right-definite case), and S in $H^2_{p,q}(-\infty,\infty)$ (*i.e.* left-definite case) respectively.

§8.1 The solutions of Hermite's equation

We begin with the following result:

Theorem 8.1.1

Consider Hermite's differential equation (8.0.1) above; and let $0 \le \arg z < 2\pi$, $\lambda \in \mathbb{C}$ and $x \in (-\infty,\infty)$. Then one solution of (8.0.1) is of the form

$$Y(x,\lambda) = 2^{1/2-\lambda/2}\int_{\infty}^{(0+)} z^{-\lambda/2-1/2}\exp[-2xz - z^2]dz \qquad (8.1.1)$$

where the contour of integration C encircles the origin, $z = 0$, of the z-plane in a positive (i.e. anti-clockwise) direction from $+\infty$ to $+\infty$.

Because of the symmetry of the equation (8.0.1) on $(-\infty,\infty)$, another solution $Z(x,\lambda) = Y(-x,\lambda)$ may be obtained by replacing x in (8.1.1) by -x, i.e.

$$Z(x,\lambda) = 2^{1/2-\lambda/2}\int_{\infty}^{(0+)} z^{-\lambda/2-1/2}\exp[\ 2xz - z^2]dz \qquad (8.1.2)$$

$$(x \in (-\infty,\infty),\ \lambda \in \mathbb{C},\ 0 \le \arg z < 2\pi).$$

Proof: Note that $z = 0$ is a branch-point of order $(\lambda+1)/2$ of the integrand. For the integrand to be single-valued, it is necessary to make the cut in the z-plane, from 0 to $+\infty$, with $0 \le \arg z < 2\pi$; and define

$$z^{-(\lambda+1)/2} := \exp[-((\lambda+1)/2)\log z] = \exp[-(\lambda+1)/2(\log|z|+i\arg z)] \qquad (8.1.3)$$

so that log z is real at the beginning of the contour.

Under these circumstances, the integrand is then single-valued, and analytic in x and z. Hence, on using a standard result in Titchmarsh (1978; §2.83), it follows from

$$Y(x,\lambda) = 2^{1/2-\lambda/2}\int_{\infty}^{(0+)} z^{-(\lambda+1)/2}\exp[-2xz - z^2]dz$$

that

$$Y'(x,\lambda) = -2^{3/2-\lambda/2}\int_{\infty}^{(0+)} z^{(1-\lambda)/2}\exp[-2xz - z^2]dz$$

and

$$e^{-x^2}Y'(x,\lambda) = -2^{3/2-\lambda/2}\int_{\infty}^{(0+)} z^{(1-\lambda)/2}\exp[-x^2 - 2xz - z^2]dz \ .$$

Again we apply the same argument to obtain

$$(e^{-x^2}Y'(x,\lambda))' = -2^{(3-\lambda)/2}\int_{\infty}^{(0+)} z^{(1-\lambda)/2}(-2x-2z)\exp[-x^2 - 2xz - z^2]dz$$

(and on integrating the right-hand side by parts; see Caratheodory (1954; §229))

$$= 2^{(3-\lambda)/2}e^{-x^2}\int_{\infty}^{(0+)} ((1-\lambda)/2)z^{-(\lambda+1)/2}\exp[-2xz - z^2]dz$$

$$= 2^{(1-\lambda)/2}(1-\lambda)e^{-x^2}\int_{\infty}^{(0+)} z^{-(\lambda+1)/2}\exp[-2xz - z^2]dz$$

$$= (1-\lambda)e^{-x^2}Y(x,\lambda) \ ,$$

i.e.

$$-(e^{-x^2}Y(x,\lambda)')' = (\lambda-1)e^{-x^2}Y(x,\lambda) \ ,$$

i.e. Y satisfies Hermite's differential equation (8.0.1). A similar proof holds also for the solution Z.

Note that if z is real then the integral (8.1.1) representing the solution Y converges at 0 (if $\mathrm{re}\lambda < 1$) and at $+\infty$, for let $\lambda = \mu + i\nu \in \mathbb{C}$ $(\mu,\nu \in \mathbb{R})$ and $z := t$ is real, then

$$\int_0^\infty \left| z^{-(\lambda+1)/2} \exp[-2xz - z^2]dz \right| = \int_0^\infty e^{-2xt-t^2} . t^{-(\mu+1)/2} dt$$

$$= e^{x^2} \{ \int_0^1 + \int_1^\infty \} e^{-(x+t)^2} t^{-(\mu+1)/2} dt .$$

The first integral on the right-hand side converges at 0 if $\mu := \mathrm{re}\lambda < 1$; and for each $x \in (-\infty,\infty)$ and all μ, the second integral also converges at $+\infty$. Thus, if z is real then the integral (8.1.1) is convergent at 0 (provided $\mathrm{re}\lambda < 1$) and at $+\infty$.

It is sometimes convenient to express the solutions of Hermite's differential equation in terms of parabolic cylinder functions. We do this as follows:

Put $y(x) = e^{x^2/2}U(\sqrt{2}x)$; equation (8.0.1) then becomes

$$U''(\sqrt{2}x) + (\lambda/2 - x^2/2)U(\sqrt{2}x) = 0 \qquad (8.1.4)$$

$$(x \in (-\infty,\infty),\ \lambda \in \mathbb{C}),$$

which is of parabolic cylinder type, and which, on replacing $\sqrt{2}x$ by x, reduces to the standard form

$$U''(x) + (\lambda/2 - x^2/4)U(x) = 0 . \qquad (8.1.5)$$

A solution of (8.1.5) is given by the formula

$$D_{((\lambda-1)/2)}(x) =$$

$$= (-1)^{(1-\lambda)/2}\frac{\Gamma((\lambda+1)/2)e^{-x^2/4}}{2\pi i}\int_{\infty}^{(0+)} z^{-(\lambda+1)/2}\exp[-xz - z^2/2]dz \quad (0 \le \arg z < 2\pi)$$

$$= \frac{e^{-x^2/4}}{\Gamma(1/2 - \lambda/2)}\int_0^{\infty} z^{-(\lambda+1)/2}\exp[-xz - z^2/2]dz \quad (\mathrm{re}\lambda < 1)$$

(see Magnus et al. (1966; §8.1.4)).

