ON THE CONSTRUCTION OF MIXED POISSON DISTRIBUTIONS

RACHEL SARGUTA
UNIVERSITY OF NAIROBI

20th July 2012

1 Introduction

Definition:

Mixtures are superimpositions of simpler component distributions depending on a parameter, itself being a random variable with some known distribution.

A probability distribution is said to be a mixture if its pdf can be written in the form

$$f(x) = \int_{\lambda} f(x \mid \lambda)\, g(\lambda)\, d\lambda \tag{1}$$

or

$$f(x) = \sum_{\lambda} f(x \mid \lambda)\, g(\lambda) \tag{2}$$

where $g(\lambda)$ is the mixing distribution.

Particular Case:

Mixed Poisson distribution;

$$f(x) = \int_\lambda \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda)\, d\lambda \tag{3}$$

or

$$f(x) = \sum_\lambda \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda) \tag{4}$$

where $f(x)$ is the mixed Poisson distribution.

2 Problem Statement

Consider the mixed Poisson distribution given by,

$$f(x) = \int_{\lambda} \frac{e^{-\lambda}\lambda^{x}}{x!} g(\lambda)\, d\lambda$$

where $g(\lambda)$ is a mixing distribution.

The major problem in constructing or obtaining mixture distributions with continuous mixing distributions is the evaluation of the above integrand as Albercht (1984) stated. Only a few integrands can be evaluated explicitly, therefore, alternative methods had to be sought.

3 Objectives

Main objective: To review some methods of determining Mixed Poisson distributions.

Specific objective: To obtain the Mixed Poisson distributions through

- Direct integration where possible,
- Recursive formulae,
- Laplace Transform technique and
- Use of special functions.

4 Significance

Mixed Poisson distributions are used for modeling non - homogeneous populations.

In Actuarial applications, they are handy in modeling the distribution of total claims payable by an insurer. This is because the observed data on the number of claims often exhibit a variance that noticeably exceeds their mean.

5 Literature Review

The derivation of Mixed Poisson distribution goes back to 1920 when Greenwood and Yule considered NBD as a mixture of a Poisson distribution with a Gamma mixing distribution.

Taking a mixture of the Poisson distribution with a normal distribution truncated at the left at zero, then we have a Poisson-Normal distribution, (Patil, 1964).

The Poisson-Linear Exponential distribution is obtained by formally mixing the Poisson distribution with the linear exponential family of distributions, (Sankaran, 1970).

Some generalizations of the concept have been studied by applying a generalized gamma distribution resulting in a generalized form of the NBD (Gupta, R.C. and S.H. Ong, 2005).

Recently, Zakerzadeh and Dolati, (2010) generalized the Lindley distribution to obtain a Generalized Lindley distribution. Taking this distribution as the mixing density, Mahmoudi and Zakerzadeh, (2010) obtained a Generalized Poisson - Lindley distribution.

Willmot (1993) obtained recursive formulae for several Mixed Poisson distributions such as; Negative Binomial distribution, Sichel distribution, Poisson Beta distribution, etc.

6 Constructing Mixed Poisson Distributions

6.1 Explicit Forms

The mixing distributions considered here include: Exponential, Gamma, Lindley, Generalized Lindley, Zero - Truncated Normal and Linear Exponential Family.

Illustration: Generalized Lindley Distribution

$$f(x) = \int_{\lambda} \frac{e^{-\lambda}\lambda^{x}}{x!} g(\lambda)\, d\lambda$$