Hence, in terms of parabolic cylinder functions, the solution $Y(\cdot,\lambda)$ may be expressed as

$$Y(x,\lambda) = \frac{-2^{5/4 - \lambda/4}\pi\exp[-(\lambda\pi i)/2]e^{x^2/2}}{\Gamma((\lambda+1)/2)}D_{(\lambda-1)/2}(\sqrt{2}x) \quad (8.1.6)$$

where

$$D_{(\lambda-1)/2}(\sqrt{2}x) = \begin{cases} \dfrac{(-1)^{(1-\lambda)/2}2^{(1-\lambda)/4}\Gamma((\lambda+1)/2)e^{-x^2/2}}{2\pi i} \\ \quad \times \displaystyle\int_{\infty}^{(0+)} z^{-(\lambda+1)/2}\exp[-2xz - z^2]dz \\ \qquad (0 \le \arg z < 2\pi) \\ \dfrac{2^{(1-\lambda)/4}e^{-x^2/2}}{\Gamma((1-\lambda)/2)}\displaystyle\int_0^{\infty} z^{-(\lambda+1)/2}\exp[-2xz - z^2]dz \\ \qquad (\mathrm{re}\lambda < 1)\ . \end{cases} \quad (8.1.7)$$

A similar expression obtains for $Z(\cdot,\lambda)$ on replacing x in (8.1.6) by -x.

At x = 0, $Y(\cdot,\lambda)$ reduces to

$$Y(0,\lambda) = 2^{1/2 - \lambda/2}\int_{\infty}^{(0+)} z^{-(\lambda+1)/2}\exp[-z^2]dz$$

$$= -2^{1/2 - \lambda/2}(\exp[-\lambda\pi i]+1)\int_0^{\infty} z^{-(\lambda+1)/2}\exp[-z^2]dz$$

$$= -2^{-(\lambda+1)/2}(\exp[-\lambda\pi i]+1)\Gamma(1/4 - \lambda/4) \qquad (8.1.8)$$

(reλ < 1, and by analytic continuation, for all λ) and

$$Y(0,\lambda) = Z(0,\lambda) \ . \qquad (8.1.9)$$

Similarly, for all $\lambda \in \mathbb{C}$,

$$Y'(0,\lambda) = 2^{1/2 - \lambda/2}(\exp[-\lambda\pi i]+1)\Gamma(3/4 - \lambda/4) = -Z'(0,\lambda) \qquad (8.1.10)$$

(note that the poles of $Y(0,\lambda)$ and $Y'(0,\lambda)$ are due to the poles of the Γ-function).

§8.2 Asymptotic behaviour of the solutions near $\pm\infty$

In view of the relationship between the solutions $Y(x,\lambda)$ and $Z(x,\lambda)$, and the corresponding parabolic cylinder functions $D_{(\lambda-1)/2}(\sqrt{2}x)$ and $D_{(\lambda-1)/2}(-\sqrt{2}x)$ respectively (see (8.1.6)), it is sufficient to consider the behaviour of the latter functions as $x \to +\infty$; and for this we use the asymptotic formula for $D_\nu(z)$, given in Magnus et al. (1966; §8.1.6), for large values of $|z|$ and $|z| >> \nu$ (with ν fixed), namely

$$D_\nu(z) \sim z^\nu \exp[-z^2/4]\{1 - \frac{\nu(\nu-1)}{2z^2} + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.4.z^4} \ \ldots\}$$

$$= z^\nu \exp[-z^2/4]\{\sum_{r=0}^{N} \frac{(-\nu/2)_r(1/2 - \nu/2)_r}{r!(-(1/2)z^2)^r} + O(|z^2|^{-N-1})\} \qquad (8.2.1)$$

$$(|\arg z| < 3\pi/4).$$

Now put $z = \sqrt{2}x$, $\nu = \lambda/2 - 1/2$, and let $x \to +\infty$; then, from (8.2.1),

$$D_{(\lambda-1)/2}(\sqrt{2}x) =$$

$$= 2^{\lambda/4 - 1/4}x^{\lambda/2 - 1/2}e^{-x^2/2}\{\sum_{r=0}^{N} \frac{(1/4 - \lambda/4)_r(3/4 - \lambda/4)_r}{r!(-x^2)^r} + O(|x^2|^{-N-1})\}$$

so that, from (8.1.6) and for each $\lambda \in \mathbb{C}$,

$$Y(x,\lambda) =$$

$$= \frac{-2\pi \exp[-(\lambda\pi i)/2]x^{\lambda/2-1/2}}{\Gamma((\lambda+1)/2)}\left\{\sum_{r=0}^{N}\frac{(1/4-\lambda/4)_r(3/4-\lambda/4)_r}{r!(-x^2)^r} + O(|x^2|^{-N-1})\right\}. \quad (8.2.2)$$

Also

$$Y'(x,\lambda) =$$

$$= \frac{-2^{5/4-\lambda/4}\pi \exp[-(\lambda\pi i)/2]}{\Gamma((\lambda+1)/2)}\{xe^{x^2/2}D_{(\lambda-1)/2}(\sqrt{2}x) + \sqrt{2}e^{x^2/2}D_{(\lambda-1)/2}(\sqrt{2}x)\}$$

$$= \frac{-2^{5/4-\lambda/4}\pi \exp[-(\lambda\pi i)/2]e^{x^2/2}}{\Gamma((\lambda+1)/2)}\{2xD_{(\lambda-1)/2}(\sqrt{2}x) - \sqrt{2}D_{(\lambda+1)/2}(\sqrt{2}x)\} \quad (8.2.3)$$

(on using the recurrence relations of $D_{(\lambda/2)-1}(\cdot)$; see Magnus et al. (1966; §8.1.3)). Then, as $x \to \infty$, we have (from above expansion)

$$2xD_{(\lambda-1)/2}(\sqrt{2}x) \sim 2^{\lambda/4+3/4}x^{\lambda/2+1/2}e^{-x^2/2}\left\{1 - \frac{(\lambda/2-1/2)(\lambda/2-3/2)}{4x^2} + \frac{(\lambda/2-1/2)(\lambda/2-3/2)(\lambda/2-5/2)(\lambda/2-7/2)}{32x^4} \mp \dots\right\} \quad (8.2.4)$$

and

$$-\sqrt{2}D_{(\lambda+1)/2}(\sqrt{2}x) \sim -2^{\lambda/4+3/4}x^{\lambda/2+1/2}e^{-x^2/2}\left\{1 - \frac{(\lambda/2+1/2)(\lambda/2-1/2)}{4x^2} + \frac{(\lambda/2+1/2)(\lambda/2-1/2)(\lambda/2-3/2)(\lambda/2-5/2)}{32x^4} \mp \dots\right\}. \quad (8.2.5)$$

It follows from (8.2.3), (8.2.4) and (8.2.5) that, as $x \to \infty$, so that, from (8.1.6) and for each $\lambda \in \mathbb{C}$,

$$Y(x,\lambda) =$$

$$= \frac{-2\pi \exp[-(\lambda\pi i)/2]x^{\lambda/2 - 1/2}}{\Gamma((\lambda+1)/2)}\{\sum_{r=0}^{N} \frac{(1/4 - \lambda/4)_r (3/4 - \lambda/4)_r}{r!(-x^2)^r} + O(|x^2|^{-N-1})\}. \quad (8.2.2)$$

Also

$$Y'(x,\lambda) =$$

$$= \frac{-2^{5/4 - \lambda/4}\pi \exp[-(\lambda\pi i)/2]}{\Gamma((\lambda+1)/2)}\{xe^{x^2/2}D_{(\lambda-1)/2}(\sqrt{2}x) + \sqrt{2}e^{x^2/2}D'_{(\lambda-1)/2}(\sqrt{2}x)\}$$

$$= \frac{-2^{5/4 - \lambda/4}\pi \exp[-(\lambda\pi i)/2]e^{x^2/2}}{\Gamma((\lambda+1)/2)}\{2xD_{(\lambda-1)/2}(\sqrt{2}x) - \sqrt{2}D_{(\lambda+1)/2}(\sqrt{2}x)\} \quad (8.2.3)$$

(on using the recurrence relations of $D_{(\lambda/2)-1}(\cdot)$; see Magnus et al. (1966; §8.1.3)). Then, as $x \to \infty$, we have (from above expansion)

$$2xD_{(\lambda-1)/2}(\sqrt{2}x) \sim 2^{\lambda/4 + 3/4}x^{\lambda/2 + 1/2}e^{-x^2/2}\{1 - \frac{(\lambda/2 - 1/2)(\lambda/2 - 3/2)}{4x^2} + \frac{(\lambda/2 - 1/2)(\lambda/2 - 3/2)(\lambda/2 - 5/2)(\lambda/2 - 7/2)}{32x^4} \mp \ldots\} \quad (8.2.4)$$

and

$$-\sqrt{2}D_{(\lambda+1)/2}(\sqrt{2}x) \sim -2^{\lambda/4 + 3/4}x^{\lambda/2 + 1/2}e^{-x^2/2}\{1 - \frac{(\lambda/2 + 1/2)(\lambda/2 - 1/2)}{4x^2} + \frac{(\lambda/2 + 1/2)(\lambda/2 - 1/2)(\lambda/2 - 3/2)(\lambda/2 - 5/2)}{32x^4} \mp \ldots\}. \quad (8.2.5)$$

It follows from (8.2.3), (8.2.4) and (8.2.5) that, as $x \to \infty$,

$Y'(x,\lambda)$

$$\sim \frac{-\pi(\lambda-1)\exp[-(\lambda\pi i)/2]}{\Gamma((\lambda+1)/2)} x^{\lambda/2-3/2}\{1 - \frac{(\lambda/2-3/2)(\lambda/2-5/2)}{4x^2}$$

$$+ \frac{(\lambda/2-3/2)(\lambda/2-5/2)(\lambda/2-7/2)(\lambda/2-9/2)}{32x^4} \mp \ldots\} . \qquad (8.2.6)$$

(Note that (8.2.6) may also be obtained directly by differentiating each term of (8.2.2).)

Consider now the solution $Z(x,\lambda) = Y(-x,\lambda)$; we put $z = -\sqrt{2}x = \sqrt{2}x\cdot\exp[\pi i]$. In this case $\arg z = \pi \in (\pi/4, 5\pi/4)$; then, from a formula in Magnus et al. (1966; §8.1.6), as $x \to +\infty$,

$D_{\lambda/2-1/2}(-\sqrt{2}x)$

$$\sim \frac{2^{\lambda/4-1/4}x^{\lambda/2-1/2}e^{-x^2/2}}{\exp[(1/2)(1-\lambda)\pi i]}\{1 - \frac{(\lambda/2-1/2)(\lambda/2-3/2)}{4x^2}$$

$$+ \frac{(\lambda/2-1/2)(\lambda/2-3/2)(\lambda/2-5/2)(\lambda/2-7/2)}{32x^4} \mp \ldots\}$$

$$+ \frac{2^{1/4-\lambda/4}\pi^{1/2}x^{-\lambda/2-1/2}e^{x^2/2}}{\exp[(\lambda-1)\pi i]\Gamma(1/2-\lambda/2)}\{1 + \frac{(\lambda/2+1/2)(\lambda/2+3/2)}{4x^2}$$

$$+ \frac{(\lambda/2+1/2)(\lambda/2+3/2)(\lambda/2+5/2)(\lambda/2+7/2)}{32x^4} + \ldots\} . \qquad (8.2.7)$$

Hence, if $x \to +\infty$, then (see (8.1.6), with $Y(-x,\lambda) = Z(x,\lambda)$)

$Z(x,\lambda)$

$$\sim \frac{2\pi\exp[(\pi i)/2]x^{\lambda/2-1/2}}{\Gamma(\lambda/2+1/2)}\{1 - \frac{(\lambda/2-1/2)(\lambda/2-3/2)}{4x^2}$$

$$+ \frac{(\lambda/2-1/2)(\lambda/2-3/2)(\lambda/2-5/2)(\lambda/2-7/2)}{32x^4} \mp \ldots\}$$

$$+ \frac{2^{3/2-\lambda/2}\pi^{3/2}x^{-\lambda/2-1/2}e^{x^2}}{\exp[(3\lambda\pi i)/2]\Gamma(1/2+\lambda/2)\Gamma(1/2-\lambda/2)}\{1 + \frac{(\lambda/2+1/2)(\lambda/2+3/2)}{4x^2}$$

$$+ \frac{(\lambda/2+1/2)(\lambda/2+3/2)(\lambda/2+5/2)(\lambda/2+7/2)}{32x^4} + \ldots\}$$

$$= \frac{2\pi \exp[(\pi i)/2]}{\Gamma(1/2+\lambda/2)} x^{\lambda/2-1/2} \{ \sum_{r=0}^{N} \frac{(1/4-\lambda/4)_r (3/4-\lambda/4)_r}{r!(-x^2)^r} + 0(|x|^{-2N-2}) \}$$

$$+ \frac{\pi^{1/2} \cos((\pi\lambda)/2) \cdot x^{-\lambda/2-1/2} e^{x^2}}{2^{\lambda/2-3/2} \exp[(3\lambda\pi i)/2]} \{ \sum_{r=0}^{N} \frac{(\lambda/4+1/4)_r (\lambda/4+3/4)_r}{r! x^{2r}}$$

$$+ 0(|x|^{-2N-2}) \} . \qquad (8.2.8)$$

Also

$$Z'(x,\lambda) = ke^{x^2/2} \{ xD_{\lambda/2-1/2}(-\sqrt{2}x) - \sqrt{2}D'_{\lambda/2-1/2}(-\sqrt{2}x) \}$$

where $k = -2^{5/4-\lambda/4}\pi \exp[-(\lambda\pi i)/2]/\Gamma((\lambda+1)/2)$ and, on using the recurrence relations for $D_{\lambda/2-1/2}(-\sqrt{2}x)$, i.e.

$$D'_{\lambda/2-1/2}(-\sqrt{2}x) = \frac{-x}{\sqrt{2}} D_{\lambda/2-1/2}(-\sqrt{2}x) - D_{\lambda/2+1/2}(-\sqrt{2}x) ,$$

we have

$$Z'(x,\lambda) = ke^{x^2/2} \{ 2xD_{\lambda/2-1/2}(-\sqrt{2}x) + \sqrt{2}D_{\lambda/2+1/2}(-\sqrt{2}x) \} . \qquad (8.2.9)$$