Figure 1: A General Framework for a Poisson Mixture

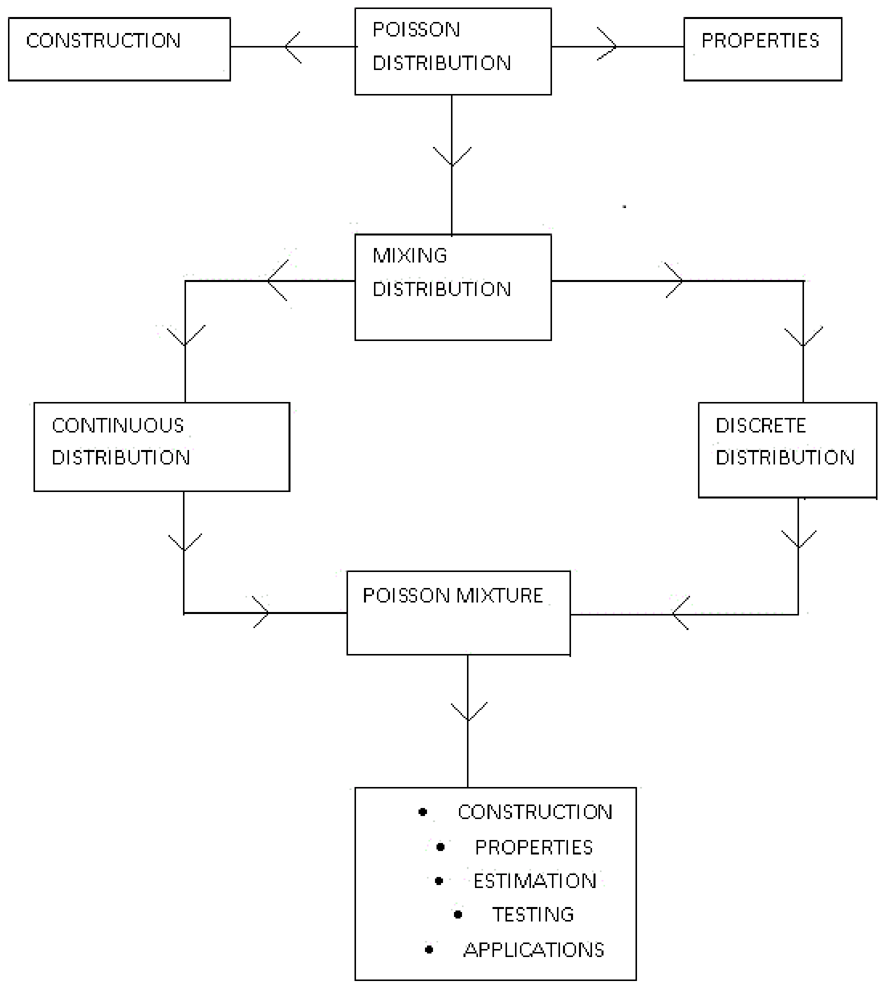


Figure 2: A Framework for Constructing Poisson Mixtures with Continuous Prior Distributions