If we now let $x \to +\infty$, then, from (8.2.7) and a similar expression for $D_{\lambda/2+1/2}(-\sqrt{2}x)$, and also from (8.2.9), we obtain

$$Z'(x,\lambda) \sim \frac{(\lambda-1)\pi x^{\lambda/2-3/2}}{\exp[-(\pi i)/2]\Gamma(1/2+\lambda/2)} \{ 1 - \frac{(\lambda/2-3/2)(\lambda/2-5/2)}{4x^2} +$$

$$+ \frac{(\lambda/2-3/2)(\lambda/2-5/2)(\lambda/2-7/2)(\lambda/2-9/2)}{32x^4} \mp \ldots \}$$

$$+ \frac{\pi^{1/2} (\cos(\pi\lambda)/2) x^{-\lambda/2-3/2} e^{x^2}}{2^{\lambda/2-3/2} \exp[(3\lambda\pi i)/2]} \{ 2x^2 + \frac{(\lambda/2-1/2)(\lambda/2+1/2)}{2}$$

$$+ \frac{(\lambda/2-1/2)(\lambda/2+1/2)(\lambda/2+3/2)(\lambda/2+5/2)}{16x^2} + \ldots \} . \qquad (8.2.10)$$

This result may also be obtained directly by taking the derivative of (8.2.8).

If we take the leading terms in (8.2.2), (8.2.6), (8.2.8) and (8.2.10), and the associated constant terms k_i, $i = 1,\dots,6$, we obtain the following:

Theorem 8.2.1

Let $Y(\cdot,\lambda)$ *and* $Z(\cdot,\lambda)$ *be the solutions of Hermite's differential equation* (8.0.1), *where* $Y(\cdot,\lambda)$ *and* $Z(\cdot,\lambda)$ *are defined by the integrals* (8.1.1) *and* (8.1.2) *respectively. Then, if* $x \to +\infty$ (*and* $\lambda \in \mathbb{C}$ *is fixed*)

$$Y(x,\lambda) = k_1 x^{\lambda/2 - 1/2}[1 + O(|x|^{-2})]$$

$$Y'(x,\lambda) = k_2 x^{\lambda/2 - 3/2}[1 + O(|x|^{-2})]$$

$$Z(x,\lambda) = (k_3 x^{\lambda/2 - 1/2} + k_4 x^{-\lambda/2 - 1/2} e^{x^2})[1 + O(|x|^{-2})]$$

$$Z'(x,\lambda) = (k_5 x^{\lambda/2 - 3/2} + k_6 x^{1/2 - \lambda/2} e^{x^2})[1 + O(|x|^{-2})] \ ;$$

similar results hold as $x \to -\infty$.

Now, following the existence theorem in Titchmarsh (1962; §1.5-1.6), we may form the solutions θ and ϕ of Hermite's equation (8.0.1) by

$$\theta(x,\lambda) = \frac{Y(x,\lambda) + Z(x,\lambda)}{2Y(0,\lambda)} \qquad (x \in (-\infty,\infty),\ \lambda \in \mathbb{C}) \qquad (8.2.11)$$

$$\phi(x,\lambda) = \frac{Y(x,\lambda) - Z(x,\lambda)}{2Y'(0,\lambda)} \qquad (x \in (-\infty,\infty),\ \lambda \in \mathbb{C}) \qquad (8.2.12)$$

such that

$$\left.\begin{aligned} \theta(0,\lambda) &= 1 \ , \quad \theta'(0,\lambda) = 0 \\ \phi(0,\lambda) &= 0 \ , \quad \phi'(0,\lambda) = 1 \ ; \end{aligned}\right\} \qquad (8.2.13)$$

and the Wronskian

$$p(0)(\theta(0,\lambda)\phi'(0,\lambda)-\theta'(0,\lambda)\phi(0,\lambda)) = 1 ,$$

i.e. θ and ϕ are linearly independent for all $\lambda \in \mathbb{C}$.

The asymptotic forms of θ and ϕ follow from those of Y and Z given in Theorem 8.2.1; for let $x \to +\infty$, then, using (8.2.11),

$$\theta(x,\lambda) = (c_1 x^{\lambda/2 - 1/2} + c_2 x^{-\lambda/2 - 1/2} e^{x^2})[1 + O(|x|^{-2})] \qquad (8.2.14)$$

$$\theta'(x,\lambda) = (c_3 x^{\lambda/2 - 3/2} + c_4 x^{-\lambda/2 + 1/2} e^{x^2})[1 + O(|x|^{-2})] \qquad (8.2.15)$$

(the constants c_i, $i = 1,\ldots,4$, may be calculated from the leading terms for Y and Z above). Similarly, on using (8.2.10), as $x \to +\infty$, we have

$$\phi(x,\lambda) = (c_5 x^{\lambda/2 - 1/2} + c_6 x^{-\lambda/2 - 1/2} e^{x^2})[1 + O(|x|^{-2})] \qquad (8.2.16)$$

$$\phi'(x,\lambda) = (c_7 x^{\lambda/2 - 3/2} + c_8 x^{-\lambda/2 + 1/2} e^{x^2})[1 + O(|x|^{-2})] . \qquad (8.2.17)$$

Similar results hold as $x \to -\infty$.

Remark 8.2.2

It is clear from these asymptotic forms that θ and $\phi \notin L^2_w(-\infty,0)$, $L^2_w(0,\infty)$, $H^2_{p,q}(-\infty,0)$, $H^2_{p,q}(0,\infty)$. It follows then as in Theorem 3.5.5 that Hermite's equation (8.0.1) is limit-point both in the right- and left-definite cases.

§8.3 The associated eigenvalues and eigenfunctions

Again consider Hermite's differential expression

$$M[y](x) := -(e^{-x^2}y'(x))' + e^{-x^2}y(x) \qquad (x \in (-\infty,\infty)).$$

Since $M[y]$ is limit-point at $\pm\infty$ in $L^2_w(-\infty,\infty)$, the general theory of

Titchmarsh and Weyl in §2.1 now gives the existence of unique m-coefficients m(·), n(·), both analytic mappings of $\mathbb{C} \to \mathbb{C}$, such that

$$\left.\begin{aligned} \psi(x,\lambda) &:= \theta(x,\lambda) + m(\lambda)\phi(x,\lambda) \in L_w^2(0,\infty) \\ \chi(x,\lambda) &:= \theta(x,\lambda) + n(\lambda)\phi(x,\lambda) \in L_w^2(-\infty,0) \,. \end{aligned}\right\} \tag{8.3.1}$$

Then ψ and χ are multiples of Y and Z respectively (note that m(·) and n(·) may be calculated using (8.2.13) as in §6.5).

Let the Green's function in this case (see (2.1.9)) be given by

$$G(x,t,\lambda) = \begin{cases} -\dfrac{\psi(x,\lambda)\chi(t,\lambda)}{\omega(\lambda)} & (-\infty < t < x < \infty) \\ -\dfrac{\chi(x,\lambda)\psi(t,\lambda)}{\omega(\lambda)} & (-\infty < x < t < \infty) \,, \end{cases} \tag{8.3.2}$$

where $\omega(\lambda)$ denotes the Wronskian of ψ and χ, and hence that of Y and Z at 0, i.e.

$$\begin{aligned} \omega(\lambda) &= p(x)(Y(x,\lambda)Z'(x,\lambda)-Y'(x,\lambda)Z(x,\lambda))_{x=0} \\ &= -2Y(0,\lambda)Y'(0,\lambda) \\ &= 2^{3/2-\lambda/2}\pi^{1/2}(\exp[-\pi\lambda i]+1)^2\Gamma(1/2-\lambda/2) \end{aligned} \tag{8.3.3}$$

(on using (8.1.8), (8.1.10) and Legendre's duplicating formula).

Let the function $\Phi : (-\infty,\infty) \times \mathbb{C} \times L_w^2(-\infty,\infty) \to \mathbb{C}$, be defined by (see (2.1.10))

$$\Phi(x,\lambda;f) = -\int_{-\infty}^{\infty} w(t)G(x,t,\lambda)f(t)dt \tag{8.3.4}$$

(i.e. all $\lambda \in \mathbb{C}$, except at the poles of $G(x,t,\cdot)$), then, following a similar analysis to that of Titchmarsh (1962; §4.2), it may be shown from the simple poles of $\omega(\lambda)$ (see (8.3.3)) that the required eigenvalues

$$\lambda = \lambda_n = 2n + 1 \quad (n \in N_0); \tag{8.3.5}$$

nd from the residues of $\Phi(x,\cdot,f)$ we obtain the corresponding eigen-

unctions

$$\phi_n(x) = \sqrt{\frac{1}{2^n n! \pi^{1/2}}}\, H_n(x) \quad (n \in N_0) \tag{8.3.6}$$

where $H_n(\cdot)$ are the Hermite polynomials; see §1.4).

We recall from Theorem 3.5.5 (and Remark 8.2.2) that the dif-

ferential expression $M[y]$ is also limit-point at $\pm\infty$ in $H^2_{p,q}(-\infty,\infty)$ (i.e.

in the left-definite case). Thus we can replace the space $L^2_w(-\infty,\infty)$

here with $H^2_{p,q}(-\infty,\infty)$ and obtain the same results.

One consequence of these results is the following property:

Theorem 8.3.1

(i) The set $\{\phi_n(\cdot),\ n \in N_0\}$, defined by (8.3.6), is orthonormal in $L^2_w(-\infty,\infty)$;

(ii) the set $\{(2n+1)^{-1/2}\phi_n(\cdot),\ n \in N_0\}$ is orthonormal in $H^2_{p,q}(-\infty,\infty)$.

Proof: This is similar to that of Theorem 7.3.1 (note that $H'_n(x) = 2nH_{n-1}(x),\ n \in N_+$), so we shall omit it.

We leave now the classical discussion of Hermite's differential equation (8.0.1), and devote the remaining sections for a discussion of the associated differential operators. We begin with the right-definite case.

§8.4 The operator T in $L^2_w(-\infty,\infty)$: the right-definite case

Again consider Hermite's differential equation

$$M[y](x) := -(e^{-x^2}y'(x))' + e^{-x^2}y(x) = \lambda e^{-x^2}y(x) \qquad (8.0.1)$$
$$(x \in (-\infty,\infty),\ \lambda \in \mathbb{C}).$$

Since M[y] is limit-point at $\pm\infty$ in $L^2_w(-\infty,\infty)$, the boundary conditions of the type (4.1.9) are not required in this case but are replaced by the integral conditions as in (4.1.1), (see also (4.1.3)).

Following (4.1.1), let Δ denote a linear manifold of $L^2_w(-\infty,\infty)$, defined as follows:

$$\Delta := \{f : (-\infty,\infty) \to \mathbb{C} : f, pf' \in AC_{loc}(-\infty,\infty), \text{ and}$$
$$f, w^{-1}M[f] \in L^2_w(-\infty,\infty)\}, \qquad (8.4.1)$$

where $p(x) = w(x) = e^{-x^2}$ are the coefficients in (8.0.1); also let T be a linear operator defined in $L^2_w(-\infty,\infty)$ by

$$Tf := w^{-1}M[f] \quad (f \in D(T)) \qquad (8.4.2)$$

where $D(T) := \Delta$ is the domain of T; then T has the following properties:

Theorem 8.4.1

(i) D(T) is dense in $L^2_w(-\infty,\infty)$;

(ii) T is a symmetric operator in $L^2_w(-\infty,\infty)$;

(iii) T is self-adjoint.

Proof: This follows from Theorem 4.1.3, with the end-points $a = -\infty$ and $b = +\infty$ both being singular. So we shall omit it.

Thus, as in §6.7 (see also the last paragraph of §5.3), it can be shown that T is the required self-adjoint, differential operator with a simple discrete spectrum

$$P\sigma(T) = \{2n + 1,\ n \in N_0\} \ ; \qquad (8.4.3)$$

and the corresponding eigenvectors in this case are the Hermite polynomials $\{H_n(\cdot),\ n \in N_0\}$. The analysis of §5.2 (see also Theorem 1.7.3) now implies the completeness of these polynomials in the space $L^2_w(-\infty,\infty)$.

Before studying the left-definite case, we look first at the properties of the resolvent function Φ (see (8.3.4)).

§8.5 The properties of Φ

As in (8.3.4), we define the function

$\Phi : (-\infty,\infty) \times (\mathbb{C} - \{2n + 1,\ n \in N_0\}) \times H^2_{\rho,\xi}(-\infty,\infty) \to \mathbb{C}$ by

$$\Phi(x,\lambda;f) = \frac{\psi(x,\lambda)}{\omega(\lambda)}\int_{-\infty}^{x} w(t)\chi(t,\lambda)f(t)dt$$

$$+ \frac{\chi(x,\lambda)}{\omega(\lambda)}\int_{x}^{\infty} w(t)\psi(t,\lambda)f(t)dt \ . \qquad (8.5.1)$$

At $\lambda = 0$, $\omega(0) = 2^{7/2}\pi$ (see (8.3.3)); hence Φ is regular at $\lambda = 0$. If we put $\Psi(x;f) := \Phi(x,0;f)$, then

$$\Psi(x;f) = \frac{\psi(x,0)}{\omega(0)}\int_{-\infty}^{x} w(t)\chi(t,0)f(t)dt$$

$$+ \frac{\chi(x,0)}{\omega(0)}\int_{x}^{\infty} w(t)\psi(t,0)f(t)dt \ . \qquad (8.5.2)$$

We prove now some properties of the function Ψ.