6.pdf

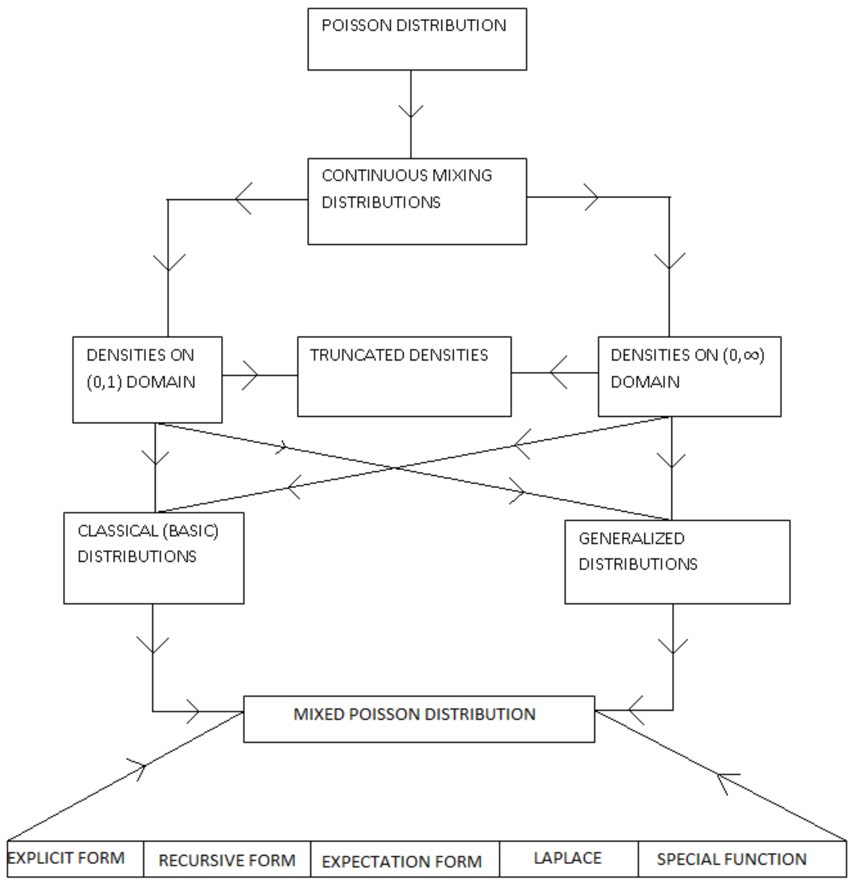


Figure 3: Direct Route

Figure 4: Expectation Route

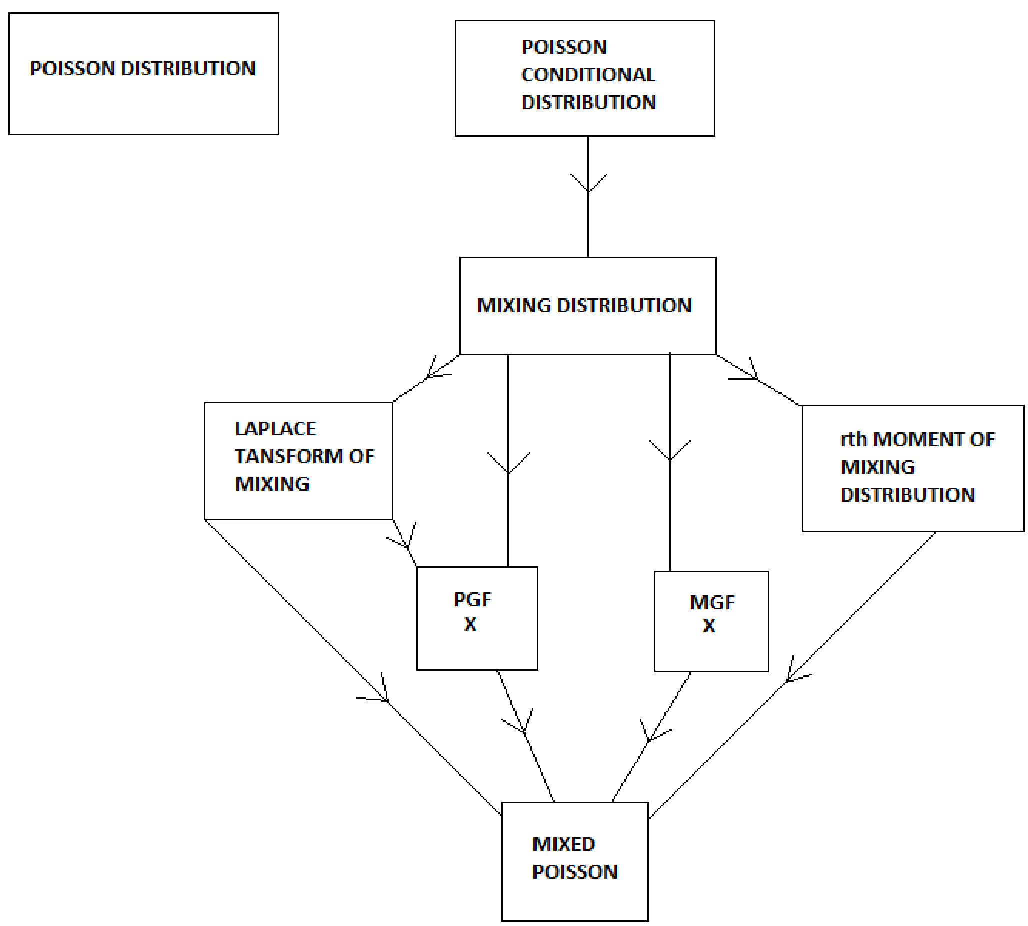
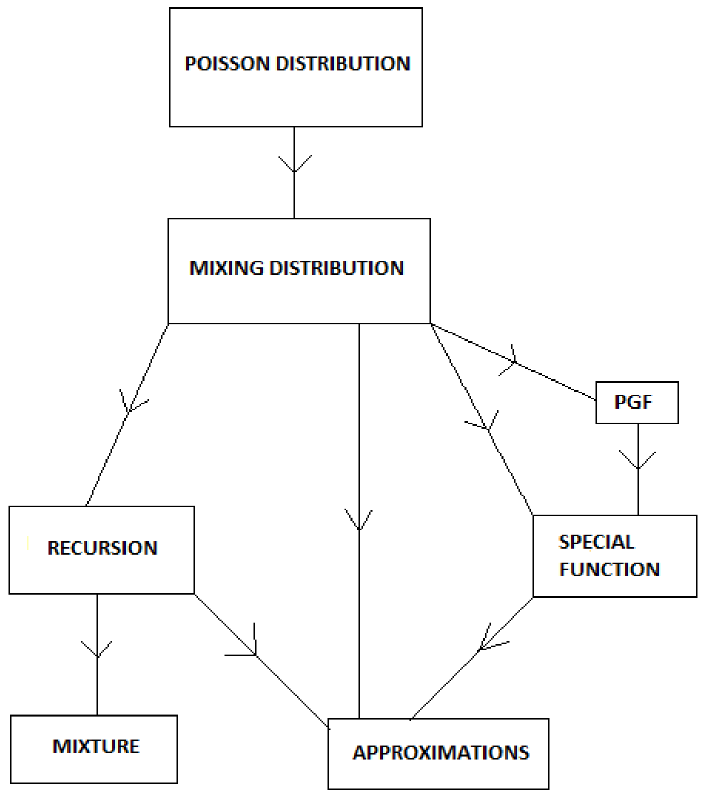