Lemma 8.5.1

Let $f \in H^2_{p,q}(-\infty,\infty)$, and Ψ be defined as above, then

$$M[\Psi(x;f)] = w(x)f(x) \quad (x \in (-\infty,\infty)). \tag{8.5.3}$$

Proof: This follows from part (a) of Theorem 4.1.2 with $\lambda = 0$ (note that in this case the interval $[a,b)$ is replaced by $(-\infty,\infty)$), and the Green's function G is given by (8.3.2).

We have seen from §8.3 that ψ and χ are multiples of the solutions Y and Z respectively; hence, except for constant factors, the asymptotic behaviour of ψ and χ may be deduced from Theorem 8.2.1. With this in mind, we prove the following lemma.

Lemma 8.5.2

Let $f \in H^2_{p,q}(-\infty,\infty)$ and $w(x) = e^{-x^2}$ $(x \in (-\infty,\infty))$; and (from Theorem 8.2.1)

$$\psi(x,0) = O(|x|^{-1/2}) \quad (x \to +\infty) \tag{8.5.4}$$

and

$$\chi(x,0) = \begin{cases} O(|x|^{-1/2}e^{x^2}) & (x \to +\infty) \\ O(|x|^{-1/2}) & (x \to -\infty) \end{cases} ; \tag{8.5.5}$$

then

(i) $$\int_{-\infty}^{x} w(t)\chi(t,0)f(t)dt = O(|x|^{-1}e^{x^2/2}) \quad (x \to +\infty)$$

(ii) $$\int_x^{\infty} w(t)\psi(t,0)f(t)dt = 0(|x|^{-1}e^{-x^2/2}) \qquad (x \to +\infty).$$

<u>Proof</u>:

(i) Let

$$\int_{-\infty}^{x} w(t)\chi(t,0)f(t)dt = (\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{x})w(t)\chi(t,0)f(t)dt .$$

For some constant $k_1 \in R_+$

$$|\int_{-\infty}^{-1} w(x)\chi(x,0)f(x)dx| \le k_1|\int_{-\infty}^{-1} e^{-x^2}|x|^{-1/2}f(x)dx|$$

$$\le k_1\{\int_{-\infty}^{-1} e^{-x^2}|x|^{-1}dx\int_{-\infty}^{-1} e^{-x^2}|f(x)|^2dx\}^{1/2}$$

$$\le k_1\{\int_{-\infty}^{-1} e^{-x^2}|x|^{-1}dx\}^{1/2}||f||_H$$

$$(f \in H^2_{p,q}(-\infty,\infty));$$

hence

$$\int_{-\infty}^{-1} w(x)\chi(x,0)f(x)dx = 0(\{\int_{-\infty}^{-1} e^{-x^2}|x|^{-1}dx\}^{1/2})$$

$$= 0(1) . \qquad (8.5.6)$$

Also

$$\int_{-1}^{1} w(x)\chi(x,0)f(x)dx = O(|\int_{-1}^{1} w(x)|\chi(x,0)|f(x)dx|)$$

$$= O(1) \qquad (8.5.7)$$

(recall from the proof of Theorem 8.1.1 that $Z(x,\lambda)$ (and hence $\chi(x,0)$) exists at 0 if $\mathrm{re}\lambda < 1$); and, as $x \to +\infty$,

$$\int_{1}^{x} w(t)\chi(t,0)f(t)dt = O(|\int_{1}^{x} e^{-t^2}e^{t^2}|t|^{-1/2}f(t)dt|)$$

$$= O(|\int_{1}^{x} |t|^{-1/2}f(t)dt|)$$

with

$$|\int_{1}^{x} |t|^{-1/2}f(t)dt| = |\int_{1}^{x} e^{t^2/2}e^{-t^2/2}|t|^{-1/2}f(t)dt|$$

$$\leq \{\int_{1}^{x} e^{t^2}|t|^{-1}dt\int_{1}^{x} e^{-t^2}|f(t)|^2dt\}^{1/2}$$

(on using the Cauchy-Schwarz inequality)

$$\leq \{\int_{1}^{x} e^{t^2}|t|^{-1}dt\}^{1/2}\|f\|_H$$

(because $f \in H^2_{p,q}(-\infty,\infty)$); hence

$$\int_{1}^{x} |t|^{-1/2}f(t)dt = O(\{\int_{1}^{x} e^{t^2}|t|^{-1}dt\}^{1/2}) \qquad (x \to \infty).$$

Consider now the right-hand side with t assumed positive

$$\int_1^x \frac{e^{t^2}}{t}dt = \left[\frac{e^{t^2}}{2t^2}+\frac{e^{t^2}}{2t^4}\right]_1^x + 2\int_1^x \frac{e^{t^2}dt}{t^5} \;;$$

and we claim that

$$\left|\int_1^x e^{t^2}t^{-5}dt\right| \le k_3 e^{x^2}|x|^{-6}$$

for some $k_3 \in R_+$, for on using L'Hospital's rule

$$\lim_{x\to\infty} \frac{\int_1^x e^{t^2}t^{-5}dt}{\frac{1}{2}e^{x^2}x^{-6}} = 2\lim_{x\to\infty} \frac{e^{x^2}x^{-5}}{e^{x^2}x^{-5}(2-6x^{-2})} = 1 \;.$$

Hence

$$\int_1^x e^{t^2}t^{-5}dt \sim \frac{1}{2}e^{x^2}x^{-6} \quad \text{as} \quad x \to \infty \;,$$

which proves the assertion. Thus

$$\int_1^x \frac{e^{t^2}}{t}dt = \frac{e^{x^2}}{2x^2}\left[1+\frac{1}{x^2}+O(|x|^{-4})\right] \quad (x \to \infty), \tag{8.5.8}$$

<u>i.e.</u>, on taking the leading term (as $x \to \infty$),

$$\int_1^x w(t)\chi(t,0)f(t)dt = O\left(\left|\int_1^x |t|^{-1/2}f(t)dt\right|\right)$$

$$= O(e^{x^2/2}|x|^{-1}) \;. \tag{8.5.9}$$

Putting (8.5.6), (8.5.7) and (8.5.9) together gives the proof of part (i).

(ii) For this part, if $x \to +\infty$, then

$$\int_x^\infty w(t)\psi(t,0)f(t)dt = O(|\int_x^\infty e^{-t^2}|t|^{-1/2}f(t)dt|) \qquad (8.5.10)$$

and, on using Cauchy-Schwarz inequality,

$$|\int_x^\infty e^{-t^2}|t|^{-1/2}f(t)dt| \le \{\int_x^\infty e^{-t^2}|t|^{-1}dt\int_x^\infty e^{-t^2}|f(t)|^2dt\}^{1/2}$$

$$\le \{\int_x^\infty e^{-t^2}|t|^{-1}dt\}^{1/2}||f||_H \qquad (8.5.11)$$

(because $f \in H^2_{p,q}(-\infty,\infty)$. As in part (i), and assuming without loss of generality that $x \ge 1$, we have

$$\int_x^\infty e^{-t^2}t^{-1}dt = [\frac{-e^{-t^2}}{2t^2}]_x^\infty - \int_x^\infty e^{-t^2}t^{-3}dt \ ;$$

and, since $-\int_x^\infty e^{-t^2}t^{-3}dt \sim \frac{1}{2}\, e^{-x^2}x^{-4}$ $(x \to +\infty)$, we have

$$\int_x^\infty e^{-t^2}t^{-1}dt = \frac{e^{-x^2}}{2x^2}[1 + O(|x|^{-2})] \qquad (x \to +\infty).$$

Hence, from (8.5.11) and (8.5.10),

$$\int_x^\infty w(t)\psi(t,0)f(t)dt = O(e^{-x^2/2}|x|^{-1}) \qquad (x \to +\infty)$$

and this completes the proof (note that similar results may be obtained if $x \to -\infty$).

Theorem 8.5.3

Let $f \in H^2_{p,q}(-\infty,\infty)$ *and let the function* $\Psi(\cdot;f)$ *be defined by* (8.5.2); *then, for all* $g \in H^2_{p,q}(-\infty,\infty)$,

$$\lim_{x\to\pm\infty} p(x)\Psi'(x;f)g(x) = 0 \; . \qquad (8.5.12)$$

Proof: Let $g \in H^2_{p,q}(-\infty,\infty)$, and recall that $p(x) = q(x) = e^{-x^2}$ $(x \in (-\infty,\infty))$; then

$$g(x) = g(0) + \int_0^x g'(t)dt \; ,$$

and

$$|g(x)| \le |g(0)| + |\int_0^x g'(t)dt|$$

$$\le |g(0)| + \{\int_0^x e^{t^2} dt \int_0^x e^{-t^2} |g'(t)|^2 dt\}^{1/2}$$

(on using the Cauchy-Schwarz inequality)

$$\le |g(0)| + \{\int_0^x e^{t^2} dt\}^{1/2} ||g||_H . \qquad (8.5.13)$$

Now

$$\int_0^x e^{t^2} dt = \int_0^1 e^{x^2} dx + \int_1^x e^{t^2} dt$$

$$= 0(|\int_1^x e^{t^2} dt|) \quad (x \to +\infty);$$

also

$$\int_1^x e^{t^2}\,dt = \left\{\frac{e^{t^2}}{2t} + \frac{e^{t^2}}{4t^3}\right\}_1^x + \frac{3}{4}\int_1^x \frac{e^{t^2}}{t^4}\,dt$$

and

$$\int_1^x e^{t^2} t^{-4}\,dt \sim \frac{1}{2}\, e^{x^2} x^{-5} \qquad (\text{as } x \to +\infty)$$

(on using L'Hospital's rule as in Lemma 8.5.2). Hence

$$\int_1^x e^{t^2}\,dt = \frac{e^{x^2}}{2x}\{1 + \frac{1}{2x^2} + O(|x|^{-4})\} \qquad (x \to +\infty)$$

so that

$$g(x) = O(e^{x^2/2}|x|^{-1/2}) \qquad (x \to +\infty). \tag{8.5.14}$$

Now, from Theorem 8.2.1 (with ψ and χ being multiples of Y and Z respectively), and taking the leading terms

$$\psi'(x,0) = O(|x|^{-3/2}) \quad \text{and} \quad \chi'(x,0) = O(|x|^{1/2}e^{x^2}) \qquad (\text{as } x \to +\infty), \tag{8.5.15}$$

then, from (8.5.2), (8.5.14) and part (i) of Lemma 8.5.2, we have

$$\frac{p\psi'(\cdot,0)g}{\omega(0)}\int_{-\infty}^x w\chi(\cdot,0)f = O(e^{-x^2}|x|^{-3/2}e^{x^2/2}|x|^{-1/2}|x|^{-1}e^{x^2/2}) \qquad (x \to +\infty)$$

$$= O(|x|^{-3}) \qquad (x \to +\infty). \tag{8.5.16}$$

Similarly, on using part (ii) of Lemma 8.5.2,

$$\frac{p\chi'(\cdot,0)g}{\omega(0)}\int_x^\infty w\psi(\cdot,0)f = O(e^{-x^2}|x|^{1/2}e^{x^2}|x|^{-1/2}e^{x^2/2}e^{-x^2/2}|x|^{-1})$$

$$= O(|x|^{-1}) \qquad (x \to +\infty). \qquad (8.5.17)$$

Putting (8.5.16) and (8.5.17) together gives

$$p(x)\Psi'(x;f)g(x) = O(|x|^{-1}) + O(|x|^{-3})$$

$$= o(1) \qquad (x \to +\infty),$$

i.e.

$$\lim_{x\to+\infty} p(x)\Psi'(x;f)g(x) = 0 \ .$$

A similar result may be obtained as $x \to -\infty$, and hence the proof.

The results of Lemmas 8.5.1 and 8.5.2 may also be used to establish the following properties of Ψ:

Theorem 8.5.4

Let $f \in H^2_{p,q}(-\infty,\infty)$, then

(i) $\Psi(\cdot;f) \in L^2_w(-\infty,\infty)$; and

(ii) $\Psi(\cdot;f) \in H^2_{p,q}(-\infty,\infty)$.

Proof:

(i) From (8.5.4) and part (i) of Lemma 8.5.2, we have

$$\frac{\psi(x,0)}{\omega(0)}\int_{-\infty}^{x} w(t)\chi(t,0)f(t)dt = O(|x|^{-1/2}|x|^{-1}e^{x^2/2}) \qquad (x \to +\infty)$$

$$= O(|x|^{-3/2}e^{x^2/2}) \qquad (8.5.18)$$

$$(x \to +\infty);$$

similarly, from (8.5.5) and part (ii) of Lemma 8.5.2,

$$\frac{\chi(x,0)}{\omega(0)}\int_x^\infty w(t)\psi(t,0)f(t)dt = O(|x|^{-3/2}e^{x^2/2}) \quad (x \to +\infty). \qquad (8.5.19)$$

It follows then from (8.5.2) that

$$\Psi(x;f) = O(|x|^{-3/2}e^{x^2/2}) \quad (x \to +\infty) \qquad (8.5.20)$$

(a similar result obtains if $x \to -\infty$); hence, for some $k \in R_+$,

$$\int_{-\infty}^{\infty} w(x)|\Psi(x;f)|^2 \le k\int_{-\infty}^{\infty} |x|^{-3}dx < \infty$$

and hence the proof (see also part (c) of Theorem 4.1.2 for an alternative proof, with $f \in H^2_{p,q}(-\infty,\infty)$ and hence in $L^2_w(-\infty,\infty)$).