Figure 5: Approximation Route



$$g(\lambda) = \frac{\theta^2 (\theta\lambda)^{\alpha-1} (\alpha+\lambda) e^{-\theta\lambda}}{(\theta+1)\Gamma(\alpha+1)}; \lambda > 0 \qquad (5)$$

$$f(x) = \frac{\theta^{\alpha+1}\Gamma(x+\alpha)}{x!(\theta+1)^{x+\alpha+1}\Gamma(\alpha+1)}\left[\alpha + \frac{(x+\alpha)}{(\theta+1)}\right]; x = 0, 1, .. \qquad (6)$$

A result obtained by Mahmoudi & Zakerzadeh, (2010).

6.2 Recursive Relations

The recursive relations for various mixed Poisson distributions were obtained by evaluating

$$f(x) = \int_{\lambda} \frac{e^{-\lambda}\lambda^{x}}{x!} g(\lambda)\, d\lambda$$

using integration by parts.

A number of Mixed Poisson distributions were obtained for several mixing distributions such as: Rectangular, Inverse Gaussian, GIG, Gamma & Generalized Gamma, Beta, etc.

Illustration: Gamma with two parameters

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta\lambda}\lambda^{\alpha-1}; \lambda > 0, \alpha > 0, \beta > 0 \quad (7)$$

The recursive relation is

$$f\left(x+1\right)=\left(\frac{x+\alpha}{x+1}\right)\left(\frac{1}{1+\beta}\right)f\left(x\right);x=0,1,2,... \tag{8}$$

A result obtained by Panjer & Willmot, (1992).

6.3 Laplace Transforms

The Mixed Poisson distribution is expressed as:

$$f(x) = \frac{1}{x!}(-1)^x L_\lambda^{(x)}(1) \tag{9}$$

where $L_\lambda^{(x)}(1)$ is the xth derivative of the Laplace Transform of the mixing distribution.

Illustration: Exponential with one parameter

$$g(\lambda) = \mu e^{-\mu\lambda}; \lambda \geq 0 \tag{10}$$

$$L(s) = \int_0^\infty e^{-\lambda s} \mu e^{-\mu\lambda} d\lambda$$

$$L(s) = \frac{\mu}{s+\mu} \tag{11}$$

$$L^{(x)}(1) = \frac{(-1)^x \, x! \mu}{(\mu+1)^{x+1}}$$

$$f(x) = \left(\frac{\mu}{\mu+1}\right)\left(\frac{1}{\mu+1}\right)^x ; x = 0, 1, 2, ... \qquad (12)$$

This result is similar to that obtained by direct integration.

6.4 Special Functions

The mixed Poisson distribution is obtained by comparing its integrand with a special function; in this case with a Confluent Hypergeometric distribution of the second kind given by:

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-xt} t^{a-1}}{(1+t)^{a-c+1}} dt \tag{13}$$

Illustration: Lomax Distribution

$$g(\lambda) = \frac{\alpha \beta^\alpha}{(\lambda + \beta)^{\alpha+1}}; \alpha > 0, \beta > 0, \lambda > 0 \tag{14}$$

Now

$$f(x) = \int_0^\infty \frac{e^{-\lambda} \lambda^x}{x!} \frac{\alpha \beta^\alpha}{(\lambda + \beta)^{\alpha+1}} d\lambda$$

$$f(x) = \frac{\alpha\beta^x}{x!} \int_0^\infty \frac{e^{-\beta t} t^{(x+1)-1}}{(1+t)^{\alpha+1}} dt \tag{15}$$

Comparing (13) and (15), then

$$f(x) = \alpha\beta^x \Psi(x+1, x-\alpha+1; \beta) \tag{16}$$

7 Conclusion

Some mixed Poisson distributions can be obtained using more than one of the methods considered in this work. For instance, to obtain Negative Binomial distribution, the method of explicit evaluation and that of using the Laplace Transform with Gamma as the mixing distribution were used. The two methods yielded the same result. This is a clear indication that there is no restriction on what kind of method to use for a particular given mixing distribution, that is, any method can be used wherever possible.

More work can be done using the methods of construction already used and also other methods can be studied or researched on.

Properties of Mixed Poisson distributions.