(ii) For this part, we appeal to Lemma 8.5.1 as follows:

$$\int_{-x}^{x} wf\overline{\Psi(\cdot;f)} = \int_{-x}^{x} M[\Psi(\cdot;f)]\Psi(\cdot;f) \qquad (f \in H^2_{p,q}(-\infty,\infty))$$

$$= \int_{-x}^{x} \{-(p\Psi'(\cdot;f))' + q\Psi(\cdot;f)\}\overline{\Psi(\cdot;f)}$$

(recall that $p(x) = q(x) = w(x) = e^{-x^2}$ $(x \in (-\infty,\infty))$)

$$= -[p\Psi'(\cdot;f)\overline{\Psi}(\cdot;f)]_{-x}^{x} + \int_{-x}^{x} \{p|\Psi'(\cdot;f)|^2 + q|\Psi(\cdot;f)|^2\} .$$

Let $x \to +\infty$ and then use Theorem 8.5.3 to give

$$\int_{-\infty}^{\infty} wf\overline{\Psi}(\cdot;f) = \int_{-\infty}^{\infty} \{p|\Psi'(\cdot;f)|^2 + q|\Psi(\cdot;f)|^2\}$$

$$= \|\Psi(\cdot;f)\|_H^2 ;$$

and, on applying the Cauchy-Schwarz inequality to the left-hand side,

$$\|\Psi(\cdot;f)\|_H^2 = \left|\int_{-\infty}^{\infty} wf\overline{\Psi}(\cdot;f)\right|$$

$$\leq \{\int_{-\infty}^{\infty} w|f|^2 \int_{-\infty}^{\infty} w|\Psi(\cdot;f)|^2\}^{1/2} \tag{8.5.21}$$

$$< \infty$$

(because $f \in H^2_{p,q}(-\infty,\infty) \subset L^2_w(-\infty,\infty)$ and $\Psi(\cdot;f) \in L^2_w(-\infty,\infty)$; see part (i) above); hence

$$\Psi(\cdot;f) \in H^2_{p,q}(-\infty,\infty) .$$

§8.6 <u>The operator S in $H^2_{p,q}(-\infty,\infty)$: the left-definite case</u>

With the above properties of Ψ in mind, we define a linear operator A on the space $H^2_{p,q}(-\infty,\infty)$ by

$$(Af)(x) := \Psi(x;f) \quad (x \in (-\infty,\infty),\ f \in H^2_{p,q}(-\infty,\infty)); \tag{8.6.1}$$

then A satisfies the following:

<u>Theorem 8.6.1</u>

(i) A <u>is a bounded linear operator on</u> $H^2_{p,q}(-\infty,\infty)$;

(ii) <u>the operator</u> A <u>is symmetric, and hence self-adjoint;</u>

(iii) A <u>has an inverse operator</u> A^{-1}.

Proof:

(i) Note from (8.5.21) that

$$\|\Psi(\cdot;f)\|_H^2 \le \{\int_{-\infty}^{\infty} w|f|^2 \int_{-\infty}^{\infty} w|\Psi(\cdot;f)|^2\}^{1/2}$$

$$\le \{\int_{-\infty}^{\infty} [p|f'|^2 + q|f|^2] \int_{-\infty}^{\infty} [p|\Psi'(\cdot;f)|^2 + q|\Psi(\cdot;f)|^2]\}^{1/2}$$

(recall that the coefficients $p(x) = q(x) = w(x) = e^{-x^2}$ and, from Theorem 8.5.4, $\Psi(\cdot;f) \in H^2_{p,q}(-\infty,\infty)$), i.e.

$$\|\Psi(\cdot;f)\|_H^2 \le \|f\|_H \|\Psi(\cdot;f)\|_H$$

or

$$\|Af\|_H = \|\Psi(\cdot;f)\|_H \le \|f\|_H$$

and this implies that A is bounded in $H^2_{p,q}(-\infty,\infty)$.

(ii) Let $f,g \in H^2_{p,q}(-\infty,\infty)$; then

$$(Af,g)_H = (\Psi(\cdot;f),g)_H$$

$$= \lim_{X\to\infty} \int_{-X}^{X} \{p\Psi'(\cdot;f)\overline{g}' + q\Psi(\cdot;f)\overline{g}\}$$

$$= \lim_{X\to\infty} [p\Psi'(\cdot;f)\overline{g}]_{-X}^{X} + \lim_{X\to\infty}\int_{-X}^{X} \{-(p\Psi'(\cdot;f))' + q\Psi(\cdot;f)\}\overline{g}$$

$$= \int_{-\infty}^{\infty} \{-(p\Psi'(\cdot;f))' + q\Psi(\cdot;f)\}\overline{g} \quad \text{(Theorem 8.5.3)}$$

$$= \int_{-\infty}^{\infty} M[\Psi(\cdot;f)]\overline{g}$$

$$= \int_{-\infty}^{\infty} wf\bar{g} \quad \text{(Lemma 8.5.1)}$$

$$= \int_{-\infty}^{\infty} f\cdot M[\Psi(\cdot;\bar{g})]$$

$$= \int_{-\infty}^{\infty} \{pf'\bar{\Psi}'(\cdot;g) + qf\bar{\Psi}(\cdot;g)\}$$

(on reversing the argument)

$$= (f,\Psi(\cdot;g))_H ,$$

i.e.

$$(Af,g)_H = (f,Ag)_H ,$$

i.e. A is symmetric in $H^2_{p,q}(-\infty,\infty)$ and hence self-adjoint (see part (i)).

(iii) Let $Af = 0$ ($f \in H^2_{p,q}(-\infty,\infty)$); then

$$\Psi(\cdot;f) = 0$$

and

$$0 = M[\Psi(\cdot;f)] = wf \quad \text{on} \quad (-\infty,\infty) ;$$

since $w > 0$ on $(-\infty,\infty)$, it follows that $f = 0$. Thus A has an inverse operator A^{-1}.

The existence of the inverse A^{-1} leads us to define the following operator, $S : D(S) \subset H^2_{p,q}(-\infty,\infty) \to H^2_{p,q}(-\infty,\infty)$, by

$$D(S) := \{Af : f \in H^2_{p,q}(-\infty,\infty)\}$$

and (8.6.2)

$$Sf := A^{-1}f \quad (f \in D(S));$$

then, as in §6.10, it can be shown that S is a self-adjoint unbounded operator, having a discrete spectrum.

$$P\sigma(S) = \{2n + 1, n \in N_0\} , \qquad (8.6.3)$$

with the corresponding eigenvectors being the Hermite polynomials $\{H_n(\cdot), n \in N_0\}$; and furthermore these polynomials are complete in $H^2_{p,q}(-\infty,\infty)$, and hence in $L^2_w(-\infty,\infty)$ (because $H^2_{p,q}(-\infty,\infty)$ is dense in $L^2_w(-\infty,\infty)$).

<u>Remark 8.6.2</u>

(i) The results of §6.12 concerning the alternative definition of T also hold in this case;

(ii) the comparisons given in §6.13 may also be extended to this case.

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